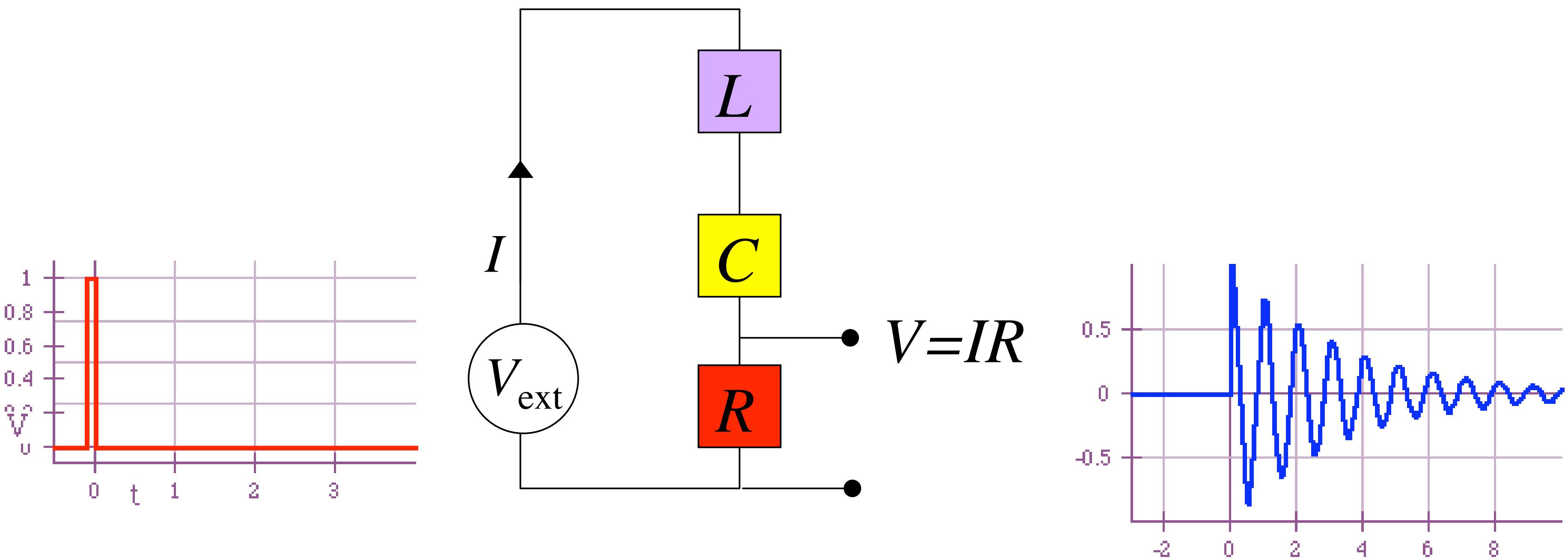
THE DRIVEN LRC CIRCUIT
(A special driving function that is an impulse function)

Reading:
Taylor 5.8
Main 11.3
Riley 13.1

Forced, damped oscillator: the pulsed LRC circuit



This response function contains information about the circuit response **at all frequencies**. In fact the Fourier transform of this function is the admittance. We can use this to determine the admittance experimentally in a single measurement!!

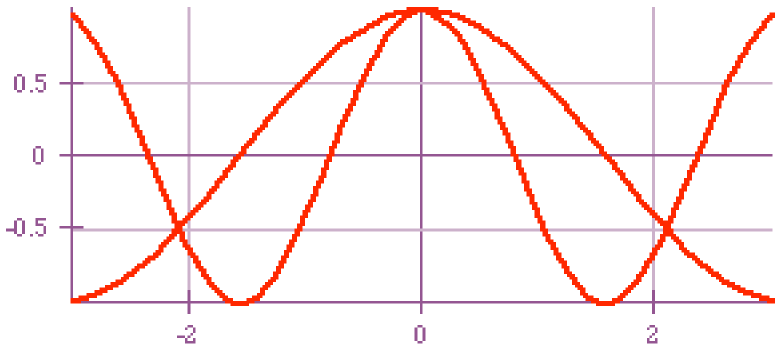
A few words before we begin. We can actually calculate the time response of a series LRC circuit to a delta function driving force rather easily without resorting to Fourier analysis (and we will, later). But the impulse function is very special, and with other, more general driving functions, the Fourier approach is much easier.

You might object - the impulse function “drives” the circuit for only a very short time and the rest of the time, the circuit is “free”. Is this really a driving force? Well, yes, it is. That voltage application for a short time is important.

You might object more - the impulse function is not a periodic function - how can it be represented by a Fourier series, then? This is only a minor problem. You can think of a non-periodic function as a periodic function with an extremely long period. The concept is the same. In the lab, you will actually drive the circuit with a “periodic impulse function”: an impulse is applied, then another a long time later, and then another along time after that, and so on.

Another objection - the data you acquire in the lab for the current in the “driven” circuit as a function of time is not continuous. How will you find the Fourier spectrum of a discrete function? This again is not a problem of principle. We simply need some numeric techniques - the technique is the FFT, the fast Fourier transform.

A delta function in time is a superposition of equal mixes of sinusoids of *all* frequencies



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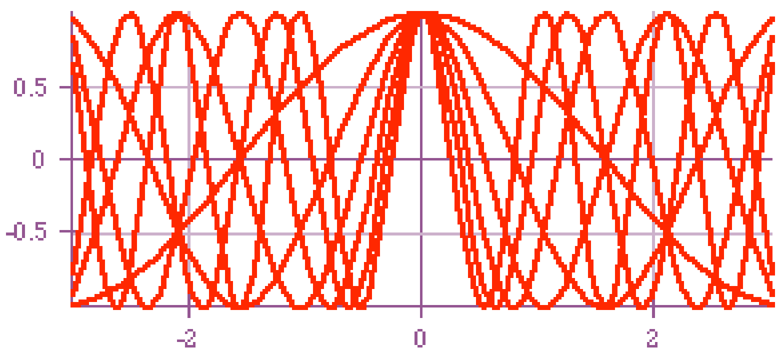

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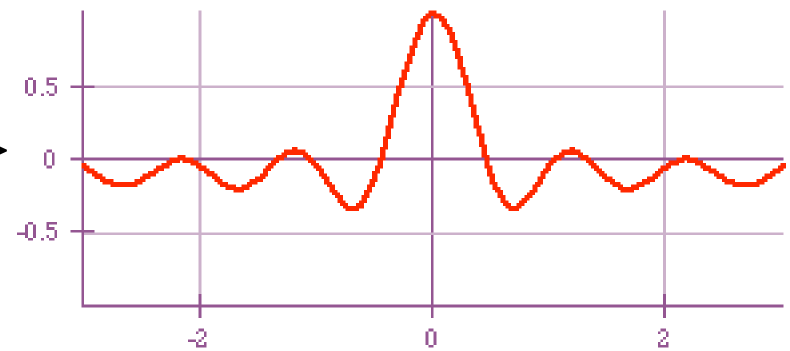
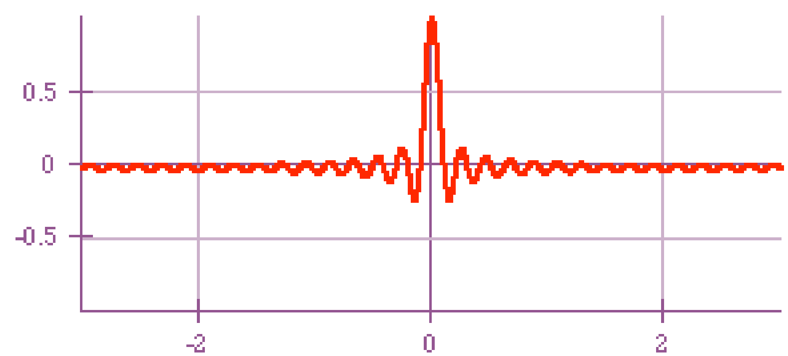
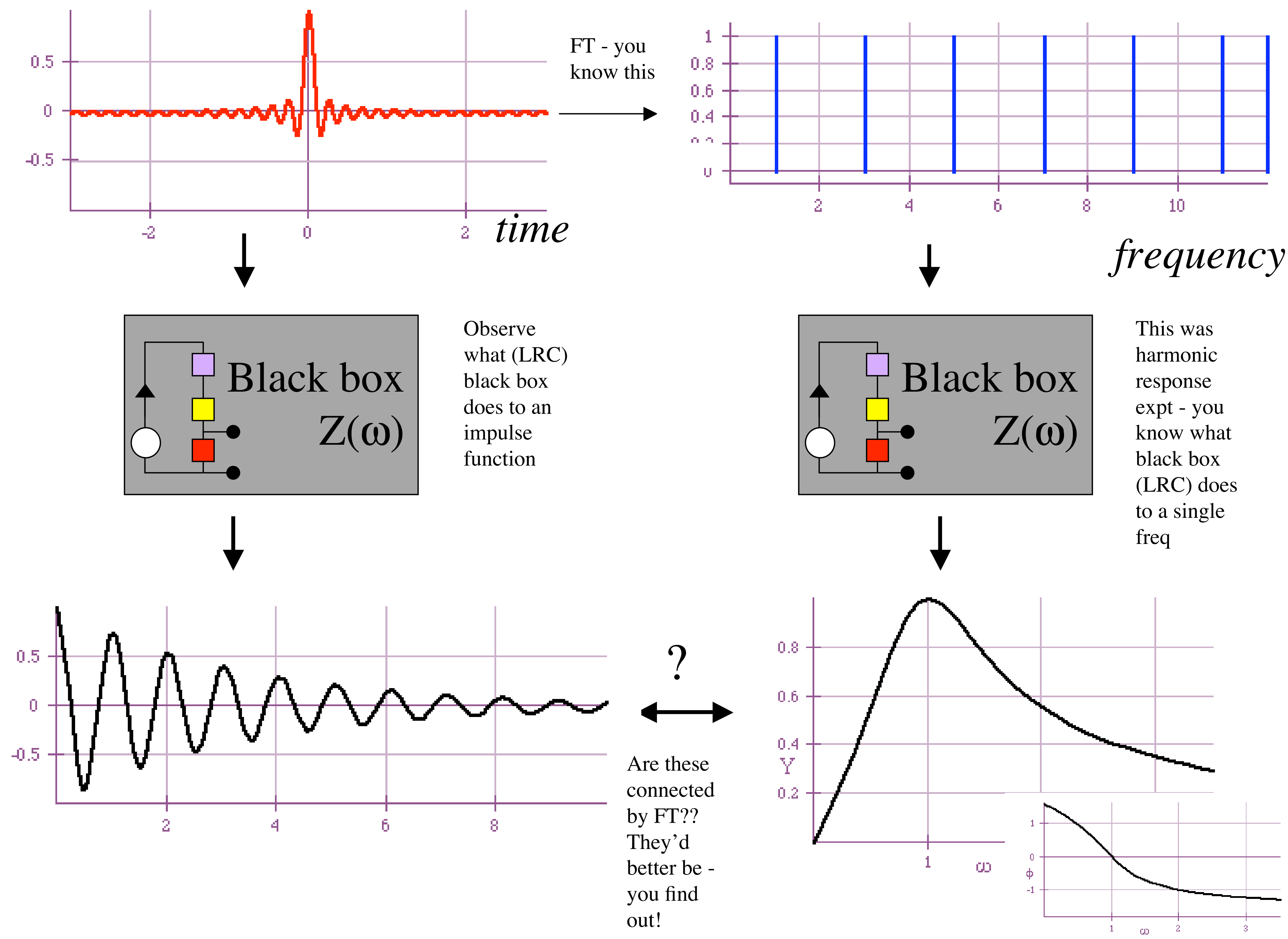


30 terms --->



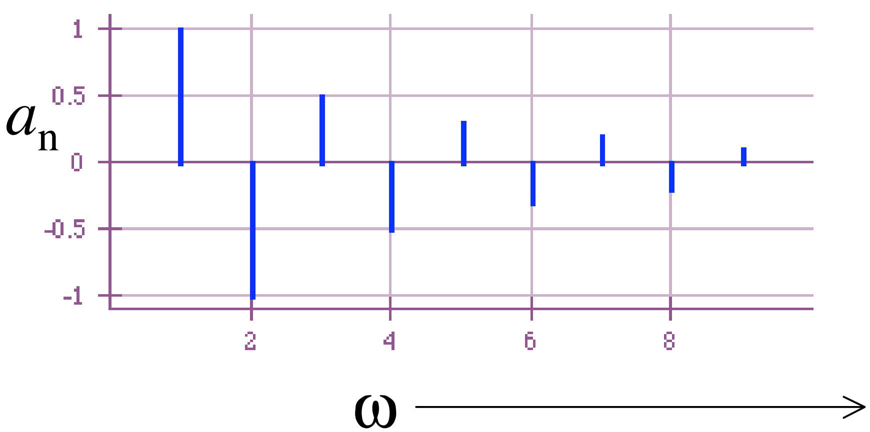

FT - you know this
time
frequency
Black box
Z(ω)
Observe what (LRC) black box does to an impulse function
This was harmonic response expt - you know what black box (LRC) does to a single freq
Black box
Z(ω)
?
Are these connected by FT?? They'd better be - you find out!
Y
ω
φ
ω

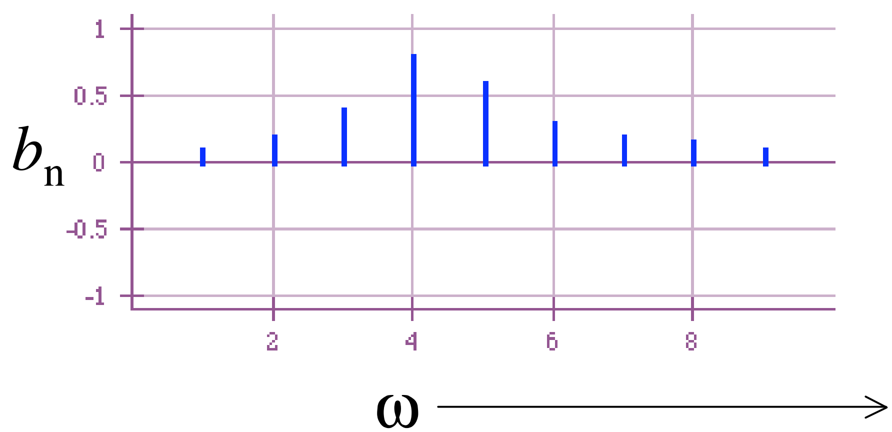
Any periodic function $f(t)$ can be written as a Fourier Series

$$\boxed{\frac{a_0}{2} + \sum_{n=1,2\ldots} a_n \cos(n\omega t) + \sum_{n=1,2\ldots} b_n \sin n(\omega t)}$$

with

$$\boxed{a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)\,dt} \qquad \boxed{b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)\,dt}$$





Any periodic function *f*(*t*) can be written as a Fourier Series

$$\boxed{\sum_{n=0}^{\infty} c_n e^{in\omega t} + c_n^* e^{-in\omega t}}$$

or

$$\boxed{\sum_{n=-\infty}^{\infty} c_n e^{in\omega t}}$$

$$c_{-n} = c_n^*$$

with

$$\boxed{c_n = \frac{1}{T}\int_0^T f(t) e^{-in\omega t} dt}$$

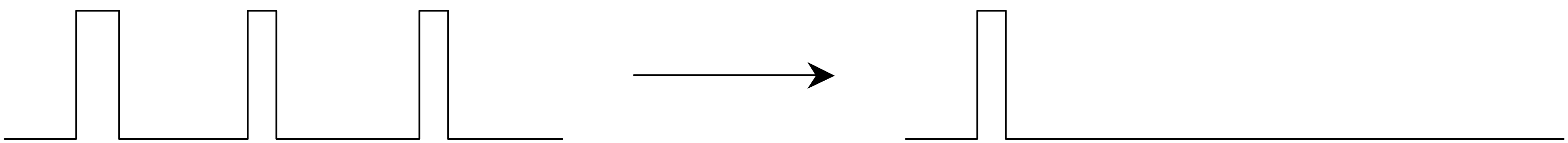
$$c_0 = \frac{a_0}{2};\ c_n = \frac{a_n - ib_n}{2};\ c_{-n} = \frac{a_n + ib_n}{2}$$

Focus on c form:

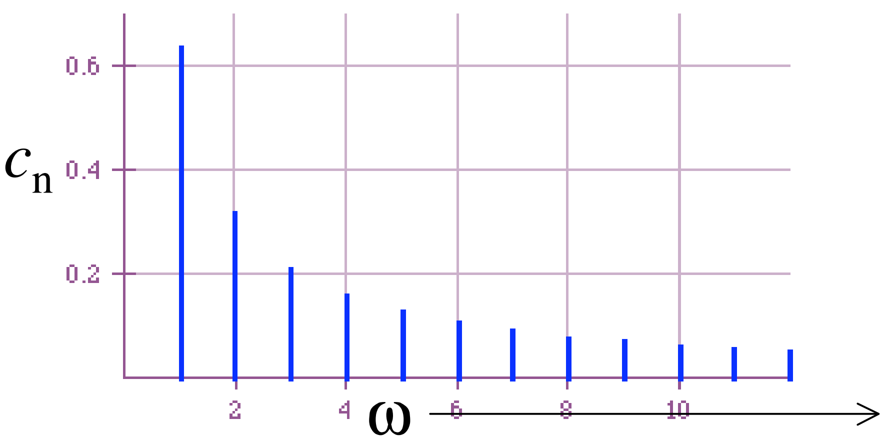
$$\boxed{f(t) = \sum_{n=-\infty}^{\infty} c_n e^{in\omega t}}$$

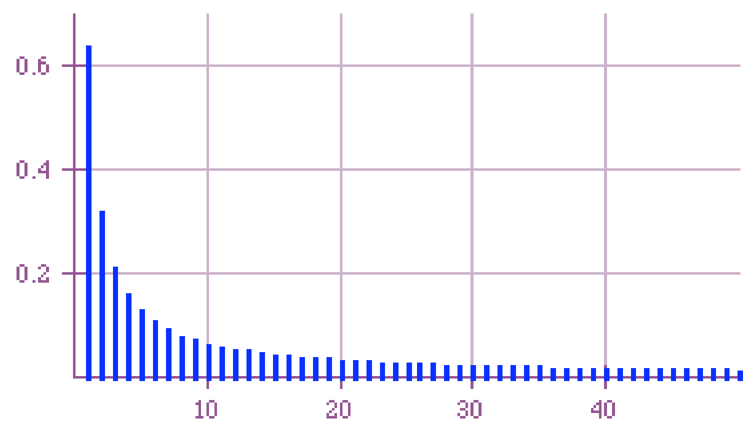
$$T \to \infty;\ \Delta\omega = \frac{2\pi}{T} \to 0$$

$$n\omega \to \omega_n$$



$$\boxed{c_n = \frac{1}{T}\int_0^T f(t) e^{-in\omega t}\,dt} \longrightarrow c_n = \frac{\Delta\omega}{2\pi}\int_{-\infty}^{\infty} f(t) e^{-i\omega_n t}\,dt$$





$$f(t)=\sum_{n=-\infty}^{\infty}\frac{\Delta\omega}{2\pi}\int_{t'=-\infty}^{\infty}f(t')e^{-i\omega_n t'}dt'\;e^{i\omega_n t}$$

$$\Delta\omega\rightarrow d\omega;\;\omega_n\rightarrow\omega;\;\sum_{n=-\infty}^{\infty}\rightarrow\int_{\omega=-\infty}^{\infty}$$

$$\boxed{f(t)=\frac{1}{\sqrt{2\pi}}\int_{\omega=-\infty}^{\infty}d\omega e^{i\omega t}\boxed{\frac{1}{\sqrt{2\pi}}\int_{t'=-\infty}^{\infty}f(t')e^{-i\omega t'}dt'}}$$

coefficient or Fourier transform (function of ω, not time); denoted "f-tilde"

$$\boxed{\tilde{f}(\omega)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)e^{-i\omega t}dt}$$

The (Fast) Fourier Transform

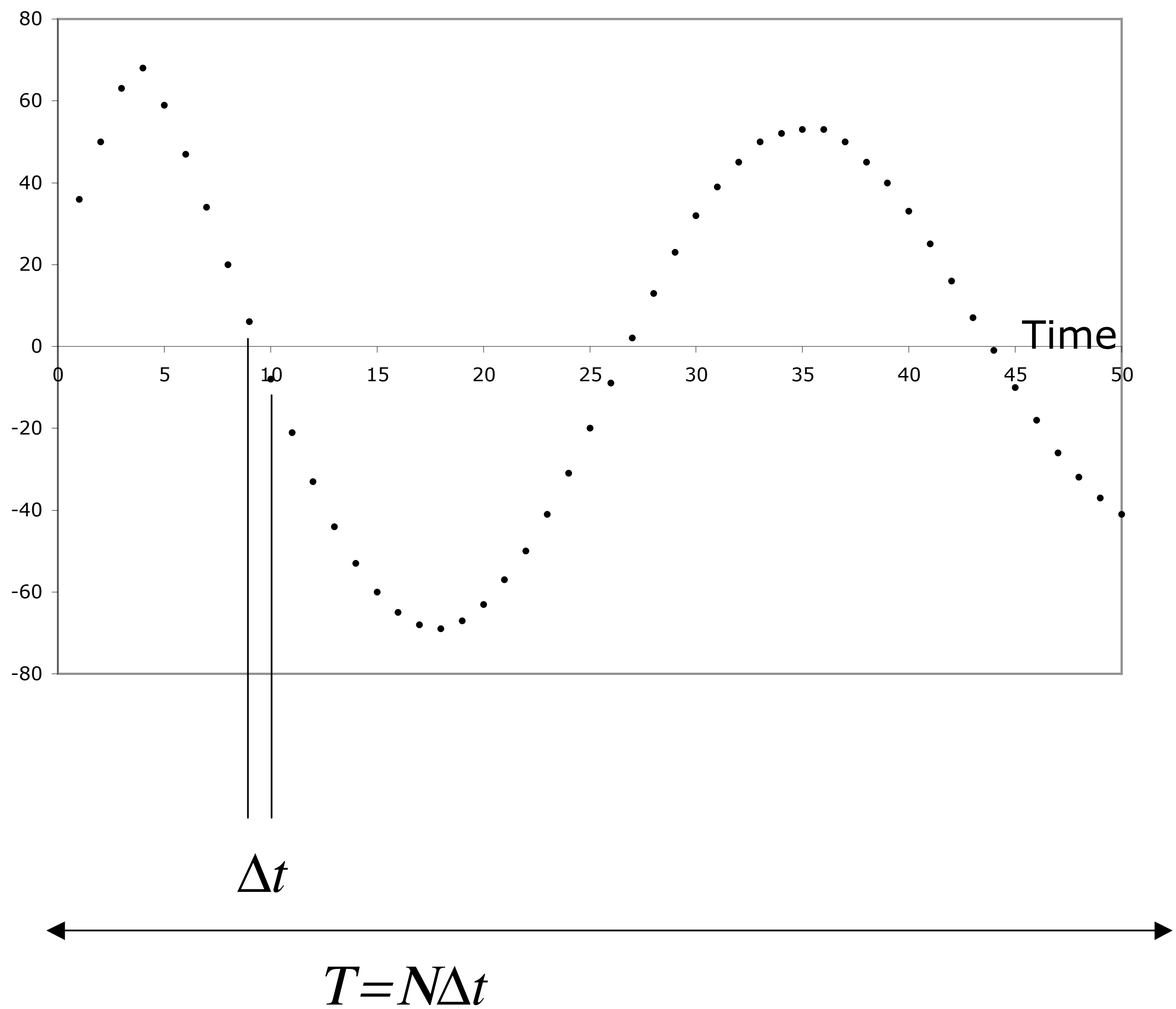
The Fourier transform (FT)

- is the analog, for non-periodic functions, of the Fourier series for periodic functions
- can be considered as a Fourier series in the limit that the period becomes infinite

The Fast Fourier Transform (FFT)

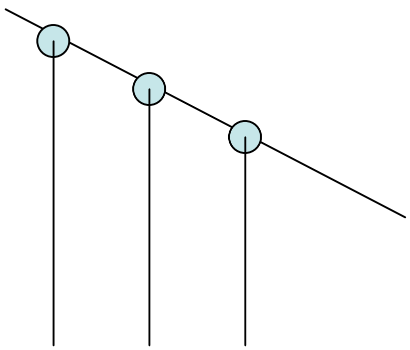
- is a computer algorithm to calculate a FT for a discrete (or digitized) function
- input is a series of 2^p(complex) numbers representing a time function; output is 2^p (complex) numbers representing the coefficients at each frequency
- has a few rules to be obeyed
- Excel (or Maple/Mathmatica) will do this for you - it’s not too hard to learn.

t	$F(t)$
1 Δt	36
2 Δt	50
3 Δt	63
4 Δt	68
5 Δt	49
6 Δt	47
7 Δt	34
8 Δt	20
9 Δt	6
10 Δt	-8
11 Δt	-21
12 Δt	-33
13 Δt	…
…	…
1024 Δt	0



$$\boxed{\frac{1}{T} = f_0;\ \frac{2\pi}{T} = \omega_0}$$

Fundamental frequency (small!)



Δt Δt

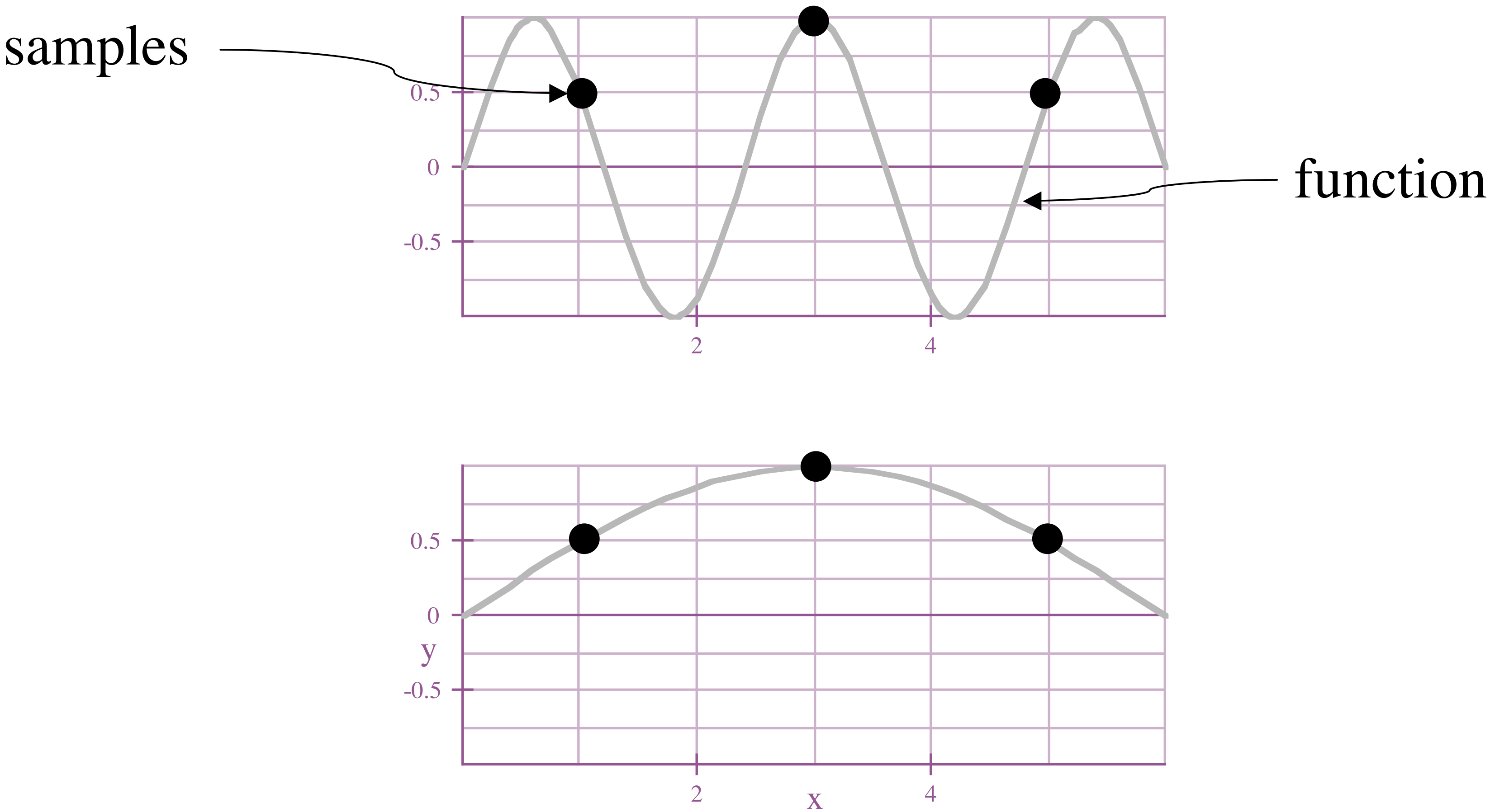
$2\Delta t$ is SMALLEST period of a sinusoidal function that is sensible to consider - faster oscillations have no meaning for this function

$$\boxed{\frac{1}{2\Delta t} = f_N;\ \frac{2\pi}{2\Delta t} = \omega_N}$$

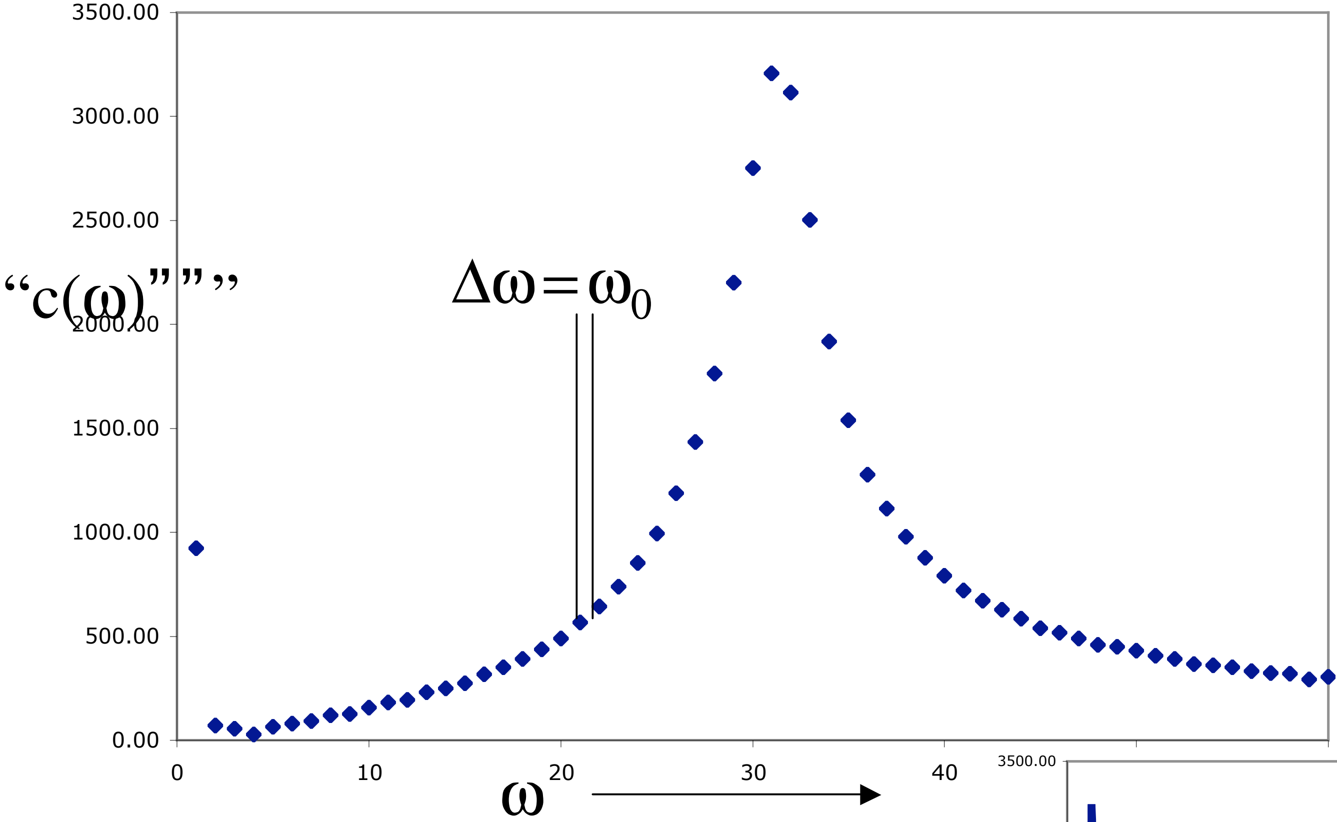
Nyquist frequency

Only HALF the frequency spectrum is unique information

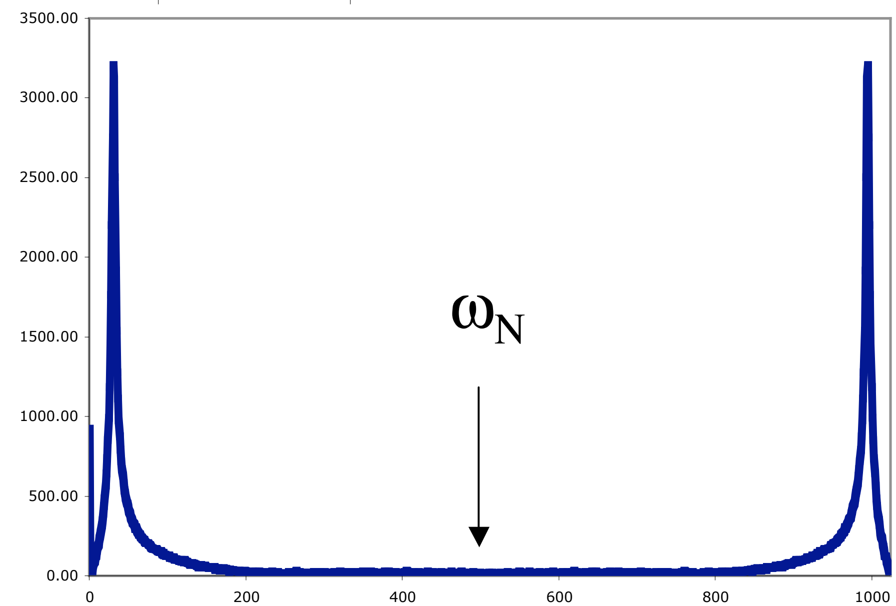
Aliasing



The FFT



3500.00
3000.00
2500.00
2000.00
1500.00
1000.00
500.00
0.00
“c(ω)”
Δω=ω0
0
10
20
30
40
ω



3500.00
3000.00
2500.00
2000.00
1500.00
1000.00
500.00
0.00
ωN
0
200
400
600
800
1000

Plan:
The “Fourier representation” for a non-periodic function like a delta function is the Fourier transform - it’s really a Fourier series but with a very (infinitely) small fundamental frequency. We’ll talk about this, and do some analytical examples later.

If we want the Fourier transform of an experimentally measured (i.e. digitized) function, we have to do something called a Fast Fourier Transform (FFT). Excel or Maple will help here, but we have to understand a few things about this.

Once we understand the concept of the FT and the FFT, we’ll experimentally measure the response of the same LRC circuit to an impulse (delta) function. We’ll FFT the response function, and discuss what we find.

Plan cont'd:
Once we've done this experiment, we'll go back and do some analytical examples of Fourier transforms, and in particular, we'll FT (analytically now) the damped harmonic oscillator response and show analytically that it really is the admittance function.

We also want to know how to calculate the time response of the LRC circuit to an arbitrary waveform, without using the easier Fourier route. We'll calculate the response to a delta function (impulse) which we already measured so we have a good idea of what to expect. Then we'll use superposition to add up a number of sequential delta functions, and find the total response.

Finally, we'll FT the expression for the general time response and we'll discover the admittance function again.