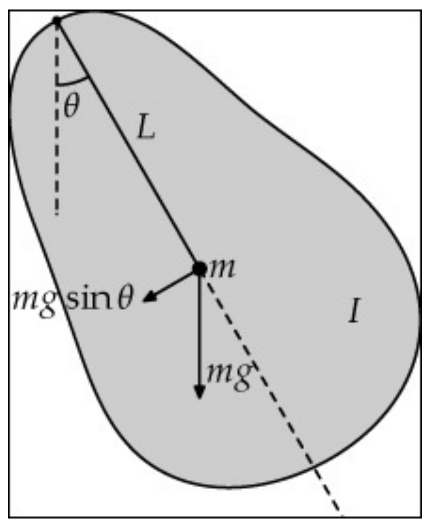
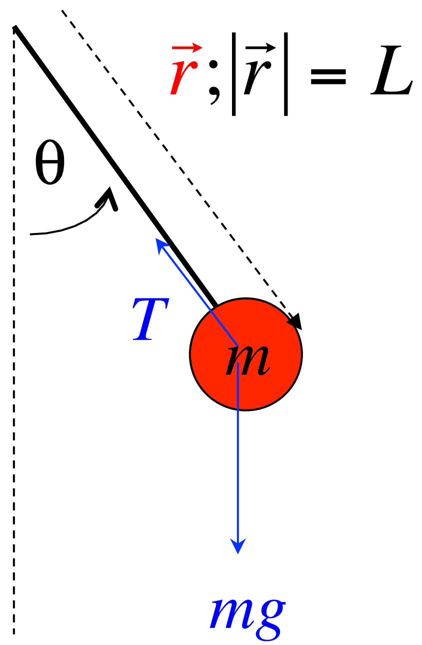
SIMPLE HARMONIC MOTION: NEWTON'S LAW

PRIOR READING:
Main 1.1, 2.1
Taylor 5.1, 5.2



http://www.myoops.org/twocw/mit/NR/rdonlyres/Physics/8-012Fall-2005/7CCE46AC-405D-4652-A724-64F831E70388/0/chp_physi_pndulm.jpg



The simple pendulum

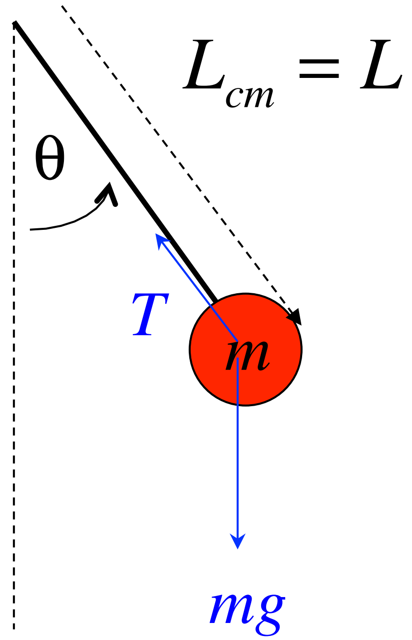
Energy approach

$$\int_0^t t' = \int_{\theta_0}^{\theta} \frac{d\theta'}{\sqrt{\frac{2}{I}\left(E - mgL_{cm}\left(1-\cos\theta'\right)\right)}}$$

$$t = \int_{\theta_0}^{\theta} \frac{d\theta'}{\sqrt{\frac{2mgL_{cm}}{mL_{cm}^2}\left(1-\cos\theta_{\max} - \left(1-\cos\theta'\right)\right)}}$$

$$t = \int_{\theta_0}^{\theta} \frac{d\theta'}{\sqrt{\frac{2g}{L_{cm}}\left(\cos\theta' - \cos\theta_{\max}\right)}}$$

The simple pendulum



$$\boxed{\omega_0 \equiv \sqrt{\frac{g}{L}} \quad \text{angular frequency}}$$

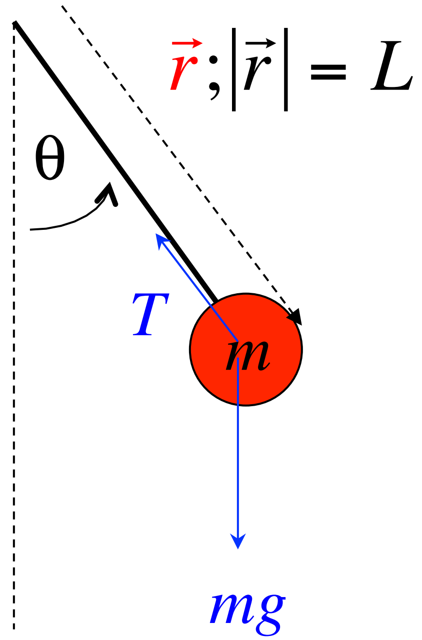
Energy approach

$$t = \int_{\theta_0}^{\theta} \frac{d\theta'}{\sqrt{\frac{2g}{L_{cm}}\left(\cos\theta' - \cos\theta_{\max}\right)}}$$

$$t = \int_{\theta_0}^{\theta} \frac{d\theta'}{\sqrt{\frac{g}{L}\left(\theta_{\max}^2 - \theta'^2\right)}} = \frac{1}{\omega_0} \arcsin \frac{\theta'}{\theta_{\max}}\Bigg|_{\theta_0}^{\theta}$$

$$\omega t = \arcsin\frac{\theta}{\theta_{\max}} - \underbrace{\arcsin\frac{\theta_0}{\theta_{\max}}}_{\phi} \qquad \Rightarrow \theta(t) = \theta_{\max}\sin(\omega t + \phi)$$

The simple pendulum



$$\vec{\tau} = I\ddot{\vec{\theta}}$$ Newton

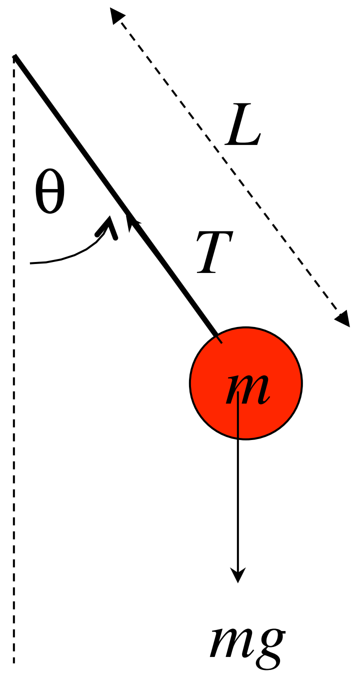
$$\vec{\tau} = \vec{r} \times \sum \vec{F} = ?$$ Known torque

$$\vec{\tau} = Lmg\sin\theta\,\hat{\otimes} \qquad I\ddot{\vec{\theta}} = mL^2\ddot{\theta}\,\hat{\bullet}$$

$$\ddot{\theta} = -\frac{g}{L}\sin\theta$$

This is NOT a restoring force proportional to displacement (Hooke's law motion) in general, but IF we consider small motion, IT IS! Expand the sin series …

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \ldots$$



The simple pendulum

in the limit of small angular displacements

$$\ddot{\theta} = -\frac{g}{L}\sin\theta \rightarrow$$

$$\ddot{\theta} = -\frac{g}{L}\theta \quad \text{or} \quad \ddot{\theta} + \frac{g}{L}\theta = 0$$

Compare with $\ddot{x} + \dfrac{k}{m}x = 0$

What is $\theta(t)$ such that the above equation is obeyed?

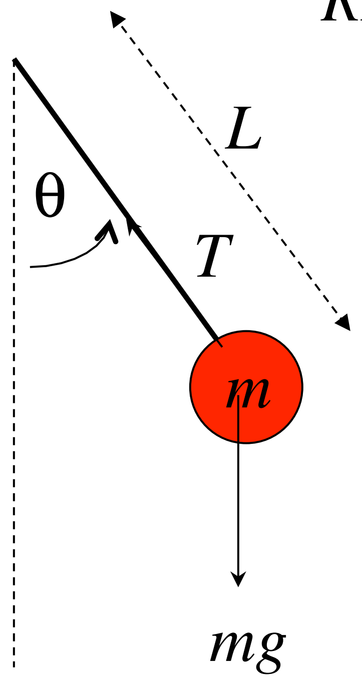
θ is a variable that describes position

t is a parameter that describes time

"dot" and "double dot" mean *differentiate w.r.t. time*

g, L are known constants, determined by the system.

REVIEW PENDULUM



$$\ddot{\theta} + \frac{g}{L}\theta = 0$$

$$\theta(t) = Ce^{pt}$$

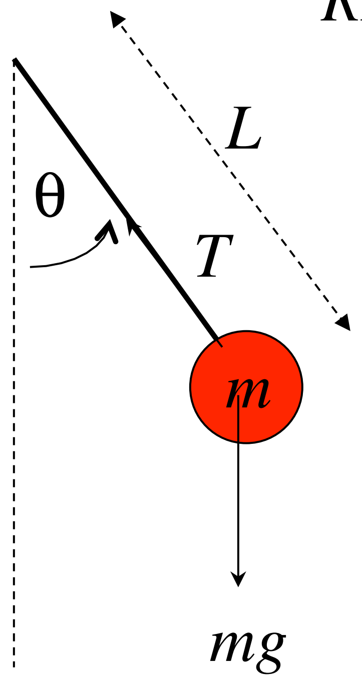
C, p are unknown (for now) constants, possibly complex

$$\boxed{\ddot{\theta}(t) = p^2 Ce^{pt} = p^2\theta(t)}$$

Substitute: $$p^2\theta + \frac{g}{L}\theta = 0 \qquad p = \pm i\sqrt{\frac{g}{L}} = \pm i\omega_0$$

p is now known (but C is not!). Note that ω_0 is NOT a new quantity! It is just a rewriting of old ones - partly shorthand, but also "ω" means "frequency" to physicists!

REVIEW PENDULUM



TWO possibilities …. general solution is the sum of the two and it must be real (all angles are real).

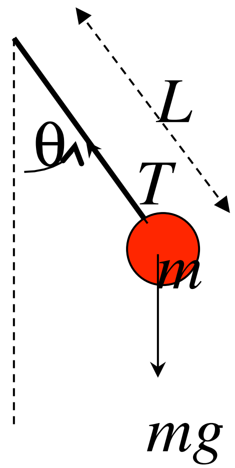
$$\boxed{\theta(t) = Ce^{i\omega_0 t} + C'e^{-i\omega_0 t}}$$

If we force $C' = C^*$ (complex conjugate of C), then $x(t)$ is real, and there are only 2 constants, Re[C], and Im[C]. A second order DEQ can determine only 2 arbitrary constants.

$$\boxed{\theta(t) = Ce^{i\omega_0 t} + C^*e^{-i\omega_0 t}}$$

Simple harmonic motion

REVIEW PENDULUM



$$\boxed{\theta(t) = C e^{i\omega_0 t} + C^* e^{-i\omega_0 t}}$$

Re[C], Im[C] chosen to fit initial conditions. Example:

θ(0) = 0 rad and dθdt(0) = 0.2 rad/sec

$$\theta(0) = C\underbrace{e^{i\omega_0 0}}_{1} + C^*\underbrace{e^{-i\omega_0 0}}_{1}$$

$$0 = C + C^* = 2\,\mathrm{Re}[C]$$

$$\Rightarrow \mathrm{Re}[C] = 0$$

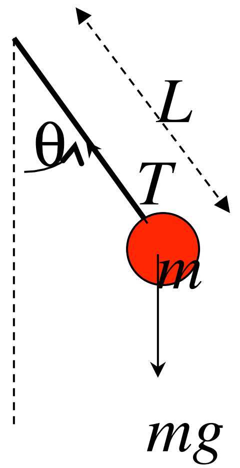
$$\dot{\theta}(0) = i\omega_0 C\underbrace{e^{i\omega_0 0}}_{1} - i\omega_0 C^*\underbrace{e^{-i\omega_0 0}}_{1}$$

$$0.2 rad/s = i\omega_0\left(C - C^*\right) = i\omega_0 2i\,\mathrm{Im}[C]$$

$$\Rightarrow \mathrm{Im}[C] = \frac{0.2}{-2\omega_0} = \frac{0.1}{\omega_0}$$

$$\boxed{C = \underbrace{0 + i\frac{0.1}{\omega_0}}_{cartesian} = \underbrace{\frac{0.1}{\omega_0}e^{i\frac{\pi}{2}}}_{polar}; \quad C^* = ?}$$

REVIEW PENDULUM



$$\theta(t) = Ce^{i\omega_0 t} + C^* e^{-i\omega_0 t}; \quad C = \frac{0.1}{\omega_0} e^{i\frac{\pi}{2}}$$

$$\boxed{\begin{aligned}
\theta(t) &= \frac{0.1}{\omega_0} e^{i\frac{\pi}{2}} e^{i\omega_0 t} + \frac{0.1}{\omega_0} e^{-i\frac{\pi}{2}} e^{-i\omega_0 t} \\
\theta(t) &= \frac{0.1}{\omega_0}\left(e^{i(\omega_0 t + \pi/2)} + e^{-i(\omega_0 t + \pi/2)} \right) \\
\theta(t) &= \frac{0.1}{\omega_0}\left(2\cos\left(\omega_0 t + \pi/2\right)\right) \\
\theta(t) &= \underbrace{\frac{0.2}{\omega_0}}_{A} \cos\left(\omega_0 t + \underbrace{\pi/2}_{\phi}\right)
\end{aligned}}$$

Remember, all these are equivalent forms. All of them have a known $\omega_0=(g/L)^{1/2}$, and all have 2 more undetermined constants that we find … how?

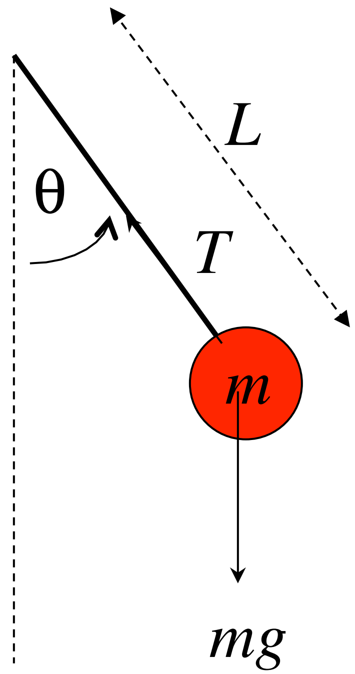
$$\theta(t) = A\cos\left(\omega_0 t + \phi\right)$$

$$\theta(t) = B_p \cos\omega_0 t + B_q \sin\omega_0 t$$

$$\theta(t) = C\exp\left(i\omega_0 t\right) + C^*\exp\left(-i\omega_0 t\right)$$

$$\theta(t) = \mathrm{Re}\left[D\exp\left(i\omega_0 t\right)\right]$$

Do you remember how the A, B, C, D constants are related? If not, go back and review until it becomes second nature!



The simple pendulum

("simple" here means a point mass; your lab deals with a plane pendulum)

$$\ddot{\theta} = -\frac{g}{L}\theta$$

$$\theta(t) = \theta_{\max}\cos(\omega_0 t + \phi)$$

simple harmonic motion

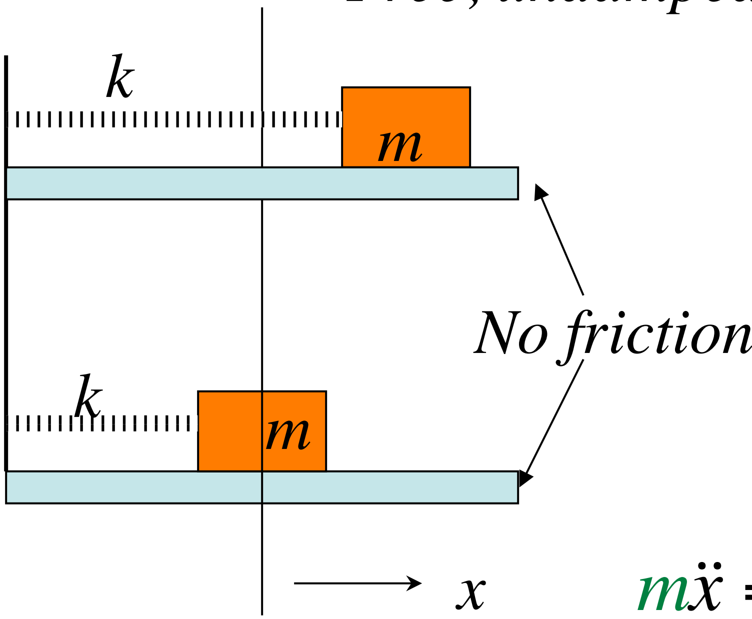
(⚡ potential confusion!! A "simple" pendulum does not <u>always</u> execute "simple harmonic motion"; it does so only in the limit of small amplitude.)

$$\omega_0 = \sqrt{\frac{g}{L}}$$

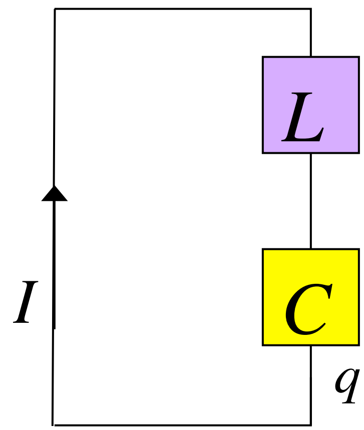
$$T = 2\pi\sqrt{\frac{L}{g}}$$

Period does not depend on θ_{max}, ϕ

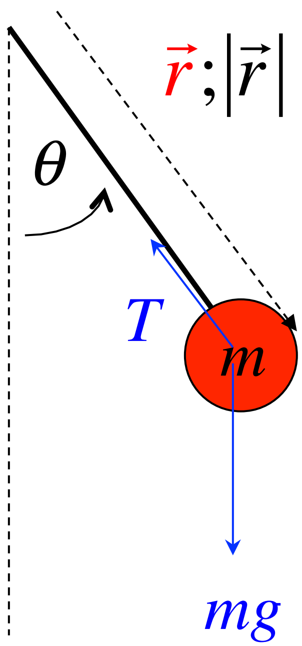
Free, undamped oscillators – other examples



$$m\ddot{x} = -kx$$



$$\ddot{q} = -\frac{1}{LC}q$$



$$\vec{r};\left|\vec{r}\right| = L$$

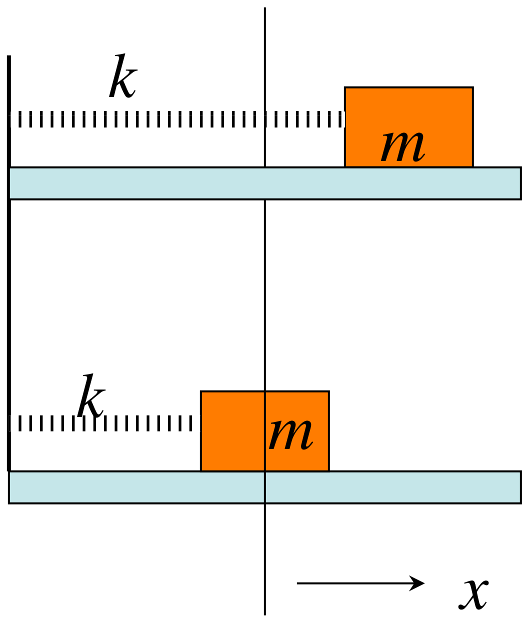
$$\ddot{\theta} \approx -\frac{g}{L}\theta$$

Common notation for all

$$\ddot{\psi} + \omega_0^2\psi = 0$$

- The following slides simply repeat the previous discussion, but now for a mass on a spring, and for a series LC circuit

REVIEW MASS ON IDEAL SPRING



$$F(x) = m\ddot{x}$$

Newton

$$\boxed{F(x) = -kx}$$

Particular type of force.
m, k known

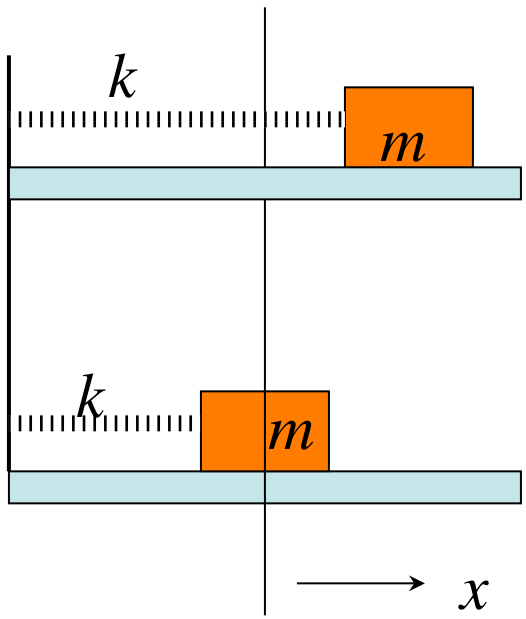
$$-kx = m\ddot{x}$$

$$\ddot{x} + \frac{k}{m}x = 0$$

Linear, 2nd order differential equation

What is $x(t)$ such that the above equation is obeyed?
x is a variable that describes position
t is a parameter that describes time
"dot" and "double dot" mean *differentiate w.r.t. time*
m, k are known constants

REVIEW MASS ON IDEAL SPRING



$$\ddot{x} + \frac{k}{m}x = 0$$

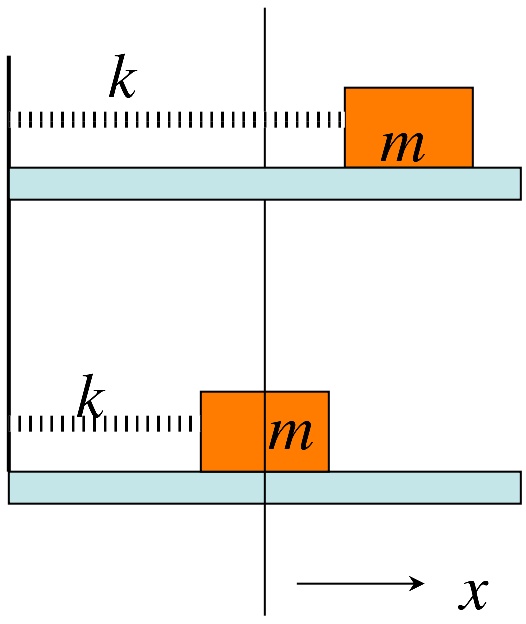
$$x(t) = Ce^{pt}$$

C, p are unknown (for now) constants, possibly complex

$$\boxed{\ddot{x}(t) = p^2 C e^{pt} = p^2 x(t)}$$

Substitute: $$p^2 x + \frac{k}{m}x = 0 \qquad p = \pm i\sqrt{\frac{k}{m}} = \pm i\omega_0$$

p is now known. Note that ω_0 is NOT a new quantity! It is just a rewriting of old ones - partly shorthand, but also "ω" means "frequency" to physicists!



$$x(t) = A\cos\left(\omega_0 t + \phi\right)$$

A, ϕ chosen to fit initial conditions:
$x(0) = x_0$ and $v(0) = v_0$

$$x_0 = A\cos\phi$$

$$v_0 = -\omega_0 A\sin\phi$$

Square and add:

$$x_0^2 + \frac{v_0^2}{\omega_0^2} = A^2\left(\cos^2\phi + \sin^2\phi\right) = A^2$$

Divide:

$$\frac{-v_0}{\omega_0 x_0} = \tan\phi$$

$$x(t) = A\cos\left(\omega_0 t + \phi\right)$$

$$x(t) = \sqrt{x_0^2 + \frac{v_0^2}{\omega_0^2}}\cos\left(\sqrt{\frac{k}{m}}t + \arctan\left(\frac{-v_0}{\omega_0 x_0}\right)\right)$$

$$x(t) = A\cos\phi\cos\omega_0 t - A\sin\phi\sin\omega_0 t$$

$$x(t) = \frac{Ae^{i\phi}}{2}e^{i\omega_0 t} + \frac{Ae^{-i\phi}}{2}e^{-i\omega_0 t}$$

$$x(t) = \mathrm{Re}\left[Ae^{i\phi}e^{i\omega_0 t}\right]$$

2 arbitrary constants (A, ϕ) because **2nd order** linear differential equation

Position: $x(t) = A\cos(\omega_0 t + \phi)$

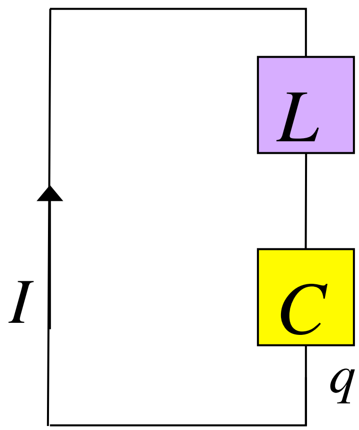
- A, ϕ are **unknown** constants - must be determined from **initial conditions**
- ω_0, in principle, is known and is a characteristic of the physical system

Velocity: $$\frac{dx}{dt} \equiv \dot{x}(t) = -\omega_0 A\sin(\omega_0 t + \phi)$$

Acceleration: $$\frac{d^2x}{dt^2} \equiv \ddot{x}(t) = -\omega_0^2 A\cos(\omega_0 t + \phi)$$
$$= -\omega_0^2 x(t)$$

This type of pure sinusoidal motion with a **single frequency** is called SIMPLE HARMONIC MOTION

THE LC CIRCUIT



$$V_L + V_C = 0$$

Kirchoff's law
(not Newton this time)

$$V_L = L\frac{dI}{dt} = L\frac{d^2q}{dt^2} = L\ddot{q}$$

$$V_C = \frac{q}{C}$$

$$\ddot{q} = -\frac{1}{LC}q$$