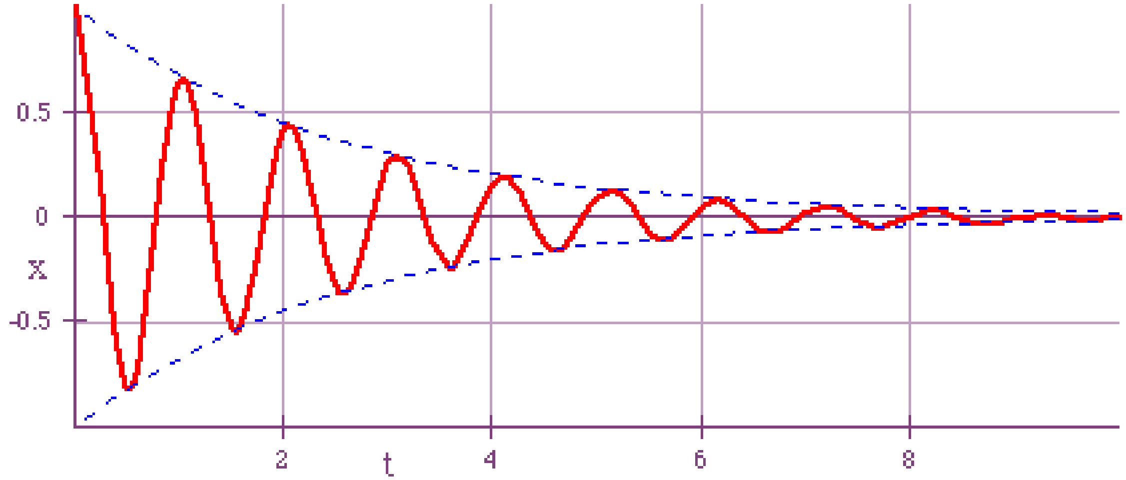
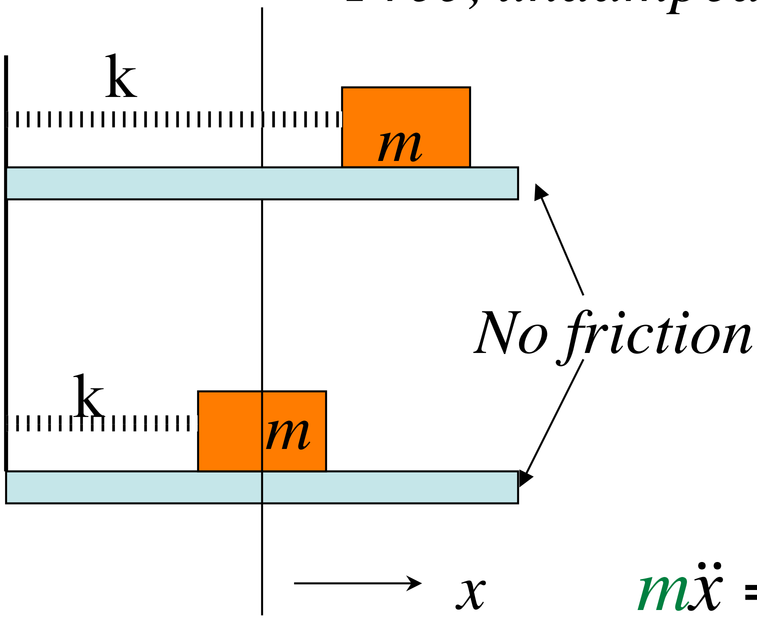
THE DAMPED HARMONIC OSCILLATOR

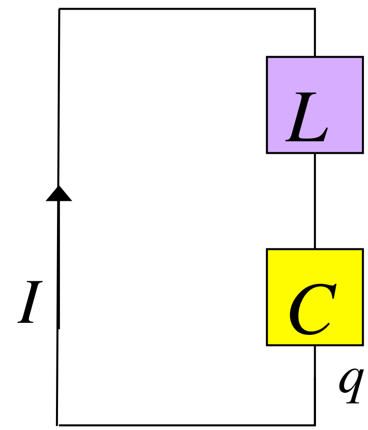
Reading:
Main 3.1, 3.2, 3.3
Taylor 5.4
Giancoli 14.7, 14.8



Free, undamped oscillators – other examples

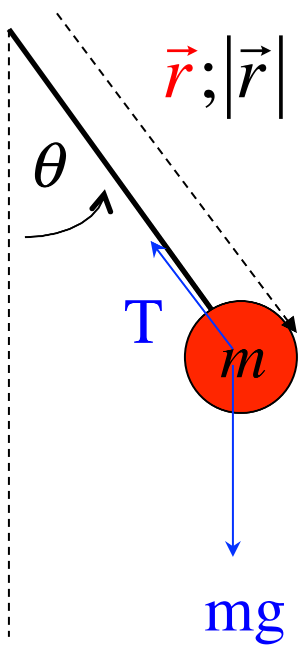


$$m\ddot{x} = -kx$$



$$\ddot{q} = -\frac{1}{LC}q$$

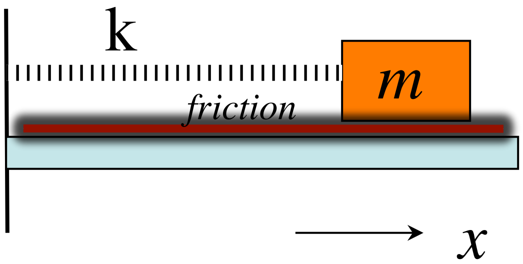
$$\vec{r};|\vec{r}| = L$$



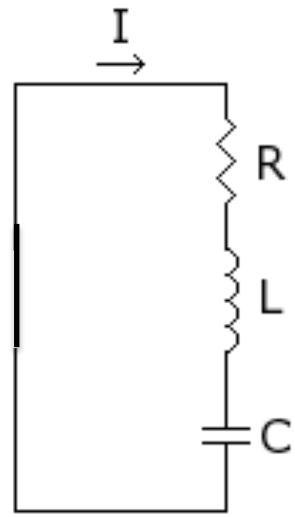
$$\ddot{\theta} \approx -\frac{g}{L}\theta$$

Common notation for all

$$\ddot{\psi} + \omega_0^2\psi = 0$$

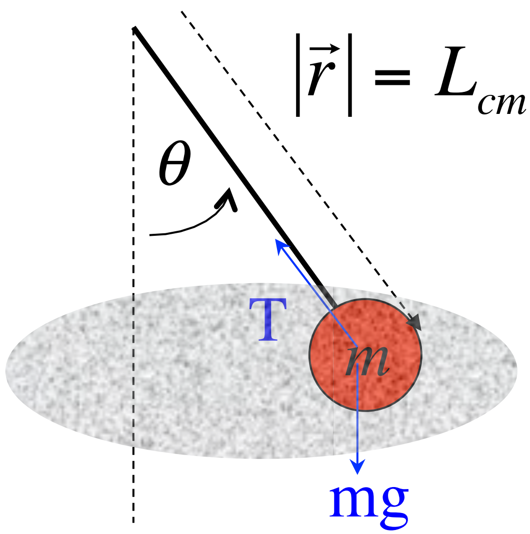


$$m\ddot{x} = -kx - b\dot{x}$$



$$L\dot{I} + \frac{1}{C}q + RI = 0$$

$$L\ddot{q} + \frac{1}{C}q + R\dot{q} = 0$$



$$\ddot{\theta} \approx -\frac{g}{L}\theta - b'\dot{\theta}$$

Common notation for all

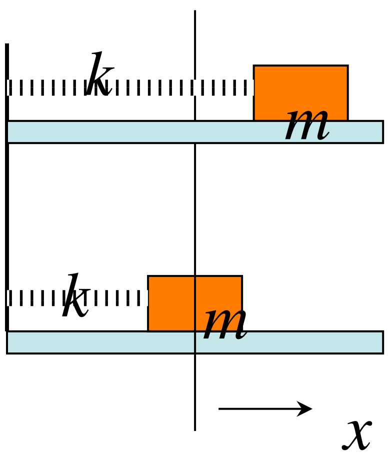
$$\ddot{\psi} + 2\beta\dot{\psi} + \omega_0^2\psi = 0$$

Natural motion of damped harmonic oscillator

$$Force = m\ddot{x}$$

$$restoring\ \ force + resistive\ \ force = m\ddot{x}$$

$$-kx$$



Need a model for this.

Try restoring force proportional to velocity

$$-b\dot{x}$$

How do we choose a model?

Physically reasonable, mathematically tractable …

Validation comes IF it describes the experimental system accurately

Natural motion of damped harmonic oscillator

$$Force = m\ddot{x}$$

$$restoring\ \ force + resistive\ \ force = m\ddot{x}$$

$$-kx - b\dot{x} = m\ddot{x}$$

Divide by coefficient of d^2x/dt^2 and rearrange:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

inverse time

β and ω_0 (rate or frequency) are generic to any oscillating system
This is the notation of TM; Main uses $\gamma = 2\beta$.

Natural motion of damped harmonic oscillator

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0$$

Try $\quad x(t) = Ce^{pt} \quad$ C, p are unknown constants

$$\dot{x}(t) = px(t), \qquad \ddot{x}(t) = p^2 x(t)$$

Substitute: $\quad \left(p^2 + 2\beta p + \omega_0^2\right)x(t) = 0$

Now p is known (and there are 2 p values) $\quad p = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$

$$x(t) = Ce^{p_+ t} + C'e^{p_- t}$$

Must be sure to make x real!

Natural motion of damped HO

Can identify 3 cases

$\beta < \omega_0$

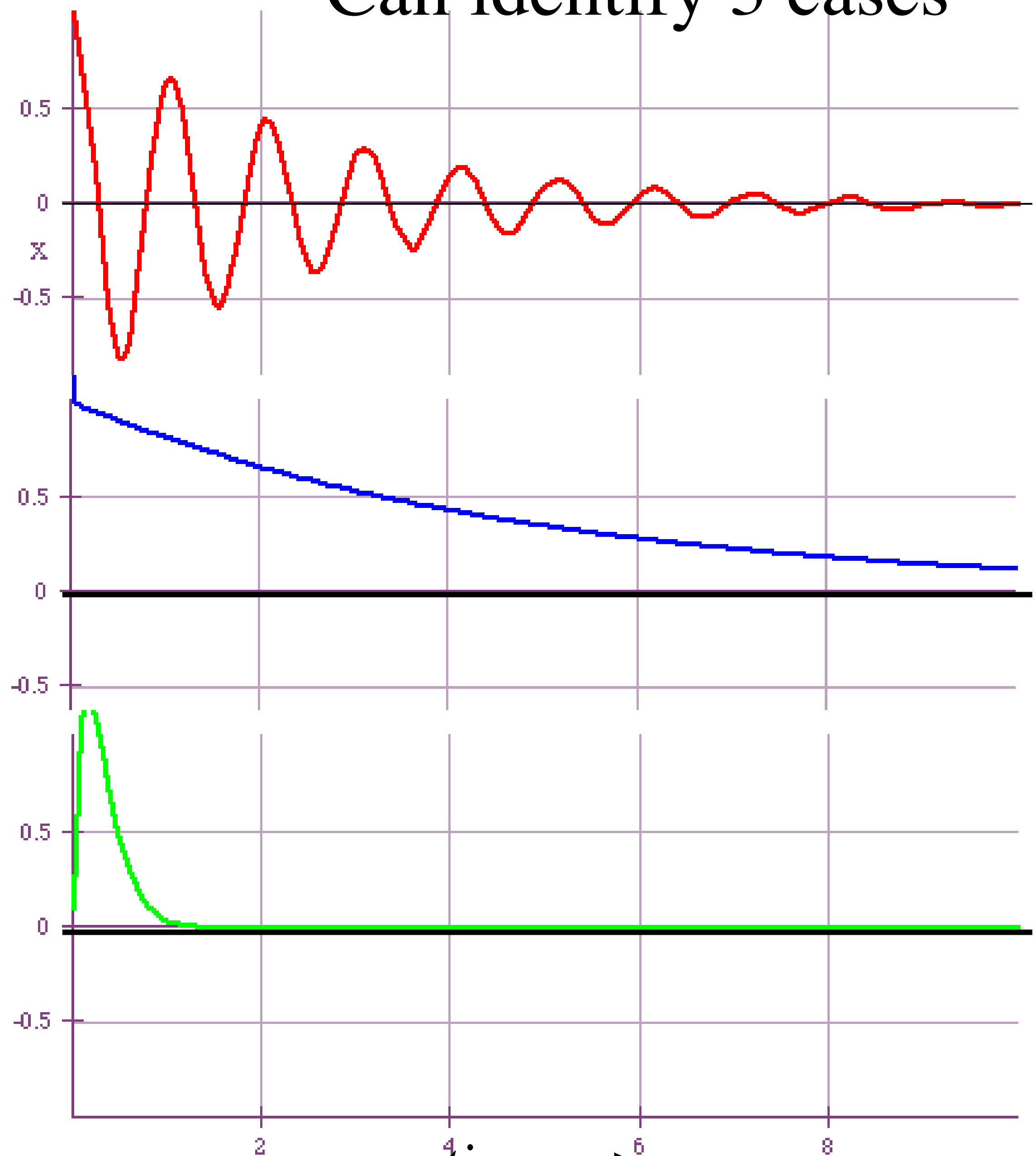
underdamped

$\beta > \omega_0$

overdamped

$\beta = \omega_0$

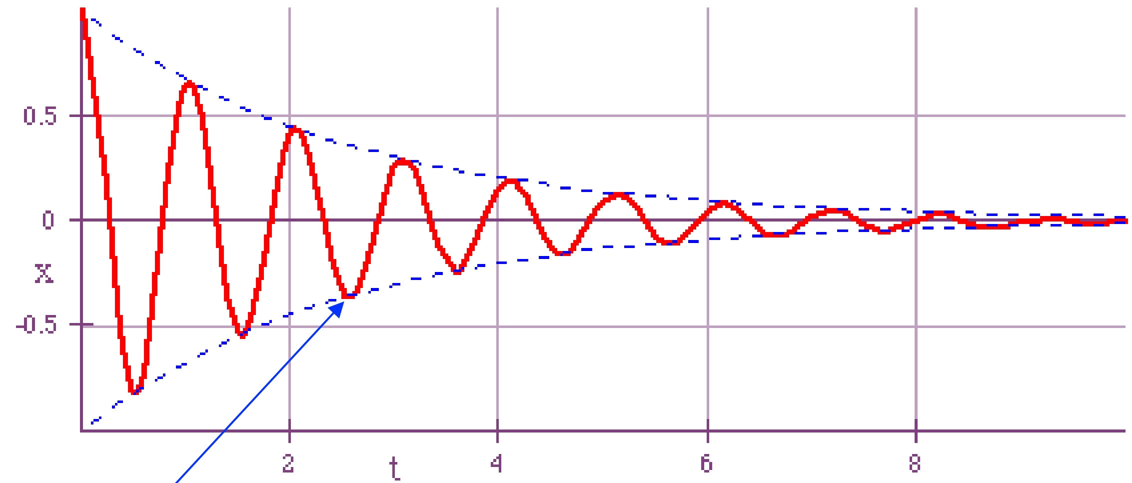
critically damped



time --->

underdamped

$$\beta < \omega_0$$



time --->

$$\omega_1 = \omega_0 \sqrt{1 - \frac{\beta^2}{\omega_0^2}}$$

$$p = -\beta \pm \sqrt{\beta^2 - \omega_0^2} = -\beta \pm i\omega_1$$

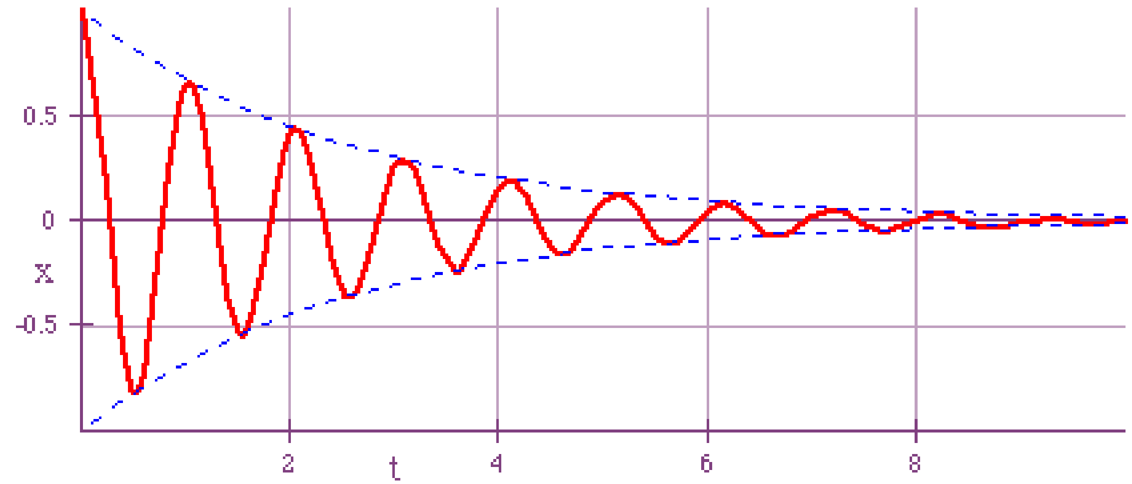
$$x(t) = Ce^{-\beta t + i\omega_1 t} + C^* e^{-\beta t - i\omega_1 t}$$ Keep $x(t)$ real

$$x(t) = Ae^{-\beta t}\left[\cos(\omega_1 t + \delta)\right]$$ complex <-> amp/phase

System oscillates at "frequency" ω_1 (very close to ω_0)
- but in fact there is not only one single frequency associated with the motion as we will see.

underdamped

$\beta < \omega_0$



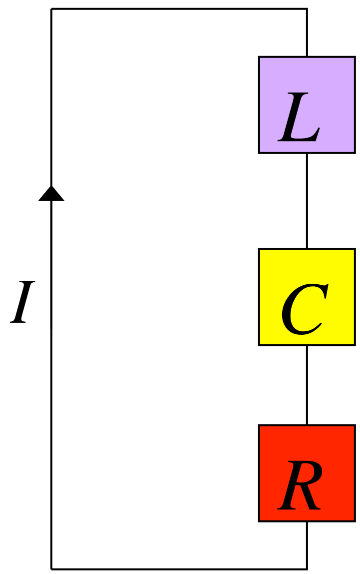
Damping time or "1/e" time is $\tau = 1/\beta > 1/\omega_0$
($\gg 1/\omega_0$ if β is very small)

How many T_0 periods elapse in the damping time?
This number (times π) is the **Quality factor** or Q of the system.

$$Q = \pi \frac{\tau}{T_0} = \frac{\omega_0}{2\beta}$$

large if β is small compared to ω_0

LRC circuit



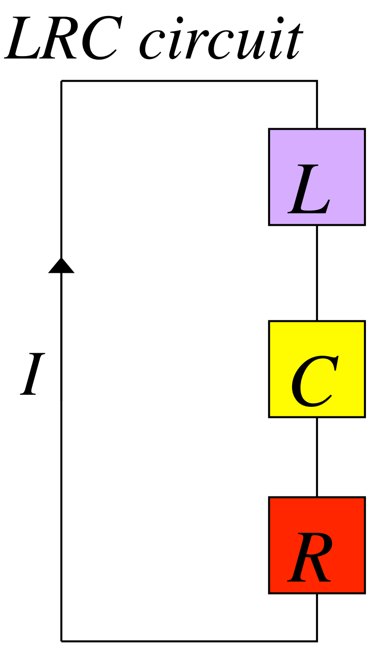
$$V_L = L\frac{dI}{dt}; V_R = IR; V_C = \frac{q}{C}$$

$$-L\frac{dI}{dt} - IR - \frac{q}{C} = 0$$

L (inductance), *C (capacitance)*, cause oscillation, *R* (*resistance*) causes damping

$$L\ddot{q} + R\dot{q} + \frac{q}{C} = 0$$

$$\ddot{q} + 2\beta\dot{q} + \omega_0^2 q = 0 \qquad \ddot{q} + \frac{R}{L}\dot{q} + \frac{1}{LC}q = 0$$



LCR circuit obeys precisely the same equation as the damped mass/spring.

Q factor:

$$Q = \frac{1}{\omega_0 RC}$$

Natural (resonance) frequency determined by the inductor and capacitor

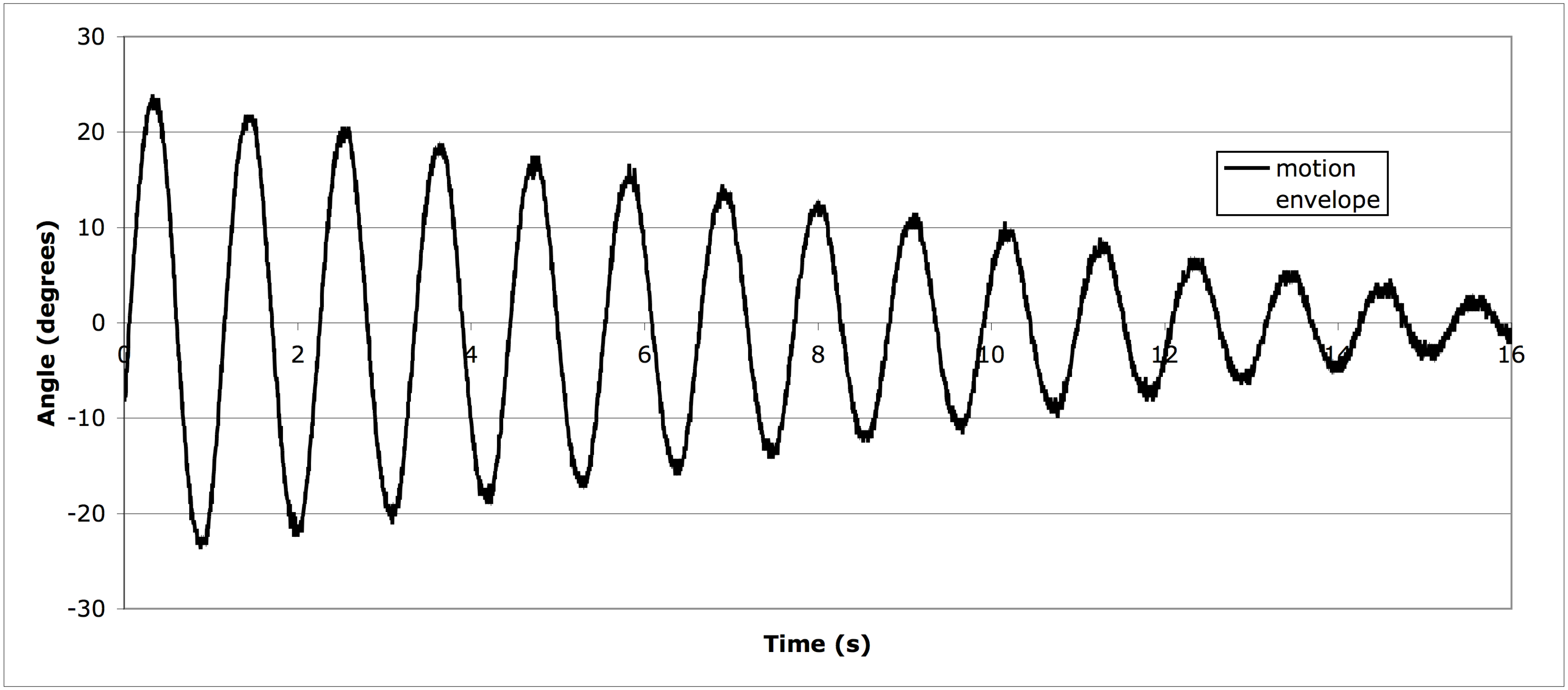
$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Damping determined by resistor & inductor

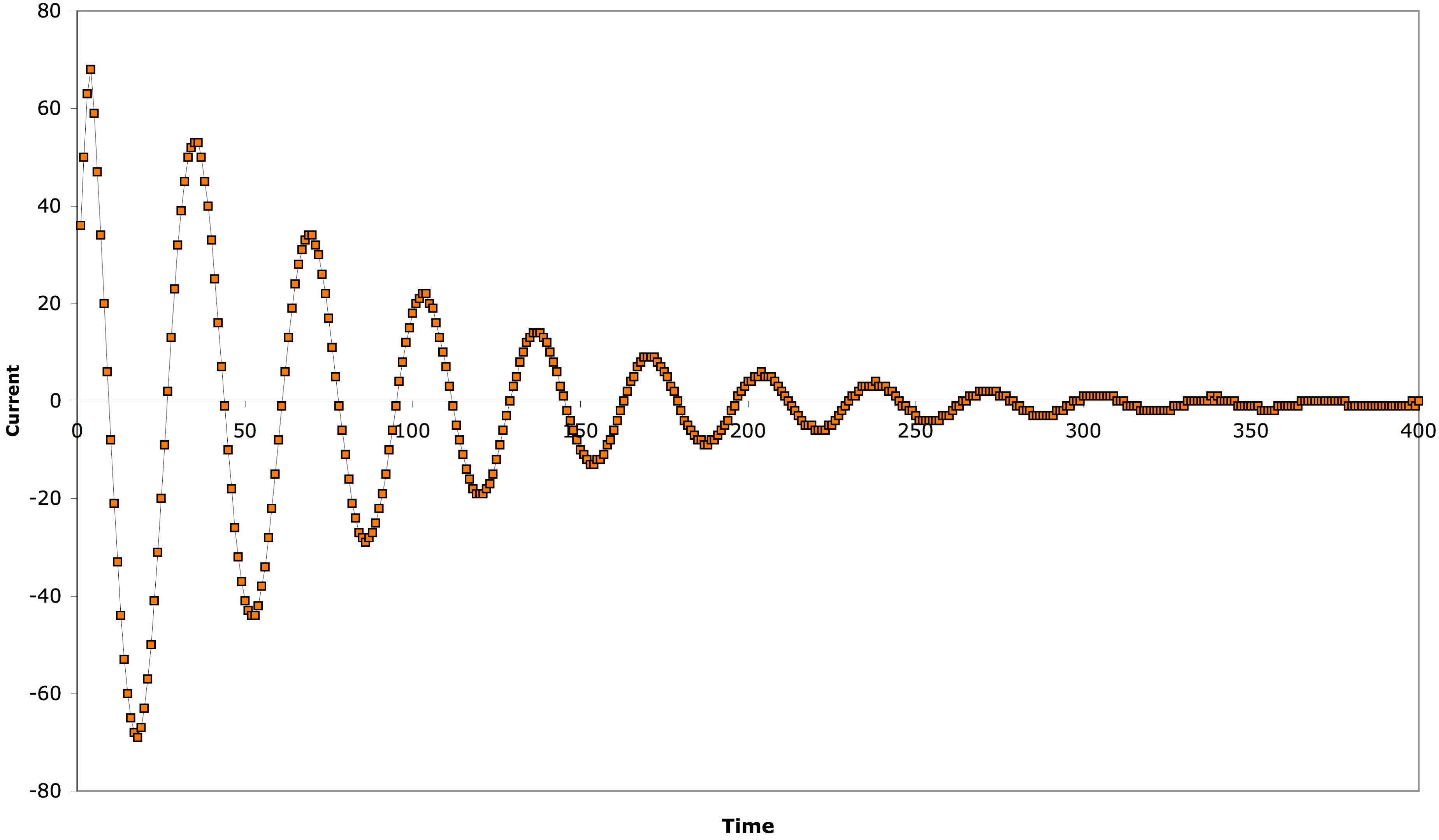
$$\beta = \frac{R}{2L}$$

Typical numbers:
$L \approx 500\mu$H; $C \approx 100$pF; $R \approx 50\Omega$
$\omega_0 \approx 10^6 s^{-1}$ ($f_0 \approx 700$ kHz)
$\tau = 1/\beta \approx 2\mu$s; $Q \approx 45$
(your lab has different parameters)

Does the model fit?



Does the model fit?



Summary so far:

- Free, undamped, linear (harmonic) oscillator
- Free, undamped, non-linear oscillator
- Free, damped linear oscillator

Next

- Driven, damped linear oscillator
- Laboratory to investigate LRC circuit as example of driven, damped oscillator
- Time and frequency representations
- Fourier series