Deconstructing periodic driving voltages (or any functions) into their sinusoidal components:

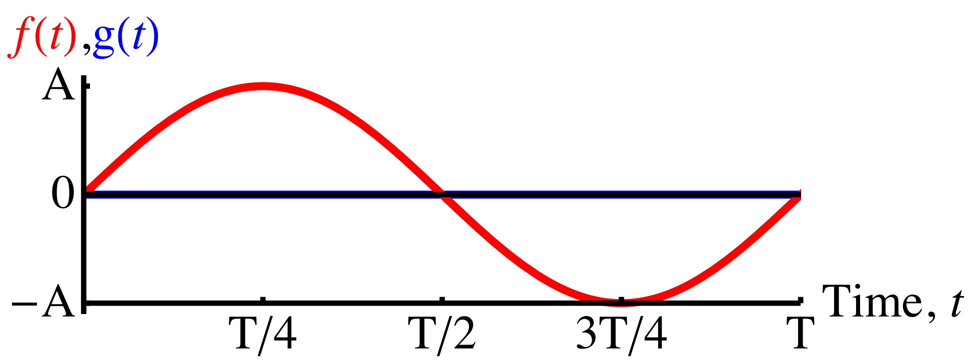
It's easy to build a periodic functions by choosing coefficients a_n and b_n or amplitudes A_n and phases ϕ_n to build periodic functions of any sort via

$$f(t) = \sum_n a_n \cos n\omega t + b_n \sin n\omega t \;\; or \;\; \sum_n A_n \cos(n\omega t + \phi_n)$$

What is less obvious is how to take a given periodic function and find out what coefficients went into making it! With a little experience, you can develop some intuition that makes it possible to solve simple cases, but we need an analytical method to do it for ANY periodic function. The technique is called Fourier analysis and it closely allied with projecting vectors onto their basis vectors to find components. It is heavily used in signal and image processing, and you will encounter related techniques is used in quantum mechanics when you work with the Schrödinger equation.

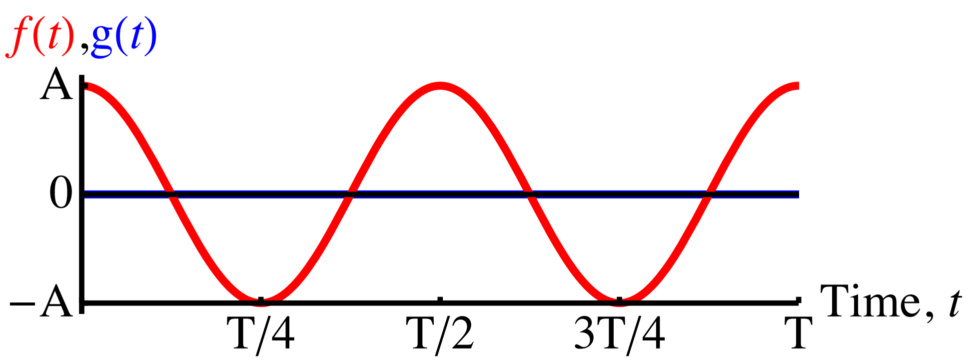
We will need to know many integrals of this kind, and we'll discuss the prefactors and the word "projection" later.

$$\frac{2}{T}\int_0^T \sin(\omega t)\,dt = \frac{-2}{\omega T}\cos(\omega t)\Big|_0^T = \frac{-1}{\pi}\left[\cos(2\pi) - \cos(0)\right] = 0$$



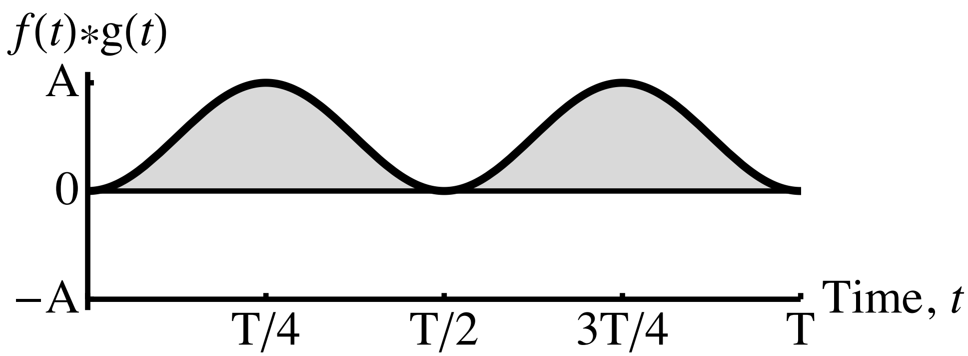
More important integrals …..

$$\frac{2}{T}\int_0^T \cos(2\omega t)\,dt = \frac{2}{2\omega T}\sin(2\omega t)\Big|_0^T = \frac{1}{2\pi}\left[\sin(2\pi)-\sin\left(0\right)\right]=0$$



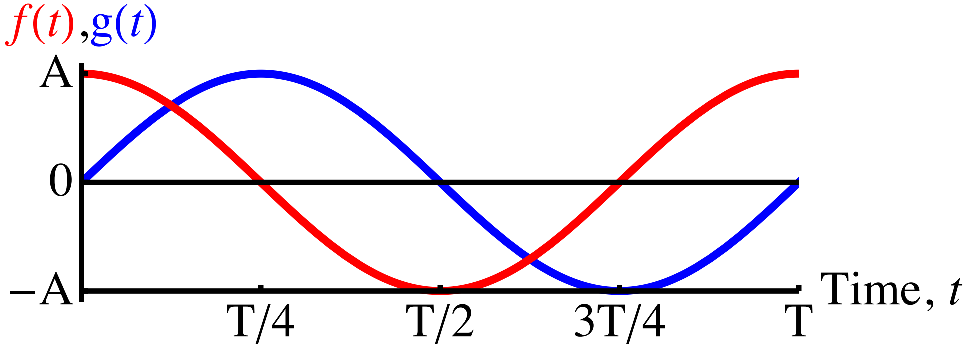
More important integrals …..

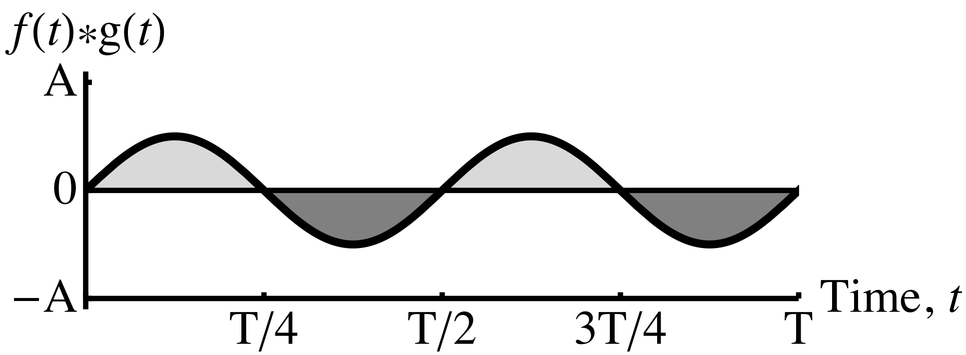
$$\frac{2}{T}\int_0^T \sin^2(\omega t)\,dt = \frac{2}{T}\int_0^T \left[\tfrac{1}{2} - \tfrac{1}{2}\cos(2\omega t)\right]dt = \frac{1}{T}\int_0^T dt = 1$$



More important integrals …..

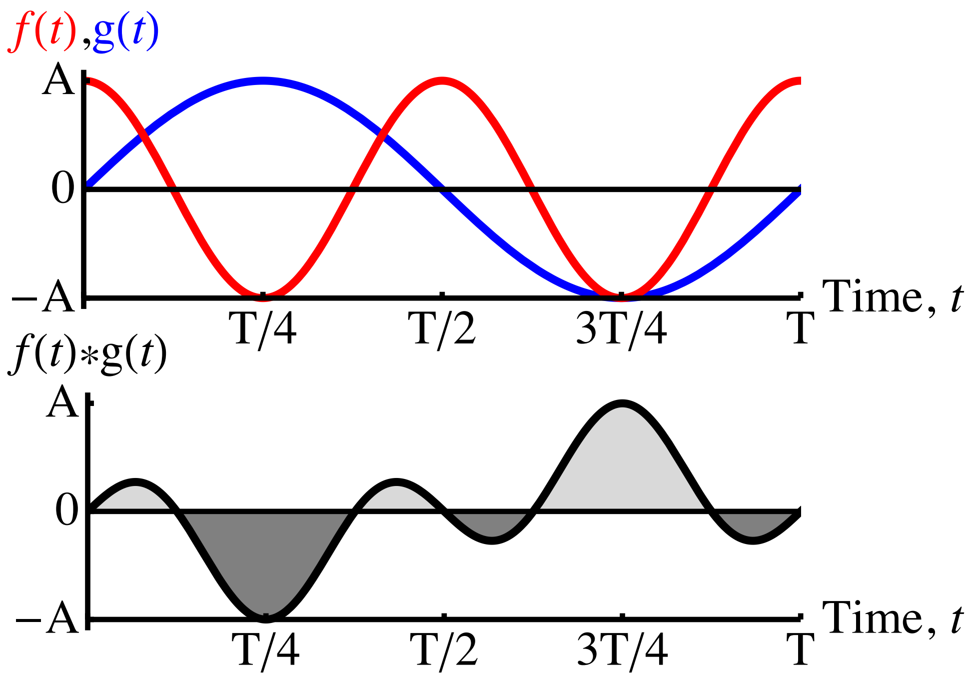
$$\frac{2}{T}\int_0^T \sin(\omega t)\cos(\omega t)\,dt = \frac{1}{T}\int_0^T \sin(2\omega t)\,dt = 0$$





Integrate the product of two harmonic functions with frequencies that are <u>integer multiples</u> of the fundamental over one period of the longer period function and you get ……

Black = red x blue



<u>Zero (0)</u> if the two harmonic functions have different frequencies. The functions are said to be “orthogonal”.

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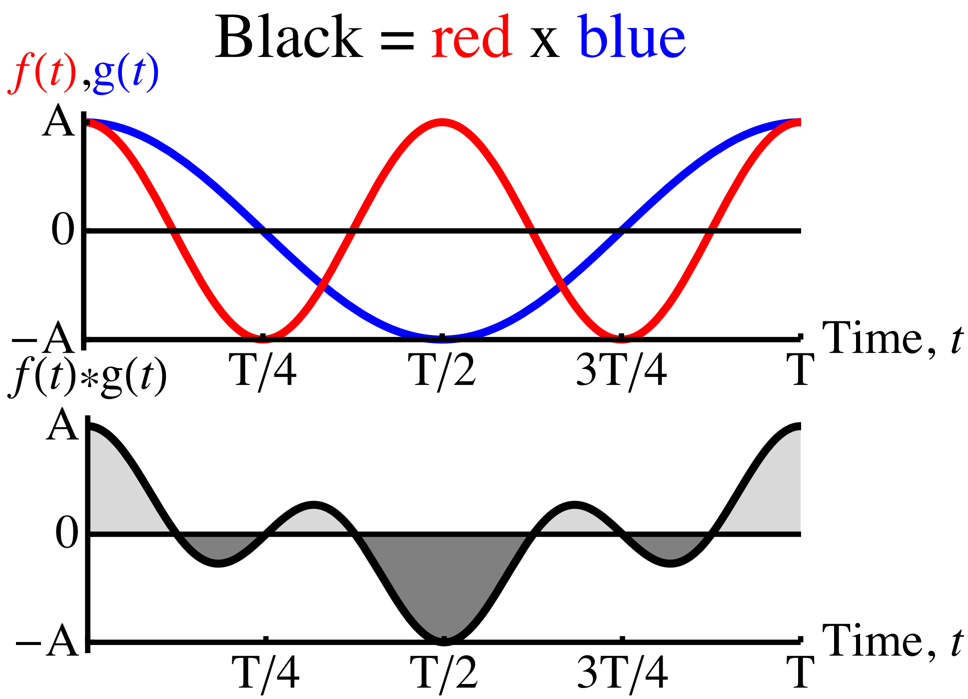
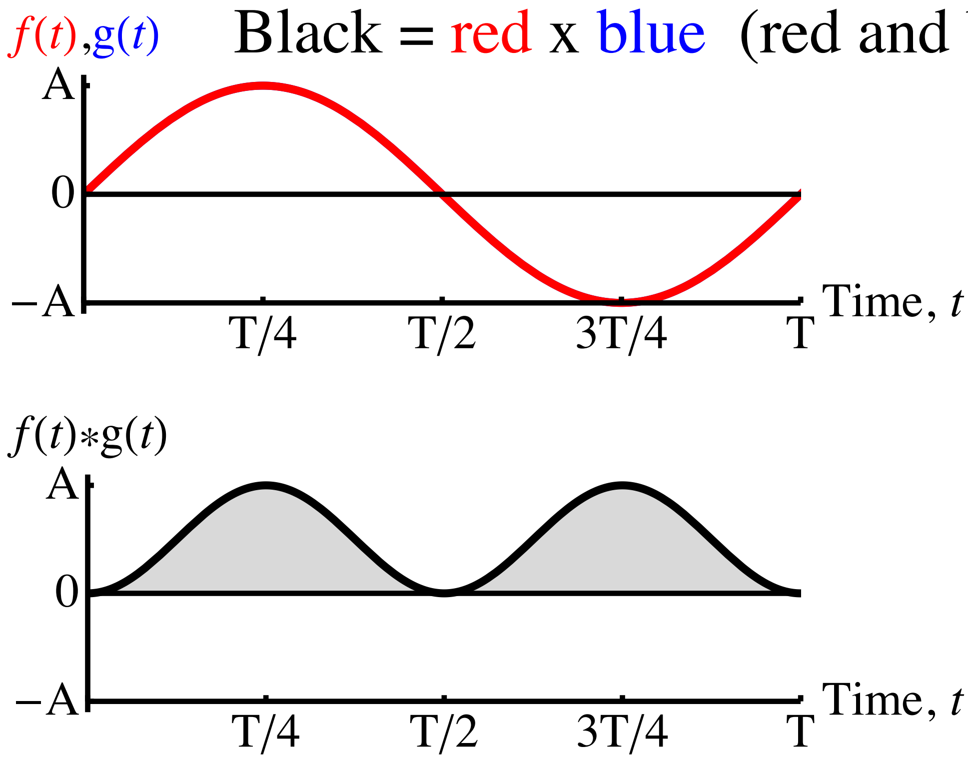


Zero (0) if the two harmonic functions have different frequencies. The functions are said to be "orthogonal".

Integrate the product of two harmonic functions with frequencies that are <u>integer multiples</u> of the fundamental over one period of the longer period function and you get ……

Black = red x blue (red and blue are same)



Not zero if the two harmonic functions have the same frequency.

Integrate the product of two harmonic functions with frequencies that are <u>integer multiples</u> of the fundamental over one period of the longer period function and you get ……

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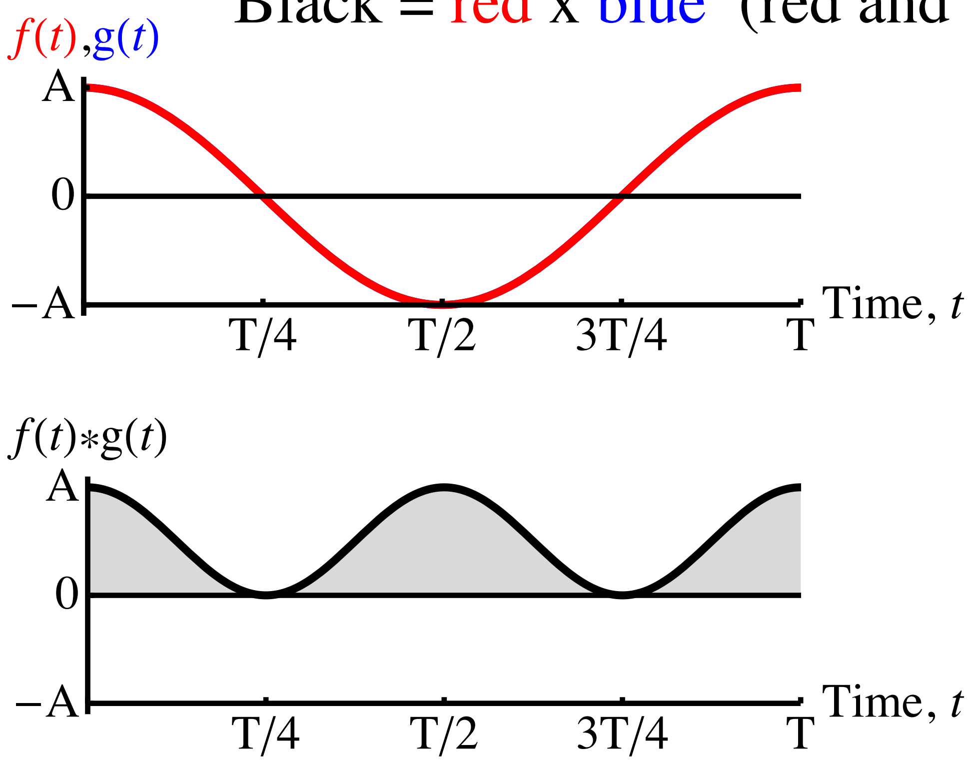


Not zero if the two harmonic functions have the same frequency……

$$\frac{2}{T}\int_0^T \sin(p\omega t)\sin(q\omega t)\,dt = \begin{cases} 0 \text{ if } p \neq q \text{ (integers)} \\ 1 \text{ if } p = q \text{ (integers)} \end{cases}$$

$$\frac{2}{T}\int_0^T \cos(p\omega t)\cos(q\omega t)\,dt = \begin{cases} 0 \text{ if } p \neq q \text{ (integers)} \\ 1 \text{ if } p = q \text{ (integers)} \end{cases}$$

We can write this more elegantly using the Kronecker delta. δ_{pq} =1 if $p = q$; = 0 if $p \neq q$.

$$\frac{2}{T}\int_0^T \sin(p\omega t)\sin(q\omega t)\,dt = \delta_{pq}$$

$$\frac{2}{T}\int_0^T cos(p\omega t)\cos(q\omega t)\,dt = \delta_{pq}$$

$$\int_0^T f(t)g(t)\,dt$$

Why do we call this integral a projection? What is projecting onto what? And what does it all mean? Up till now we have been doing this integral with $f(t)$ having the form of a sinusoidal function with a frequency that in an integer multiple of ω.

The projection of a vector A = (A_1, A_2, A_3 …) onto a vector B = (B_1, B_2, B_3 …) is found by multiplying corresponding components and adding them.

$$\vec{A}\bullet\vec{B} = A_1B_1 + A_2B_2 + A_3B_3 + \ldots$$

Can you see the similarity to thinking of two functions of a variable t as long "vectors" $f(t) = (f(t_1), f(t_2), f(t_3) \ldots)$ and $g(t) = (g(t_1), g(t_2), g(t_3) \ldots)$ and multiplying and "adding" (integrating)? Thus in some sense, the integral above is a "projection" of $f(t)$ onto $g(t)$. But it's not quite right yet ….

$$\frac{1}{T}\int_0^T f(t)g(t)\,dt$$

Because the dot product must have the dimensions of *f* x *g* and *dt* has dimensions of time, we need the factor of 1/*T* out front for dimensional purposes, at least.

Now specializing to the specific case of *g*(*t*) = sinusoidal function, we would like to say that if we choose *f*(*t*) = *g*(*t*), then the projection of a function onto itself must surely be unity! Thus we need

$$\frac{2}{T}\int_0^T f(t)g(t)\,dt$$

Check for yourself that if you project any sinusoidal function cos $(n\omega t)$ or $sin(n\omega t)$ onto itself you get unity by using the second equation above. As we saw earlier, this definition also shows that projections of sinusoidal functions onto others with (different) integer multiples of a common fundamental frequency are zero.

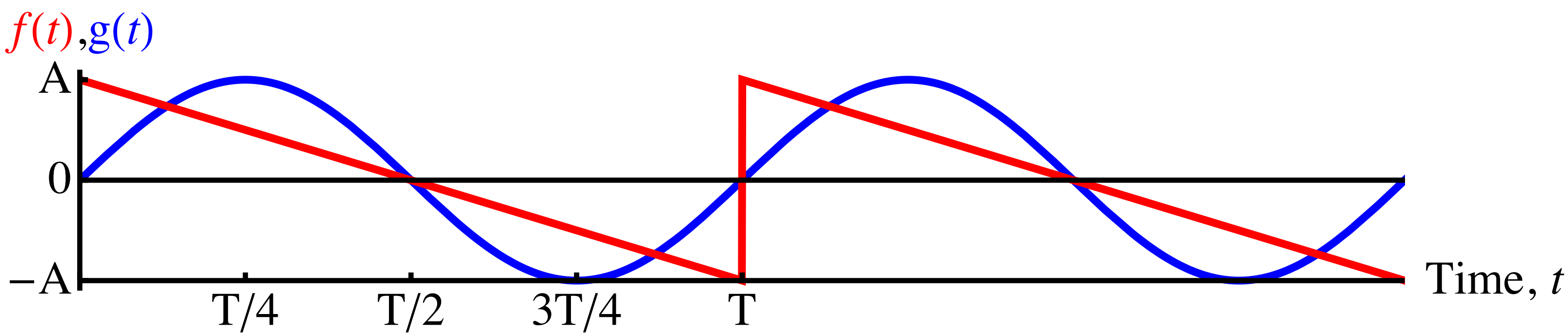
In projection language, we can say that the set of functions cos ($n\omega t$) and $sin(n\omega t)$; n integer; form and ORTHONORMAL SET or ORTHONORMAL BASIS just like the unit vectors x(hat), y (hat) and z(hat) (sometimes also called i, j and k).

And just as you can write any vector that lives in 3-d space as a sum of multiples of vectors x(hat), y(hat) and z(hat),

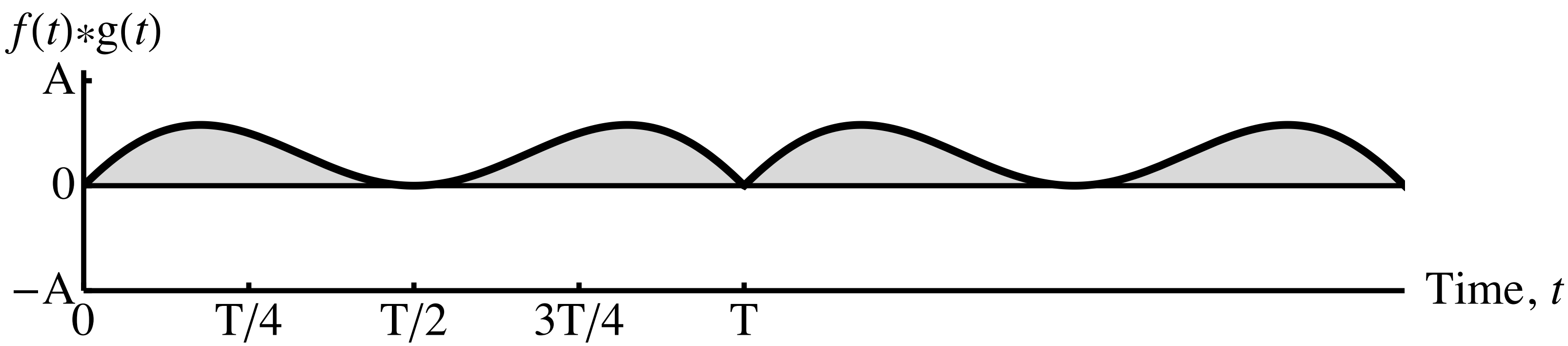
so, too, can you write any function that lives in "periodic space" as a a sum of multiples of cos($n\omega t$) and $sin(n\omega t)$.
More powerful, because of the orthonormal nature of the basis functions, you can “project out” the coefficient of any basis function by projecting the function onto that basis vector.

Let's do an example.

Example of sine series: Take the red sawtooth function

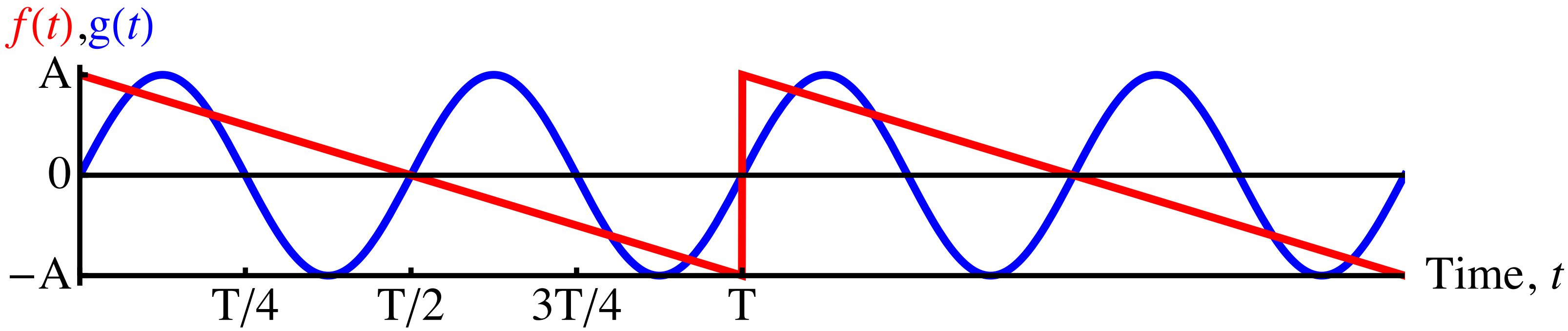


Multiply it by the sine term whose coefficient you want to find ……… (here choose $n = 1$)

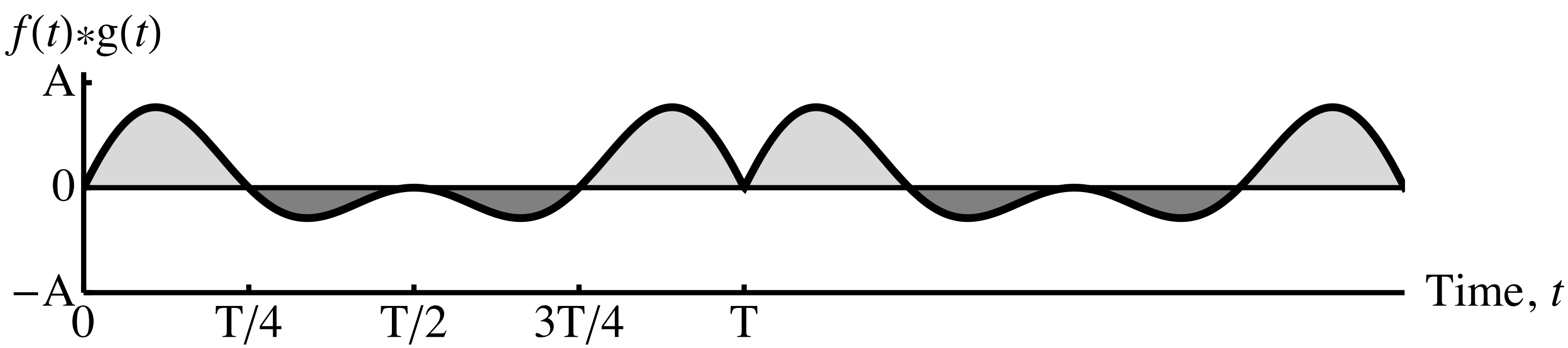


Integrate over 1 period of the fundamental. Multiply by $2/T$. That number is the coefficient of the $\sin(1\omega t)$ term.

Example of sine series: Take the red sawtooth function

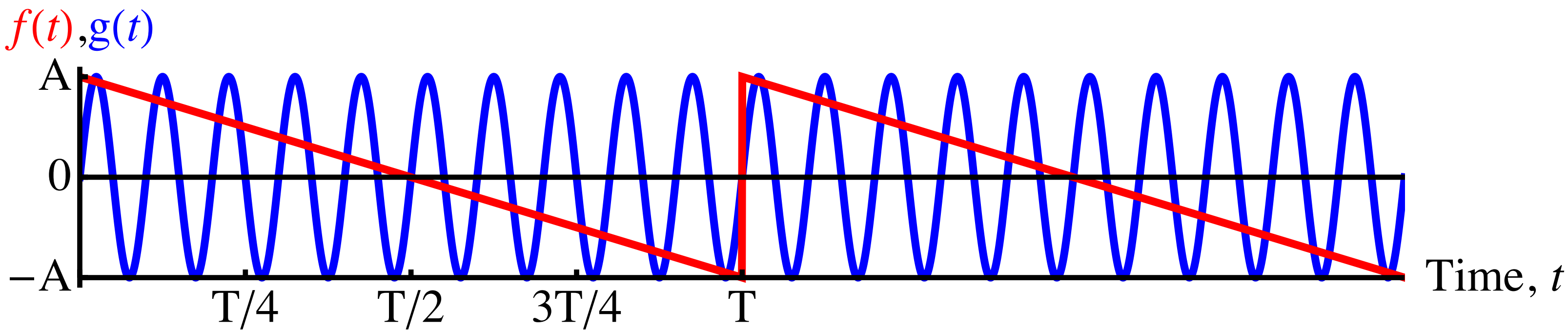


Multiply it by the sine term whose coefficient you want to find .... (here choose n =2)

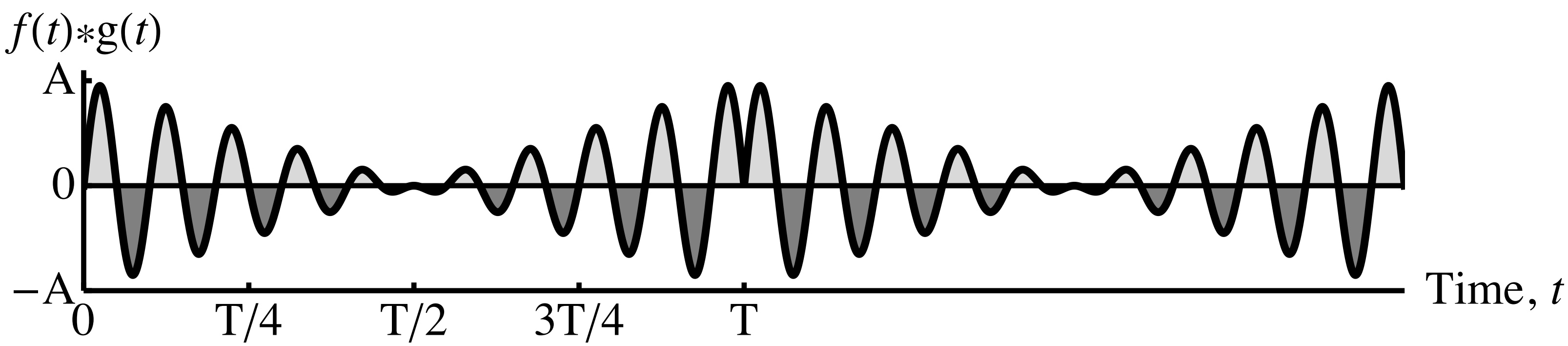


Integrate over 1 period of the fundamental. Multiply by $2/T$. That number is the coefficient of the sin(2t) term.

Take the function (example of sine series).........

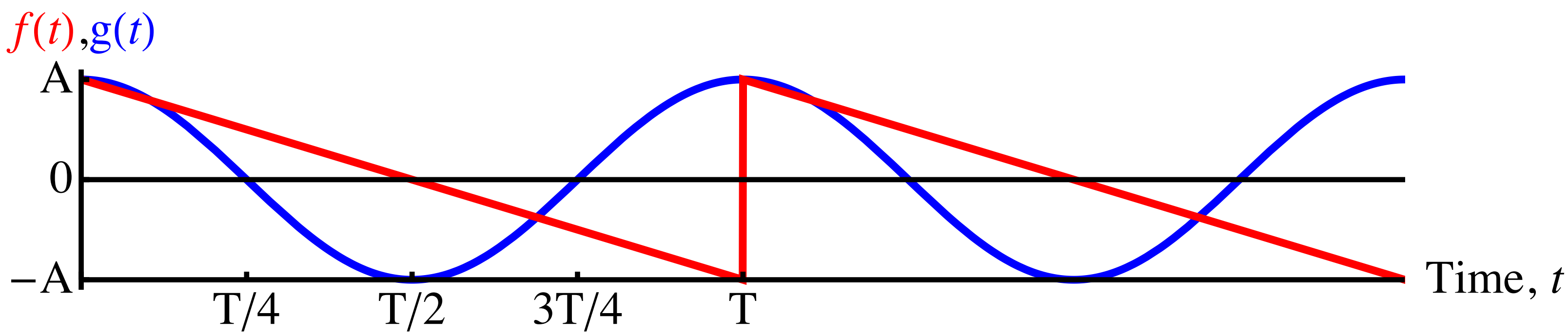


Multiply it by the sine term whose coefficient you want to find .... (here choose $n = 10$)

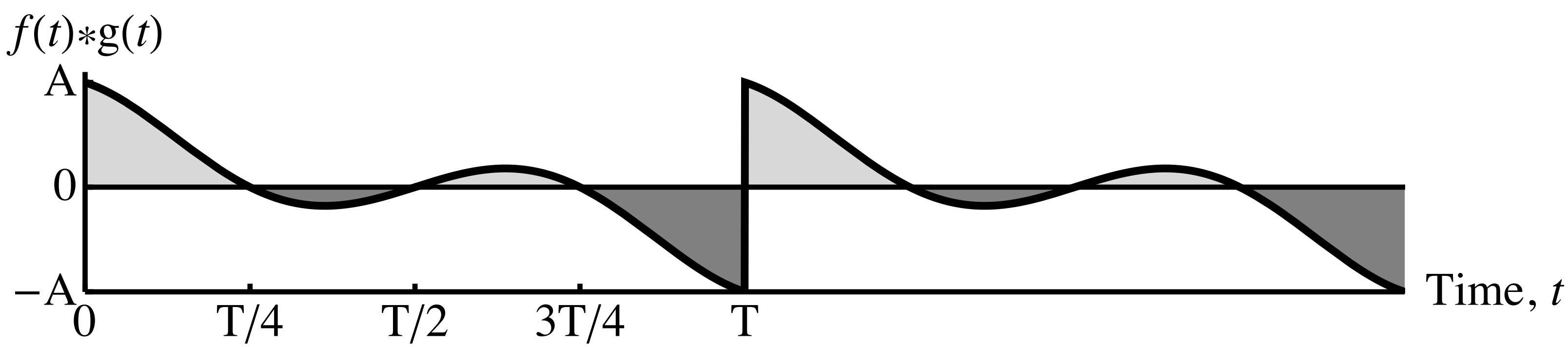


Integrate over 1 period of the fundamental. Multiply by $2/T$. That number is the coefficient of the sin($10t$) term.

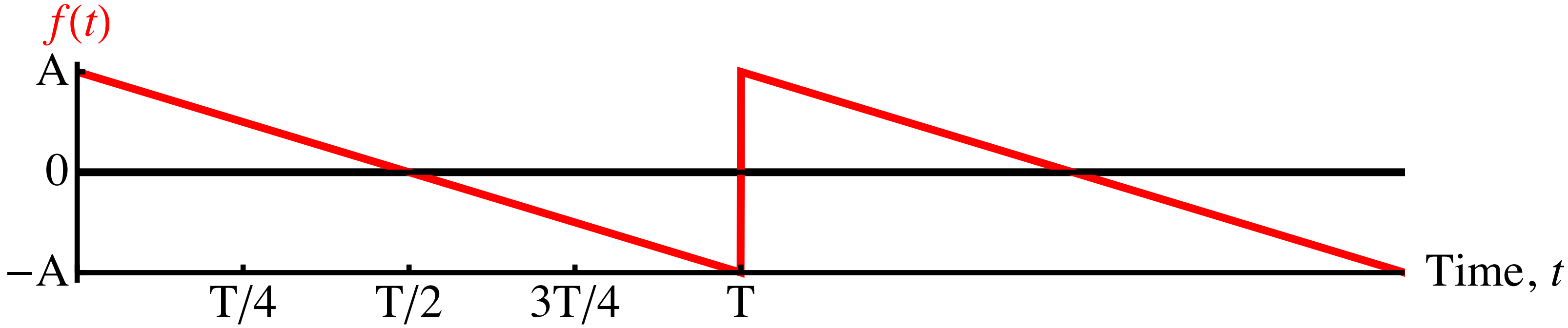
Are we sure there is no cosine contribution?



Multiply it by the cosine term whose coefficient you want to find .... (here choose $n = 1$)

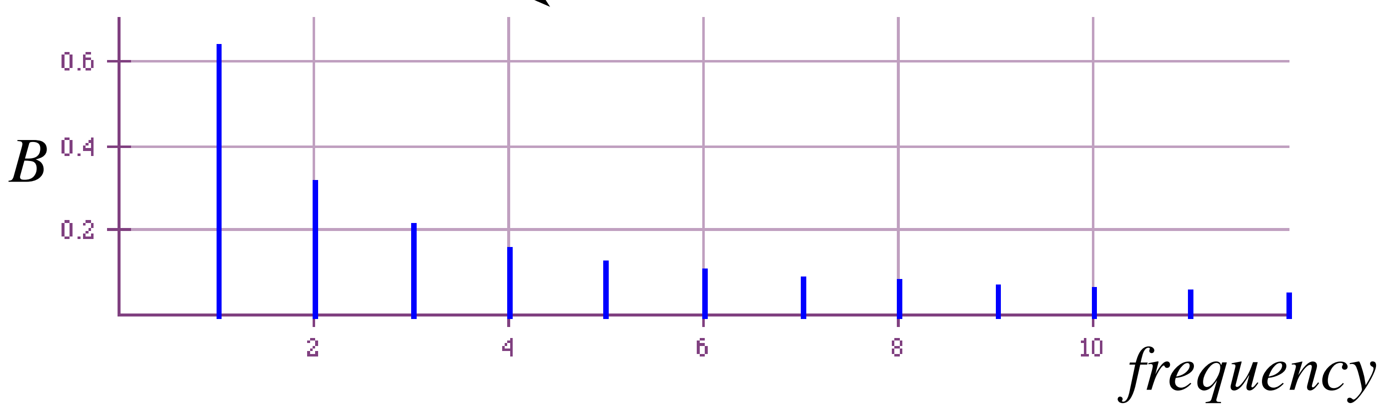


Integrate over 1 period of the fundamental. Multiply by $2/T$. That number is the coefficient of the $\cos(1t)$ term. (ZERO)



$$f(t) = \sum_n \frac{2}{n\pi} \sin\left(n\omega_f t\right)$$

$$f(t) = A - \frac{2A}{T_f} t \quad 0 < t < T_f$$



ODD functions of t have the property $f(t) = -f(-t)$. Their Fourier representation must also be in terms of odd functions, namely sines.

Suppose we have an odd periodic function $f(t)$ like our sawtooth wave and you have to find its Fourier series

$$\sum_{n=1,2\ldots} b_n \sin(n\omega t)$$

Then the unknown coefficients can be evaluated this way

Integrate over the period of the fundamental

Here's the coefficient of the $\sin(\omega_n t)$ term! Plot it on your spectrum!

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)\,dt$$

normalize properly

the function

the harmonic

EVEN functions of t have the property $f(t) = +f(-t)$. Their Fourier representation must also be in terms of even functions, namely cosines.

Suppose we have an even periodic function $f(t)$ and you have to find its Fourier series

$$\frac{a_0}{2} + \sum_{n=1,2\ldots} a_n \cos(n\omega t)$$

Then the unknown coefficients can be evaluated this way

Integrate over the period of the fundamental

Here's the coefficient of the $\cos(\omega_n t)$ term! Plot it on your spectrum!

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)\,dt$$

normalize properly — the function — the harmonic

Any periodic function $f(t)$, whether even, odd, or neither, can be written as a Fourier Series

$$\frac{a_0}{2} + \sum_{n=1,2...} a_n \cos(n\omega t) + \sum_{n=1,2...} b_n \sin(n\omega t)$$

or

$$\sum_{-\infty}^{\infty} c_n e^{in\omega t}$$

We won't do this form for now!

with

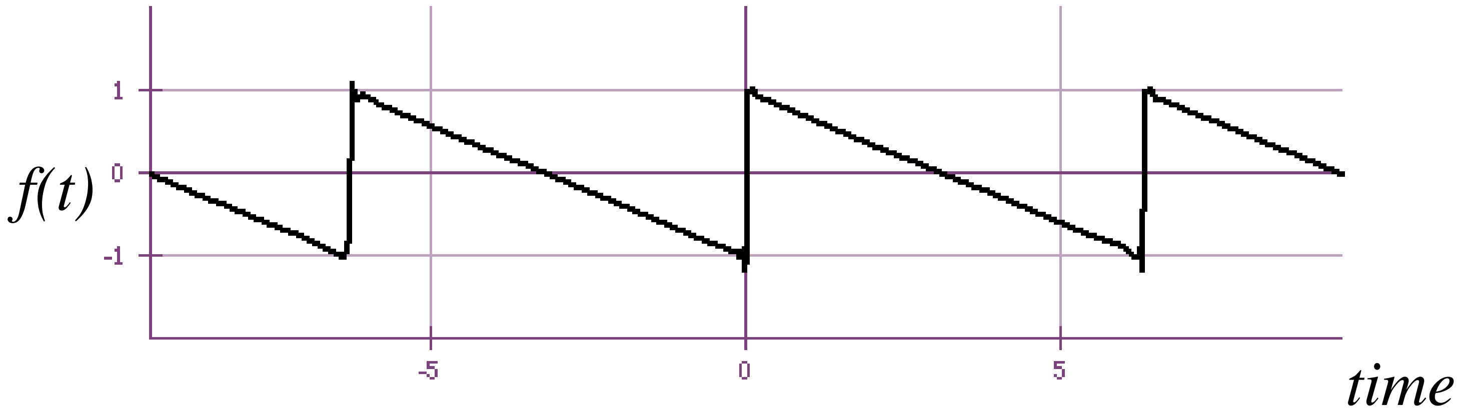
$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega t)\,dt$$

$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega t)\,dt$$

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-in\omega t}\,dt$$

$$c_0 = \frac{a_0}{2}$$

$$c_n = \frac{a_n - ib_n}{2};\ c_{-n} = \frac{a_n + ib_n}{2}$$



$$f(t) = A - \frac{2A}{T_f}t \quad ;0 < t < T_f$$

$$b_n = \frac{2}{T_f}\int_0^{T_f} f(t)\sin(n\omega_f t)\,dt$$

T_f *(the fundamental period)* is related to ω_f. How?

$$b_n = \frac{2}{T_f}\int_0^{T_f}\left[A - \frac{2At}{T_f}\right]\sin(n\omega_f t)\,dt$$

Do this integral in your head

Look up this integral

$$b_n = \frac{2}{T_f}\int_0^{T_f}\left[A\sin\left(n\omega_f t\right) - \frac{t2A\sin\left(n\omega_f t\right)}{T_f}\right]dt = -\frac{4A}{T_f^2}\int_0^{T_f} t\sin\left(n\omega_f t\right)dt$$

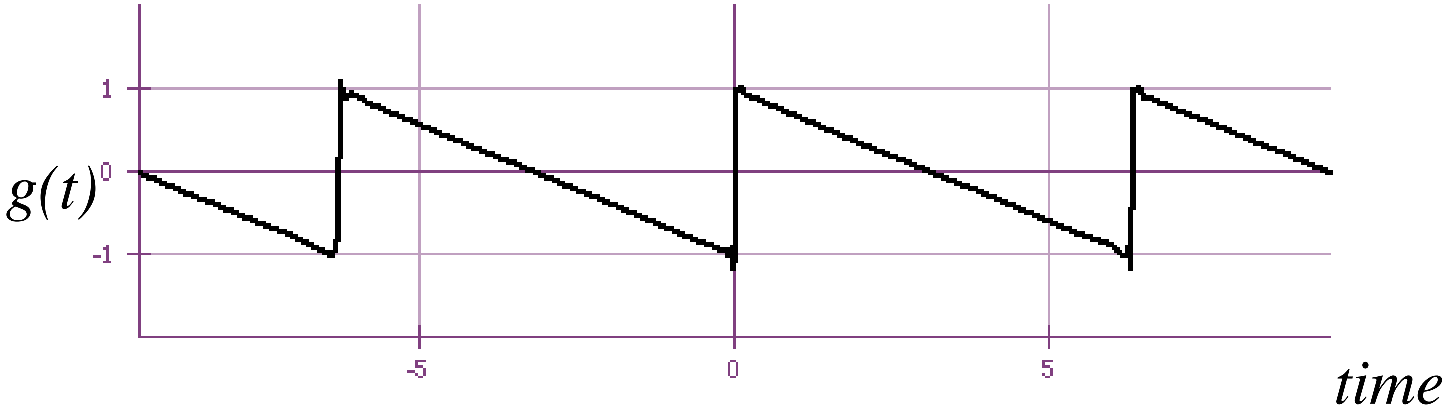
$$= -\frac{4A}{T_f^2}\left[\frac{1}{n^2\omega_f^2}\sin\left(n\omega_f t\right) - \frac{t}{n\omega_f}\cos\left(n\omega_f t\right)\right]_0^{T_f}$$

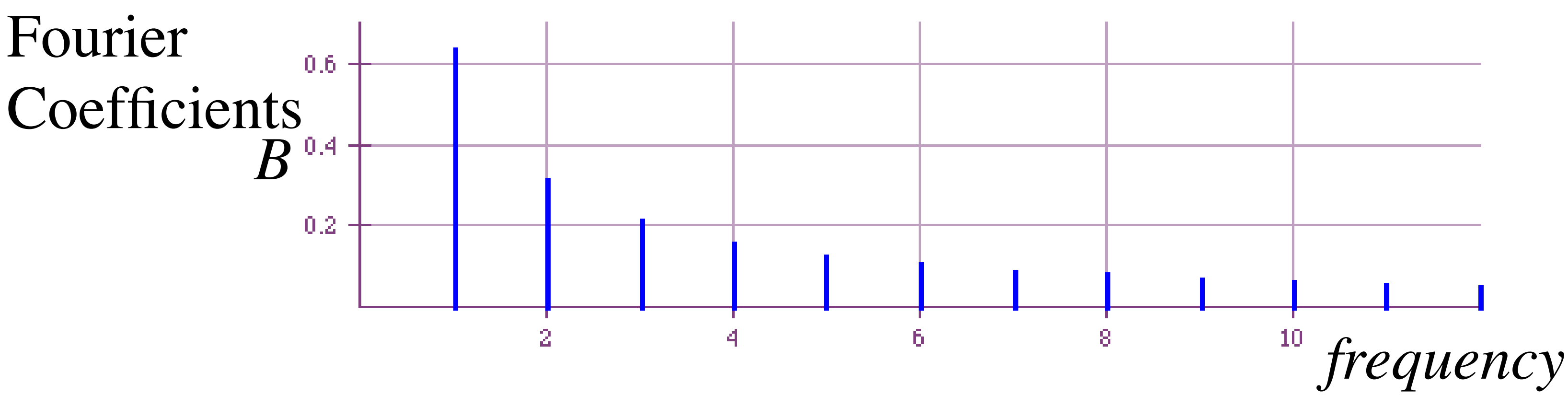
$$= \frac{4A}{T_f^2}\left[\frac{1}{n^2\omega_f^2}\left\{\sin\left(n\omega_f T_f\right) - \sin\left(0\right)\right\} - \left\{\frac{T_f}{n\omega_f}\cos\left(n\omega_f T_f\right) - 0\cos\left(0\right)\right\}\right]$$

$$= \frac{2A}{n\pi}$$

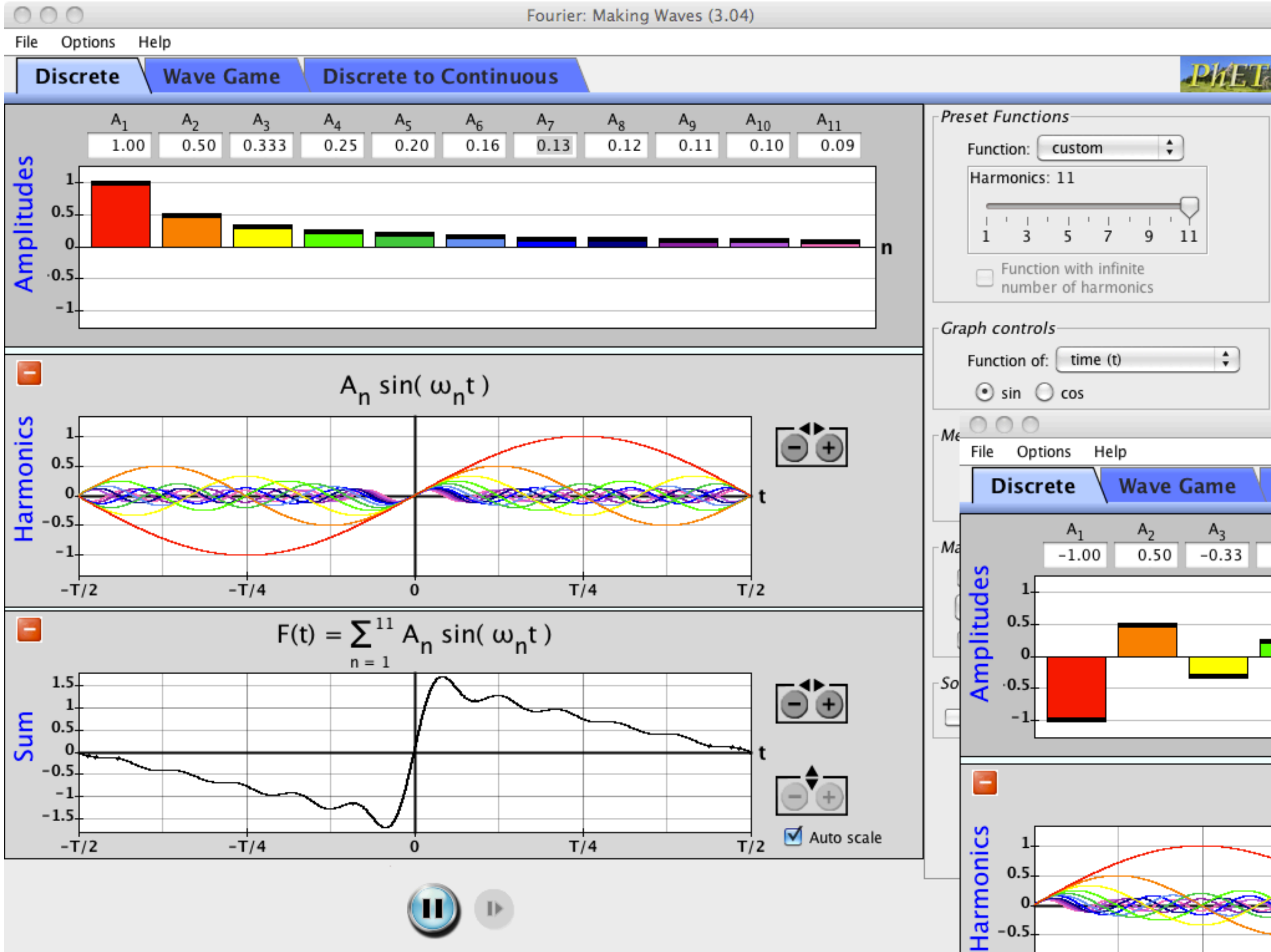
$$\omega_f T_f = 2\pi$$

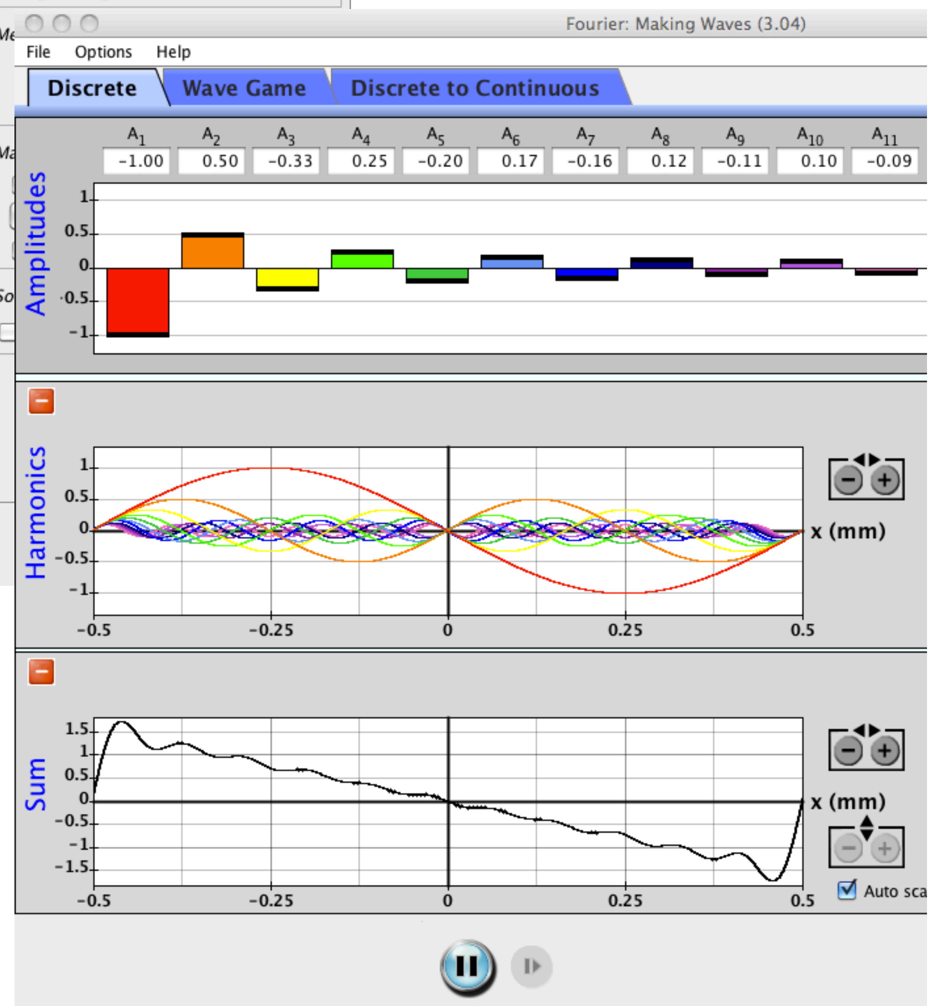
$$f(t) = \sum_n \frac{2}{n\pi} \sin\left(n\omega_f t\right)$$





$$f(t) = A - \frac{2A}{T_f} t; \quad 0 < t < T_f \text{ and repeated}$$





Different time origin
by change of phase!

http://phet.colorado.edu/en/get-phet/one-at-a-time
Fourier: Making waves

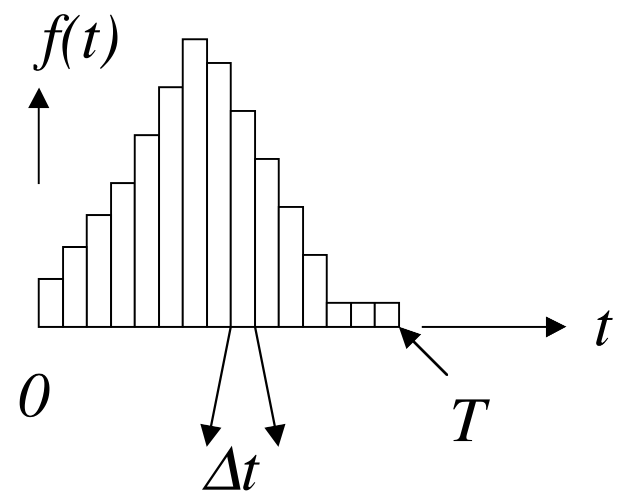
Aside: note that $\frac{1}{T}\int_0^T f(t)\,dt$ can be identified as the average value of the function $f(t)$ over the time period T.

$$\langle f(t)\rangle = \frac{1}{N}\left(f_1 + f_2 + f_3 + \ldots. + f_N\right)$$

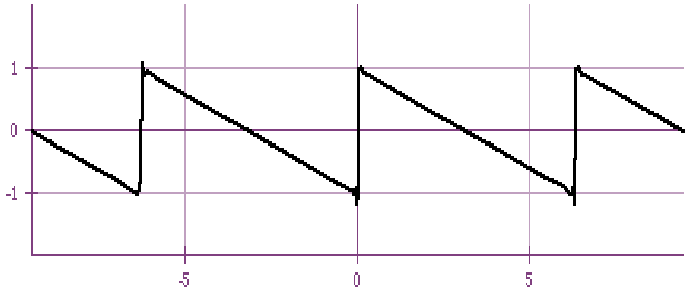
$$\langle f(t)\rangle = \frac{\Delta t}{T}\left(f_1 + f_2 + f_3 + \ldots. + f_N\right)$$

$$\langle f(t)\rangle = \frac{1}{T}\sum_{i=1}^{N} f_i \Delta t$$

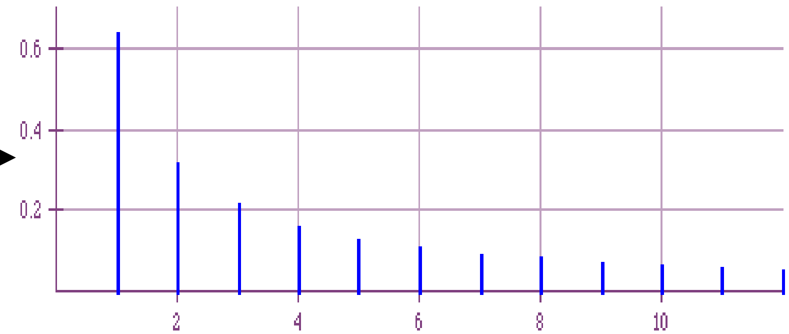
$$\langle f(t)\rangle \rightarrow \frac{1}{T}\int_0^T f(t)\,dt$$



Example of a response:



time

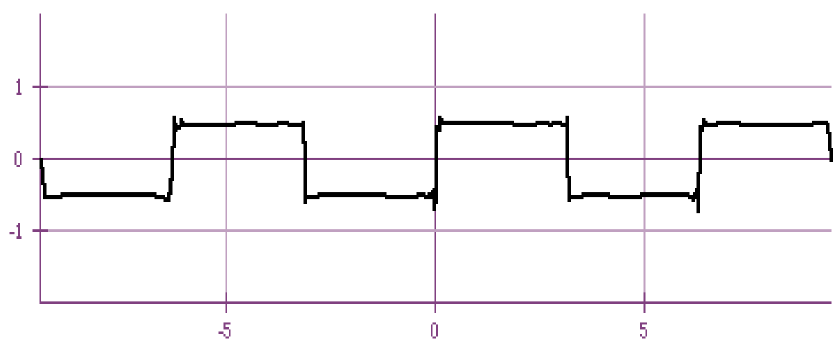


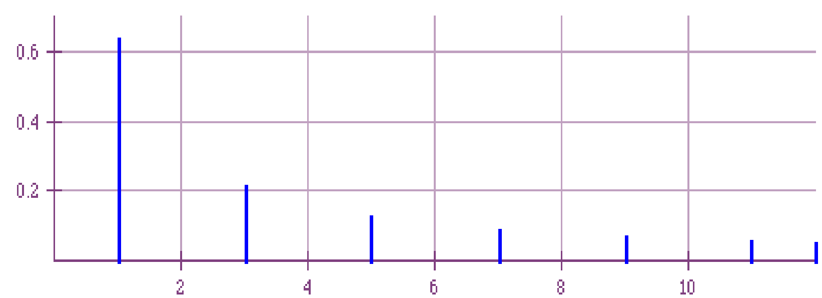
frequency

Black box
Z(ω)

Black box Z(ω)
filters even freqs
no phase shift

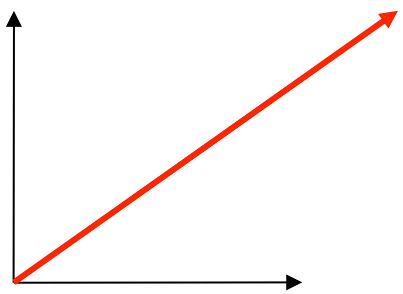
?





Sines and cosines as basis functions: projections

Vectors are familiar examples: we *project* a vector onto its *basis vectors* to find its *components*.



$$\vec{F} \bullet \hat{x} = |F| \cos\phi = F_x$$
$$\vec{F} \bullet \hat{y} = |F| \cos\left(\frac{\pi}{2} - \phi\right) = |F| \sin\phi = F_y$$

Projections involve a dot product, which can be written this way:

$$\vec{F} = (F_1, F_2, F_3 ....)$$
$$\vec{G} = (G_1, G_2, G_3 ....)$$
$$\vec{F} \bullet \vec{G} = F_1 G_1 + F_2 G_2 + ...$$

Sines and cosines as basis functions: projections

Functions can be compared to vectors – a list of values

$$f(t) \doteq (f_1, f_2, f_3 ....)$$
$$g(t) \doteq (g_1, g_2, g_3 ....)$$
$$"f(t) \bullet g(t)" = \sum f_i g_i \rightarrow \frac{2}{T} \int_0^T f(t) g(t) dt$$

The integral has to be over one period for sine and cosine functions to ensure orthogonality and the factor out front ensures proper normalization.