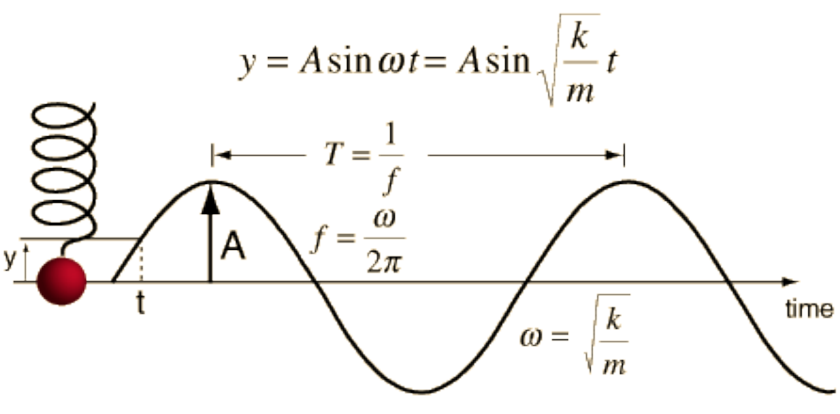
REPRESENTING SIMPLE HARMONIC MOTION

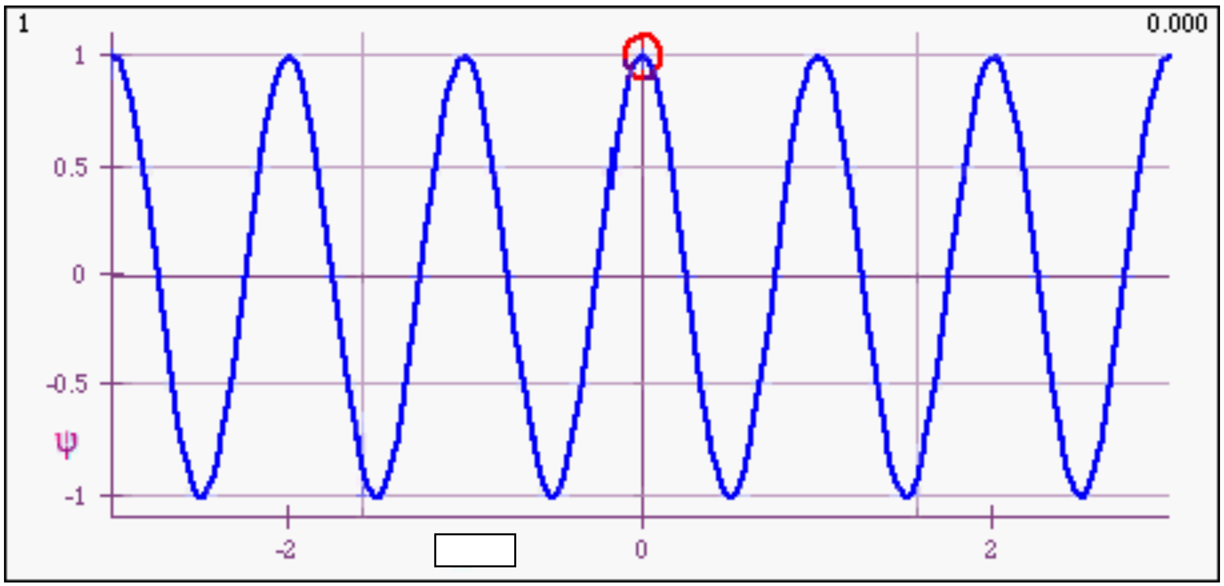
PRIOR READING:
Main 1.1 & 1.2
Taylor 5.2



http://hyperphysics.phy-astr.gsu.edu/hbase/imgmec/shm.gif

Simple Harmonic Motion

$y(t)$



Watch as time evolves

$$x(t) = A\cos\left(\omega_0 t + \phi\right)$$

amplitude

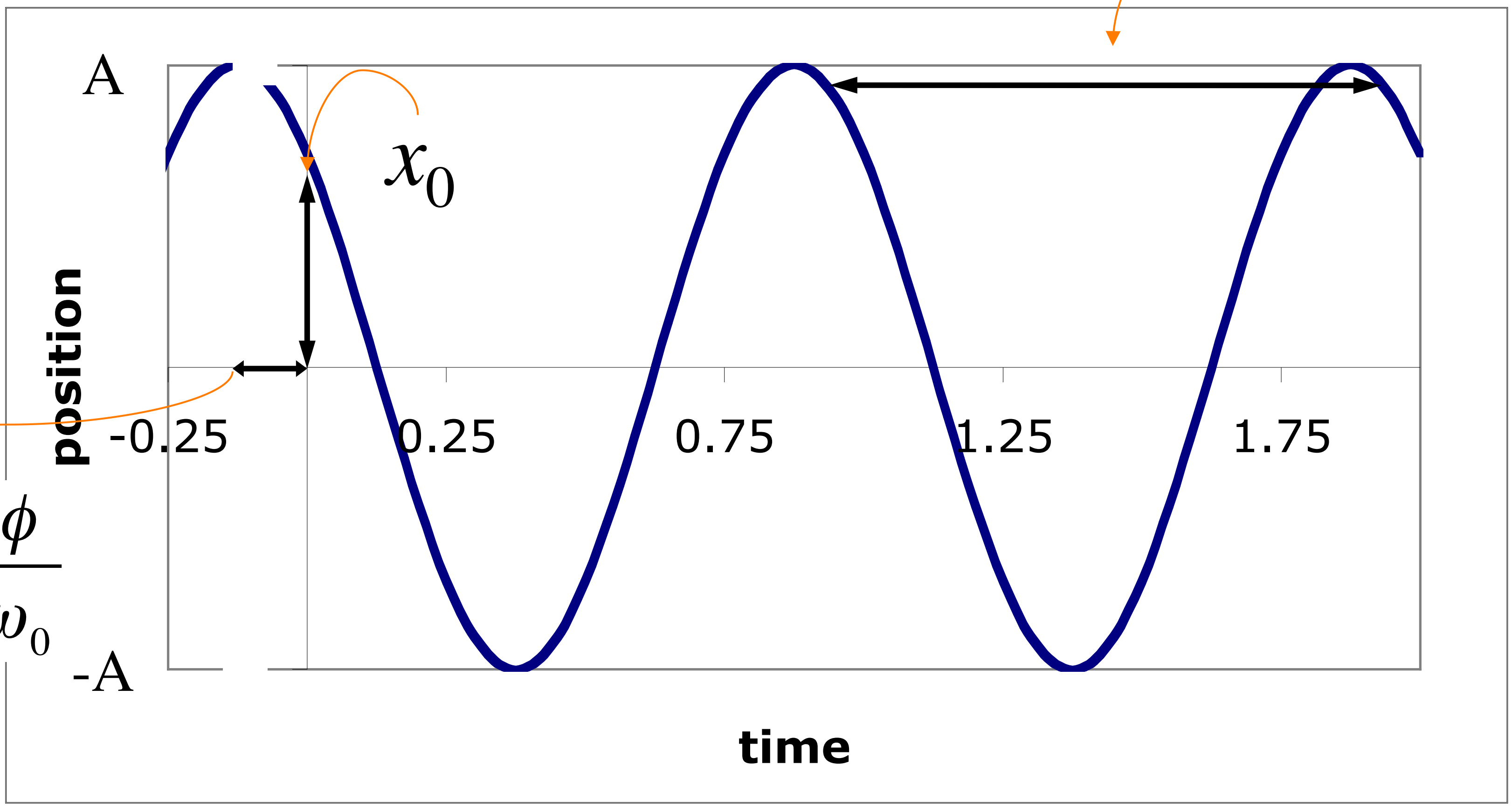
phase angle

period

$$T = \frac{2\pi}{\omega_0} = \frac{1}{f}$$

angular freq

(cyclic) freq



$$t = -\frac{\phi}{\omega_0}$$

determined by initial conditions

determined by physical system

Position (cm)

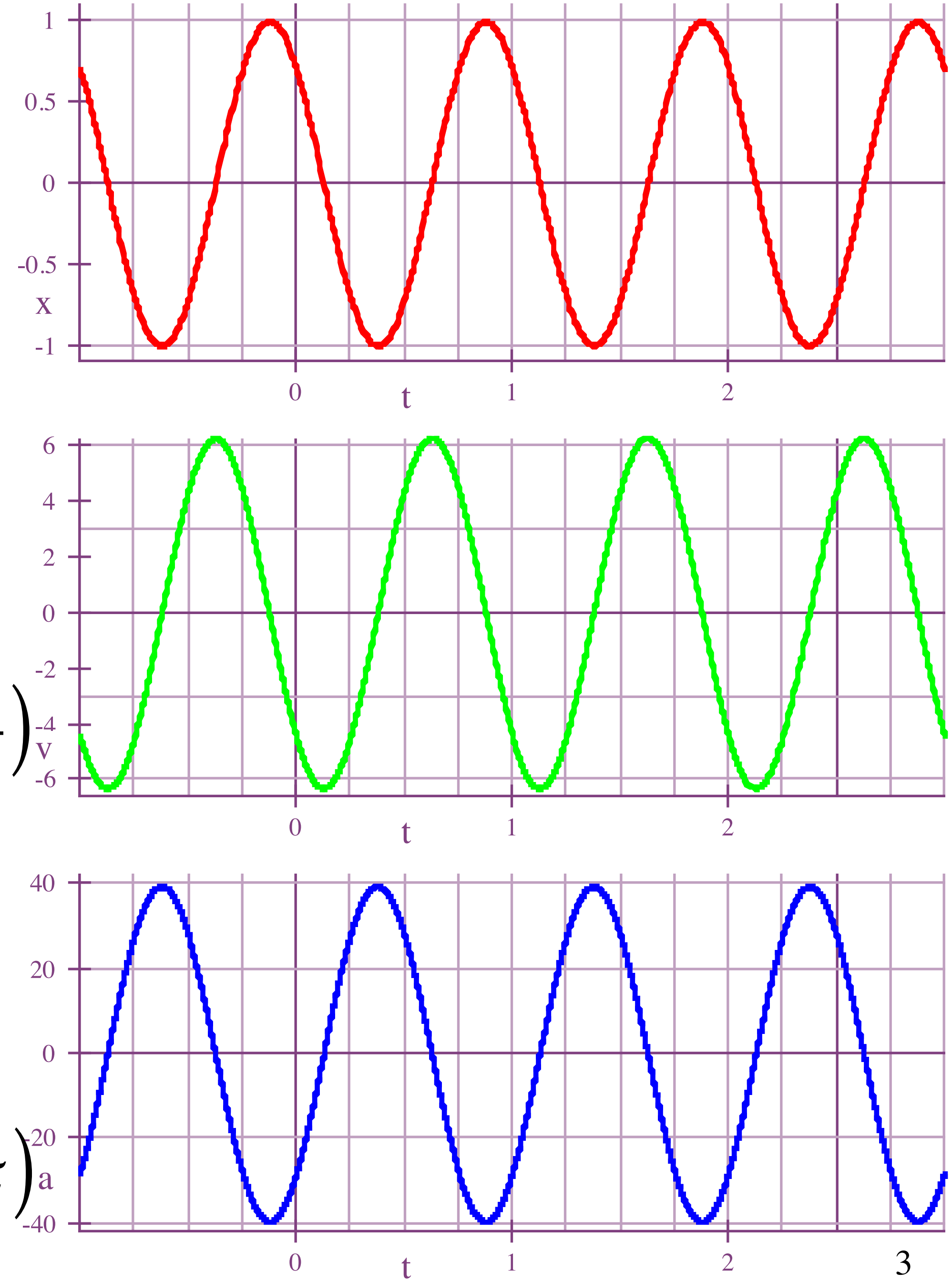
$$x(t) = A\cos\left(\omega_0 t + \phi\right)$$

Velocity (cm/s)

$$\dot{x}(t) = A\omega_0 \cos\left(\omega_0 t + \phi + \tfrac{\pi}{2}\right)$$

Acceleration (cm/s^2)

$$\ddot{x}(t) = A\omega_0^2 \cos\left(\omega_0 t + \phi + \pi\right)$$



time (s)

These representations of the position of a simple harmonic oscillator as a function of time are all equivalent - there are 2 arbitrary constants in each. Note that A, ϕ, B_p and B_q are REAL; *C and D are* COMPLEX.

$x(t)$ is real-valued variable in all cases. MAIN Eq. 1.22

$$x(t) = A\cos(\omega_0 t + \phi)$$

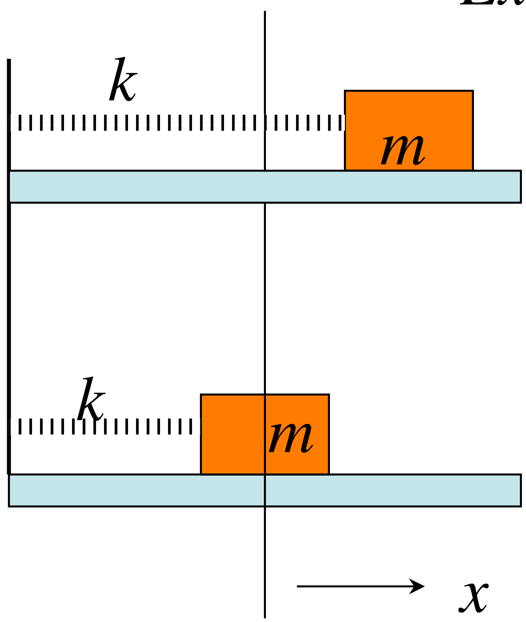
$$x(t) = B_p \cos\omega_0 t + B_q \sin\omega_0 t$$

$$x(t) = C\exp(i\omega_0 t) + C^*\exp(-i\omega_0 t)$$

$$x(t) = \mathrm{Re}\left[D\exp(i\omega_0 t)\right]$$

Engrave these on your soul - and know how to derive the relationships among A & ϕ; B_p & B_q; C; and D .

Example: initial conditions



$$x(t) = A\cos(\omega_0 t + \phi)$$

$$x(t) = B_p \cos\omega_0 t + B_q \sin\omega_0 t$$

$m = 0.01$ kg; $k = 36$ Nm^{-1}. At $t = 0$, m is displaced 50mm to the right and is moving to the right at 1.7 ms^{-1}.
Express the motion in
form A (even groups)
form B (odd groups)

$$x(t) = A\cos\left(\omega_0 t + \phi\right)$$

$$x(t) = 57.5\cos\left(\left[60s^{-1}\right]t - 0.516\right)\mathrm{mm}$$

$$x(t) = B_p \cos\omega_0 t + B_q \sin\omega_0 t$$

$$x(t) = 50mm\cos\left(\left[60s^{-1}\right]t\right) + 28.3mm\sin\left(\left[60s^{-1}\right]t\right)$$

$$\boxed{\begin{aligned} B_p &= A\cos\phi \\ B_q &= -A\sin\phi \end{aligned}}$$

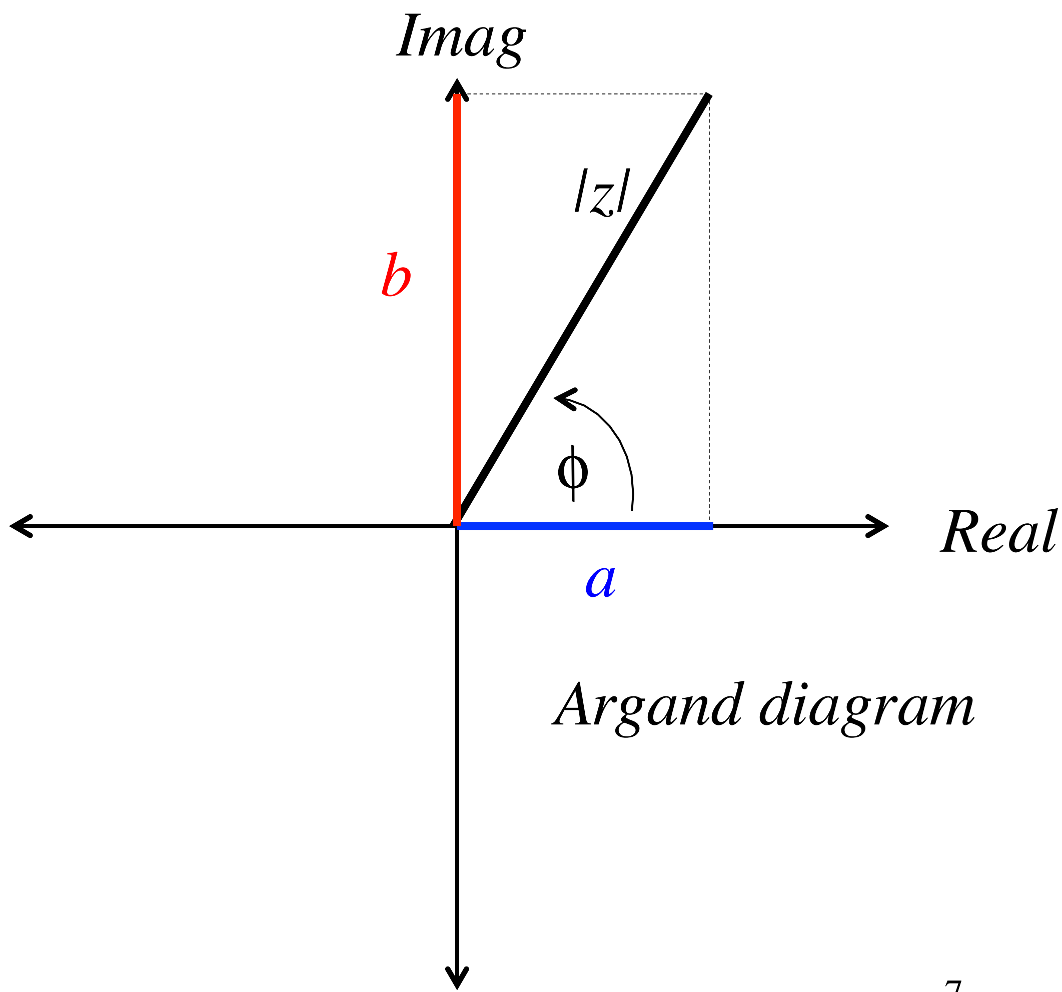
$$\boxed{\begin{aligned} A &= \sqrt{B_p{}^2 + B_q{}^2} \\ \tan\phi &= -\frac{B_q}{B_p} \end{aligned}}$$

Complex numbers

$$i = \sqrt{-1}$$

$$z = a + ib \qquad\qquad z = |z|e^{i\phi}$$

$$\left.\begin{array}{l}\mathrm{Re}(z) = a \\ \mathrm{Im}(z) = b\end{array}\right\}\text{real numbers}$$



Argand diagram

$$\boxed{|z| = \sqrt{a^2 + b^2}}$$

$$\boxed{\tan\phi = \frac{b}{a}}$$

Euler's relation

$$\boxed{\exp(i\phi) = \cos\phi + i\sin\phi}$$

$$\exp(i\omega_0 t) = \cos(\omega_0 t) + i\sin(\omega_0 t)$$

Consistency argument

$$z = a + ib \qquad\qquad z = |z|e^{i\phi}$$

If these represent the same thing, then the assumed Euler relationship says:

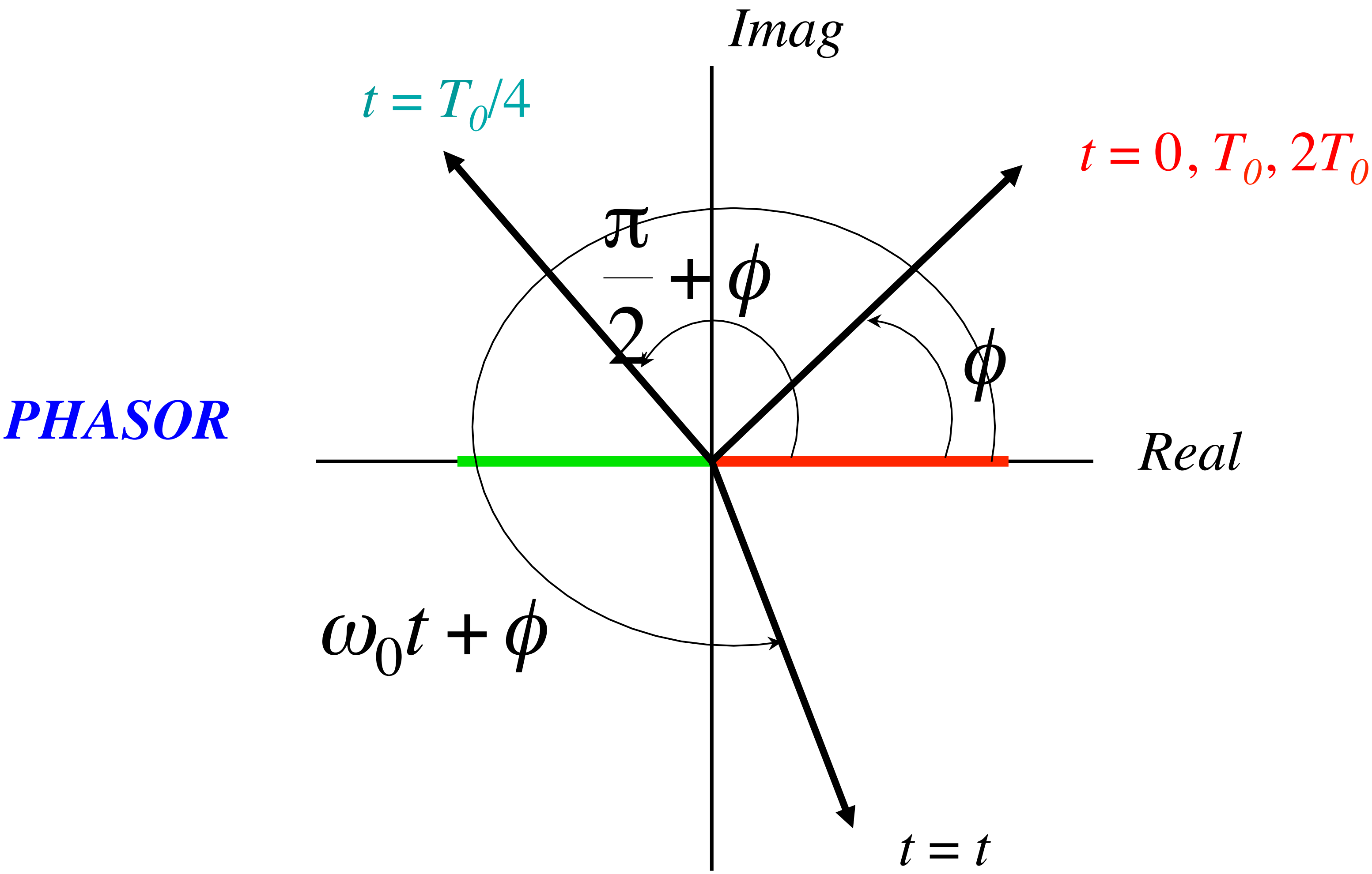
$$a + ib = |z|\cos\phi + i|z|\sin\phi$$

Equate real parts: $$a = |z|\cos\phi$$

Equate imaginary parts: $$b = |z|\sin\phi$$

$$\boxed{|z| = \sqrt{a^2 + b^2}} \qquad\qquad \boxed{\tan\phi = \frac{b}{a}}$$

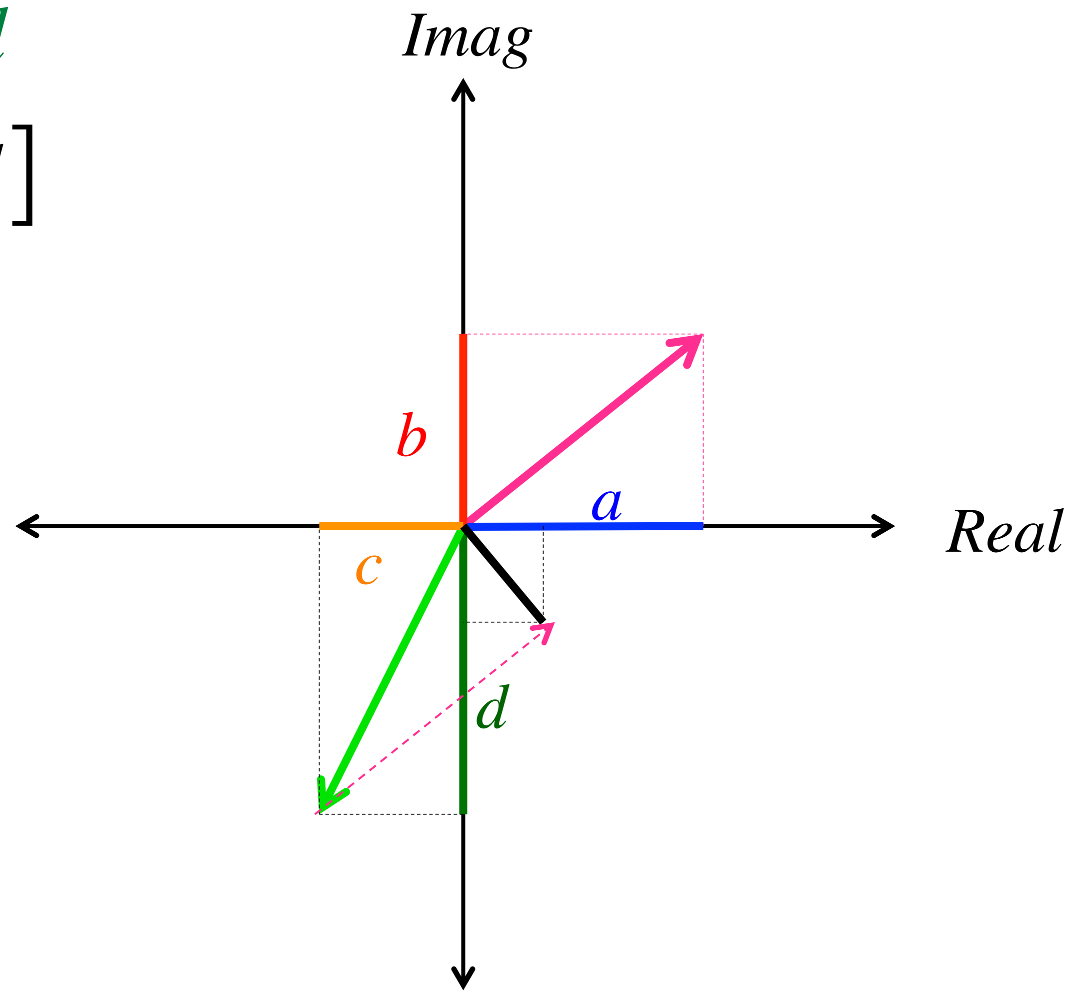
$$x(t) = \mathrm{Re}\left[Ae^{i\phi}e^{i\omega_0 t}\right] = \mathrm{Re}\left[Ae^{i(\omega_0 t+\phi)}\right]$$



Adding complex numbers is easy in rectangular form

$z = a + ib \qquad w = c + id$

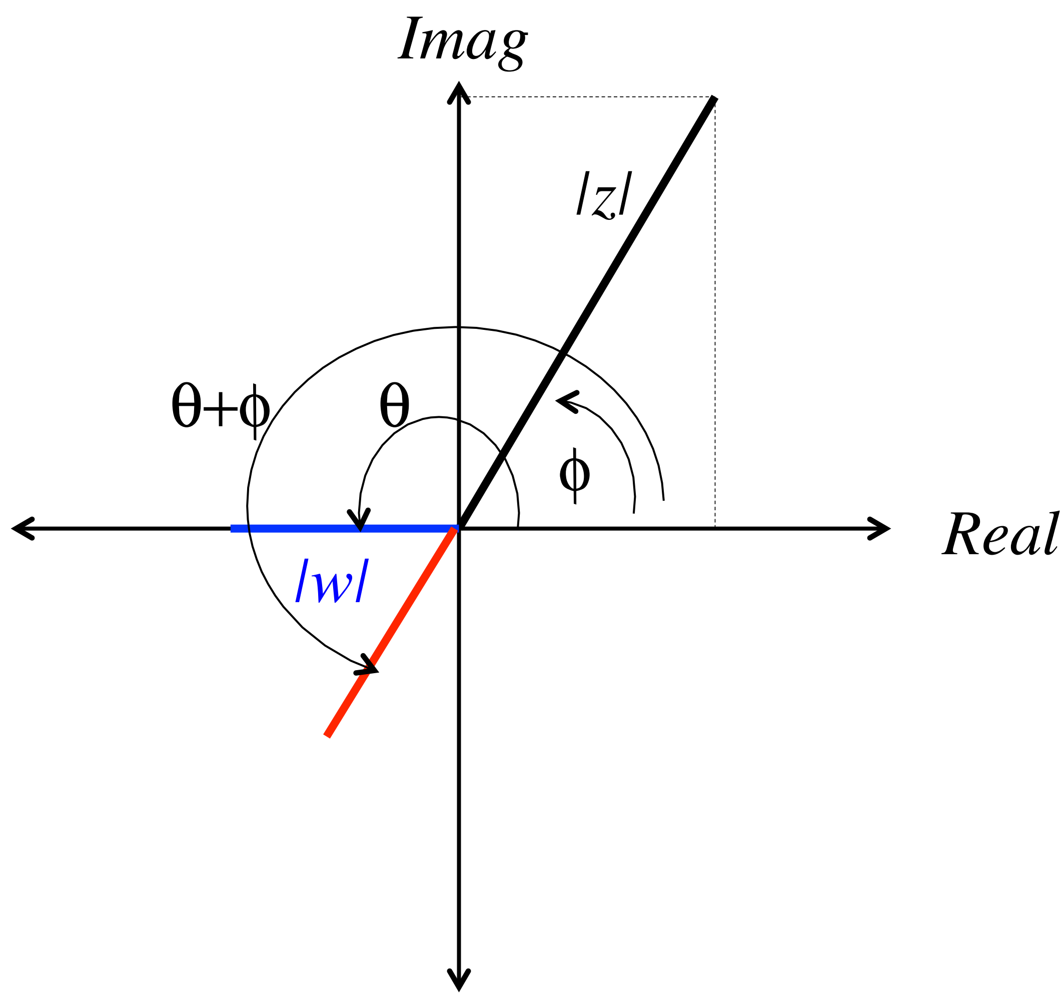
$$z + w = \left[a + c\right] + i\left[b + d\right]$$



Multiplication and division of complex numbers is easy in *polar* form

$$z = |z| e^{i\phi} \qquad w = |w| e^{i\theta}$$

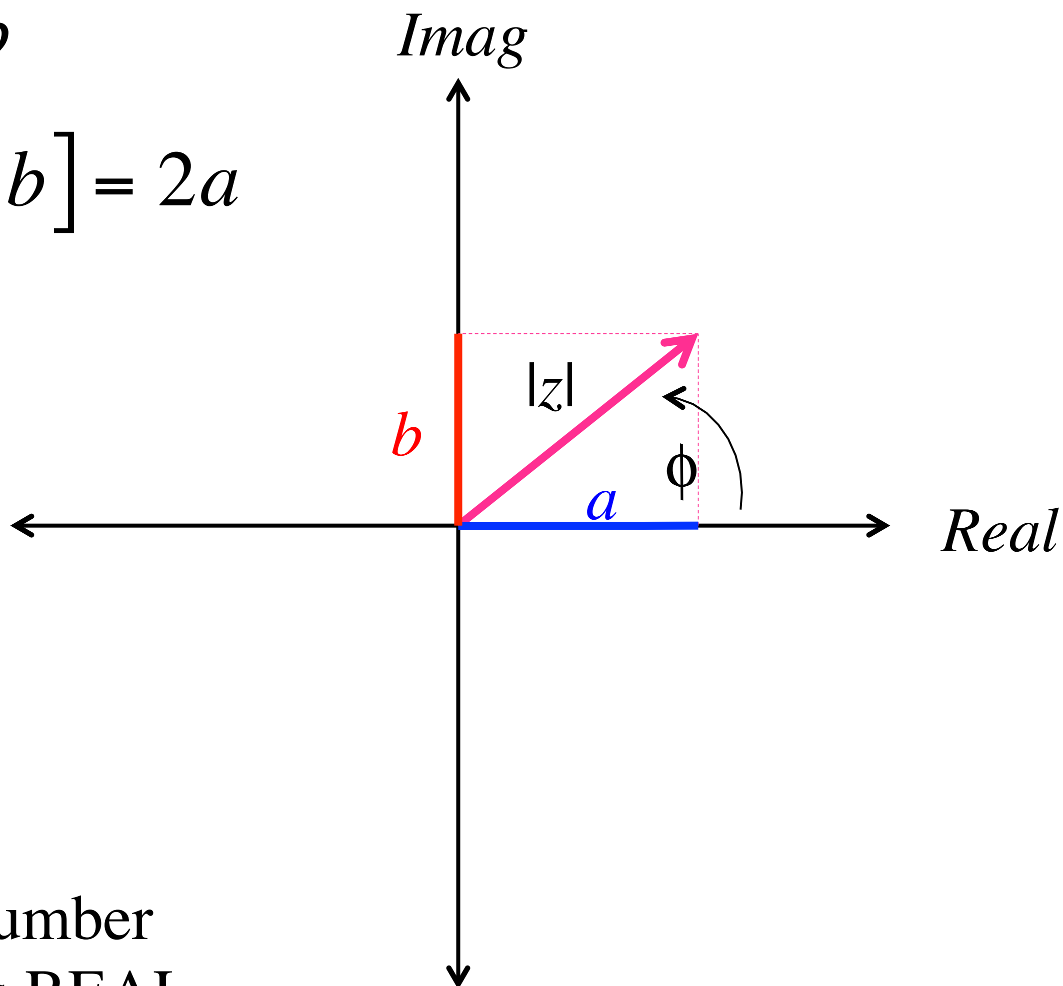
$$zw = |z||w| e^{i[\phi+\theta]}$$



Another important idea is the COMPLEX CONJUGATE of a complex number. To form the c.c., change *i* -> -*i*

$$z = a + ib \qquad z^* = a - ib$$

$$z + z^* = \left[a + a\right] + i\left[b - b\right] = 2a$$



$$z = |z| e^{i\phi} \qquad z^* = |z|\, e^{-i\phi}$$

$$zz^* = |z|\, e^{i\phi}\, |z|\, e^{-i\phi} = |z|^2$$

The product of a complex number and its complex conjugate is REAL. We say "*zz** equals mod z squared"

And finally, rationalizing complex numbers, or: what to do when there's an i in the denominator?

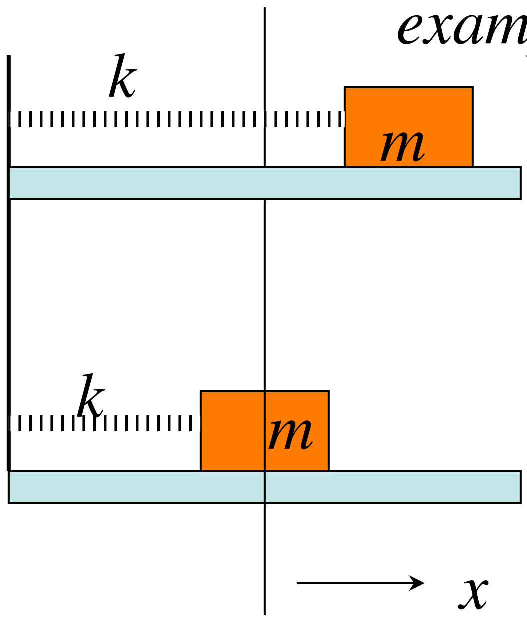
$$z = \frac{a+ib}{c+id}$$

$$z = \frac{a+ib}{c+id} \times \frac{c-id}{c-id}$$

$$z = \frac{ac+bd+i\left(bc-ad\right)}{c^2+d^2}$$

$$= \underbrace{\frac{ac+bd}{c^2+d^2}}_{\mathrm{Re}(z)} + i\underbrace{\frac{\left(bc-ad\right)}{c^2+d^2}}_{\mathrm{Im}(z)}$$

Using complex numbers: initial conditions. Same example as before, but now use the "C" and "D" forms



$$x(t) = C\exp(i\omega_0 t) + C^*\exp(-i\omega_0 t)$$

$$x(t) = \mathrm{Re}\left[D\exp(i\omega_0 t)\right]$$

$m = 0.01$ kg; $k = 36$ $\mathrm{Nm^{-1}}$. At $t = 0$, m is displaced 50mm to the right and is moving to the right at 1.7 $\mathrm{ms^{-1}}$.

Express the motion in

form C (even groups),

form D (odd groups)

$$x(t) = C\exp(i\omega_0 t) + C^*\exp(-i\omega_0 t)$$

$$x(t) = \left(25e^{-i0.516}e^{i60s^{-1}t} + 25e^{+i0.516}e^{-i60s^{-1}t}\right)\text{mm}$$

$$x(t) = \mathrm{Re}\left[D\exp(i\omega_0 t)\right]$$

$$x(t) = \mathrm{Re}\left[\left(50e^{-i0.516}\right)e^{i60s^{-1}t}\right]\text{mm}$$

$$\boxed{\begin{aligned} A\cos\phi &= B_p = 2\mathrm{Re}[C] = \mathrm{Re}[D] \\ A\sin\phi &= -B_q = 2\mathrm{Im}[C] = \mathrm{Im}[D] \end{aligned}}$$

$$\boxed{\begin{aligned} |D| &= 2|C| = A \\ \tan\phi &= \frac{\mathrm{Im}[D]}{\mathrm{Re}[D]} = \frac{\mathrm{Im}[C]}{\mathrm{Re}[C]} \end{aligned}}$$