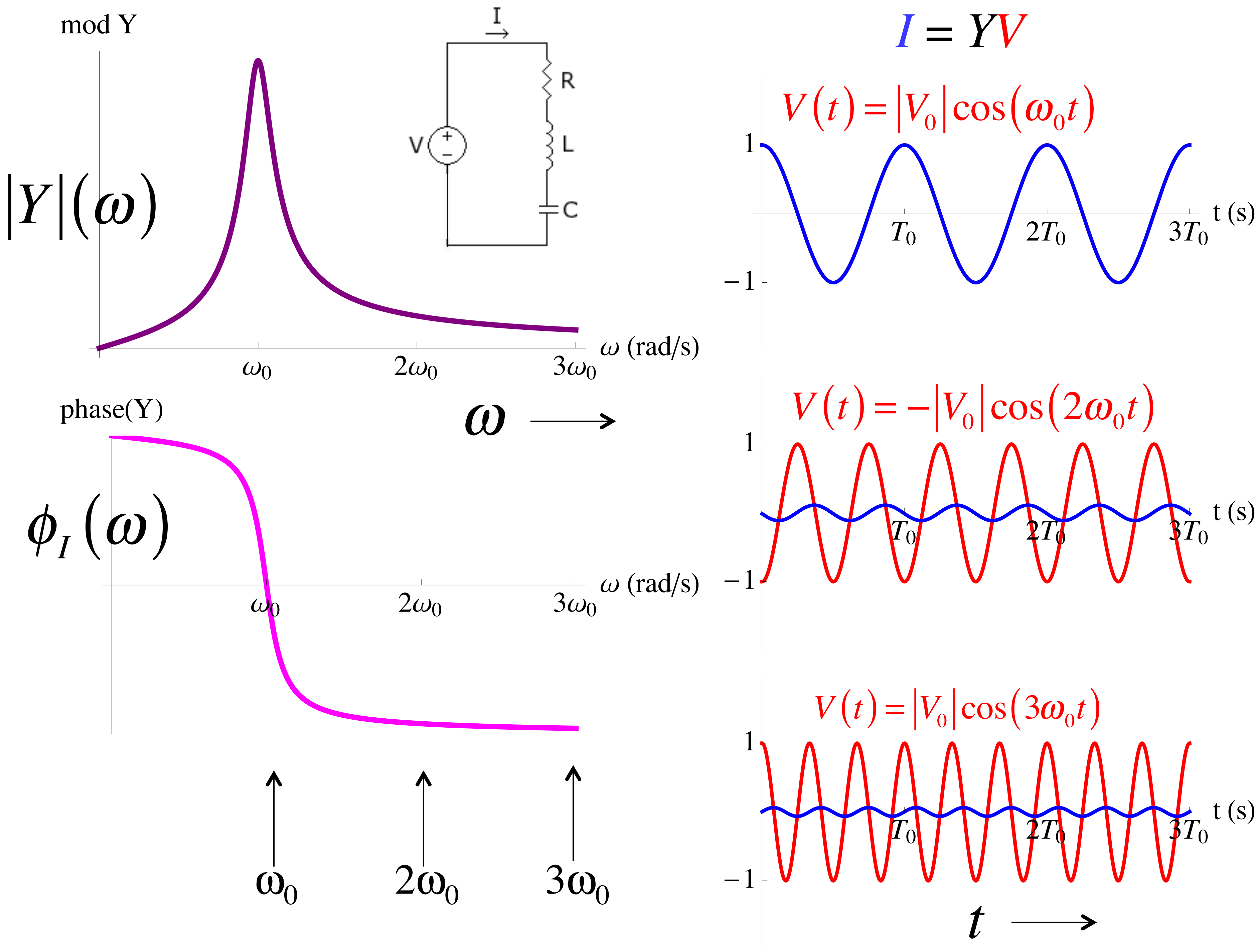
Response of a damped oscillator to a periodic driving force of arbitrary shape:

We have learned that a damped oscillator produces a sinusoidal response to a sinusoidal driving force. In the language of your lab example: The LRC circuit responds to a sinusoidal driving voltage with a sinusoidal current (at the same frequency). That current is related to the voltage by the "admittance" *Y* by

$I = V\,Y = V(1/Z)$

Y is determined by the circuit parameters *L, R, C,* and is also dependent on the frequency of the driving force. We have learned that *Y* can be written as a complex number whose amplitude tells how large the current is for a given *V* and whose phase tells how much $I(t)$ is shifted wrt $V(t)$. (see previous notes and your class group exercise for derivation)

The following page shows this graphically for a given circuit and for 3 different driving voltages.



mod Y
|Y|(ω)
I
R
L
C
V
ω (rad/s)
ω0
2ω0
3ω0
phase(Y)
ω
φ_I(ω)
I = YV
V(t) = |V0|cos(ω0 t)
V(t) = −|V0|cos(2ω0 t)
V(t) = |V0|cos(3ω0 t)
1
−1
t (s)
T0
2T0
3T0
t

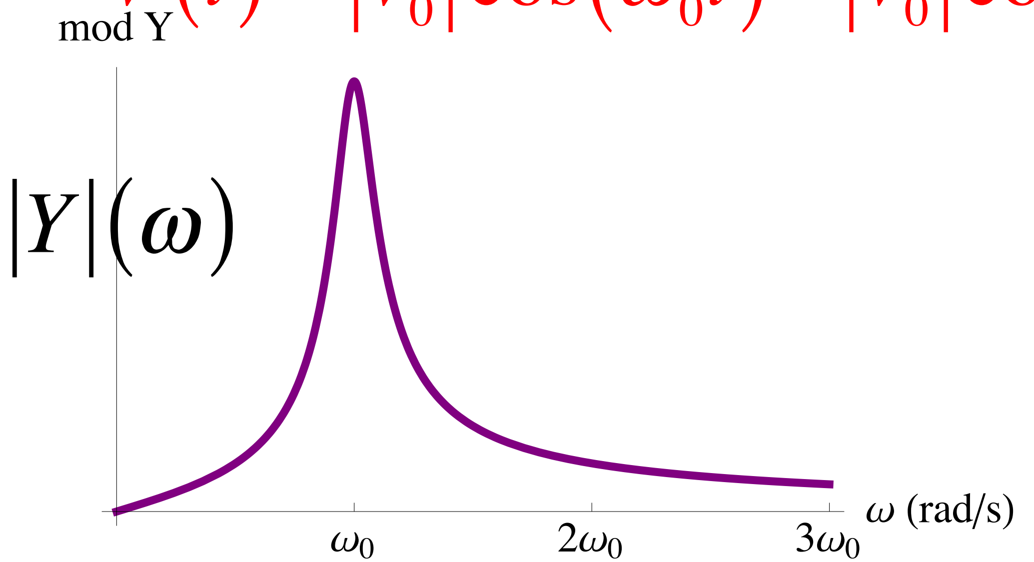
Response of a damped oscillator to a periodic driving force of arbitrary shape:

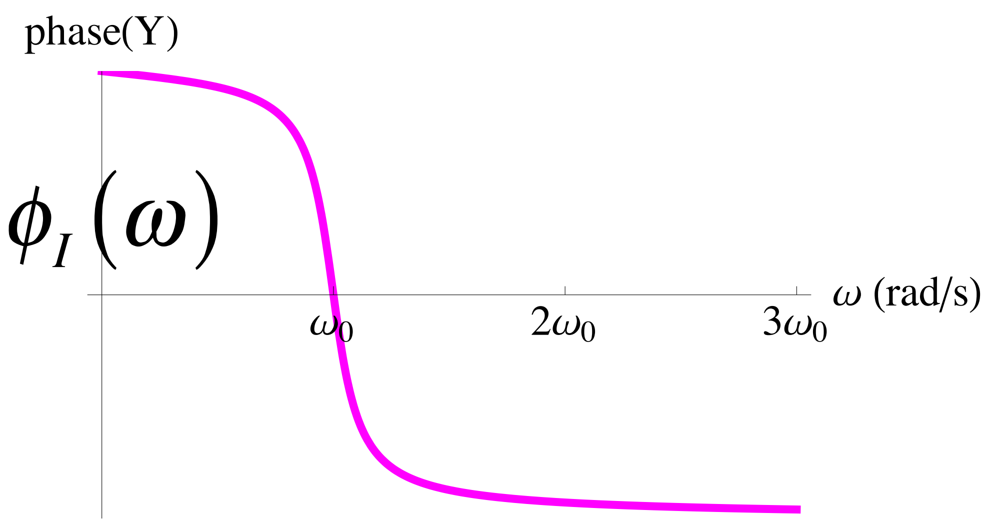
If a system is a LINEAR system, it means that if several sinusoidal driving forces are added and applied at once, then the response is just the sum of the individual responses. It seems obvious that the circuit is linear, but there are many systems that are non-linear.

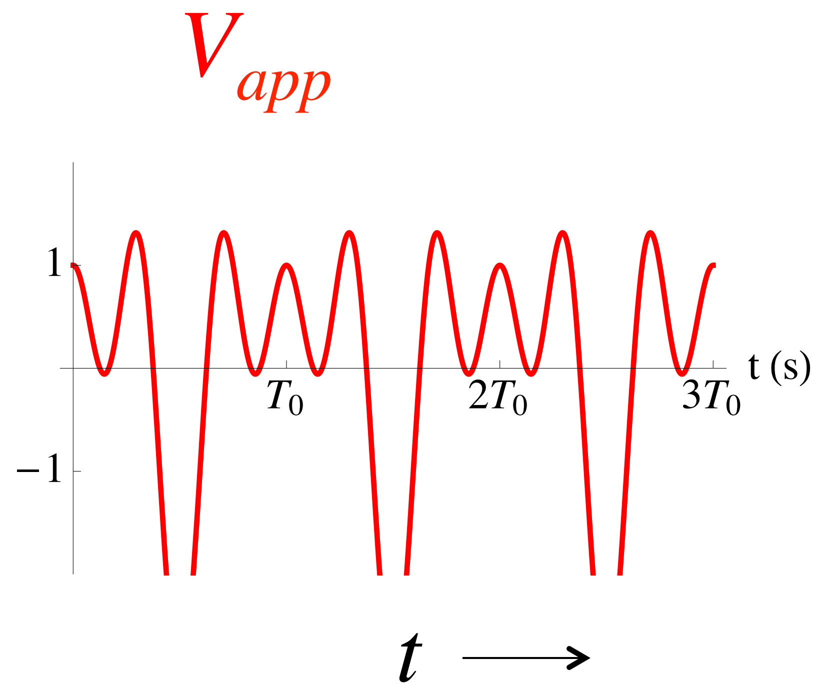
The following page shows this graphically for the same circuit the driving voltage that is the sum of the three previous ones, and the resulting current which is also the sum of the three previous currents. No longer pure sinusoids!

Notice that the shape of the current and the voltage are not the same anymore! It's not true that there's one simple scaling factor and one phase shift!

$$V(t) = |V_0|\cos(\omega_0 t) - |V_0|\cos(2\omega_0 t) + |V_0|\cos(3\omega_0 t)$$

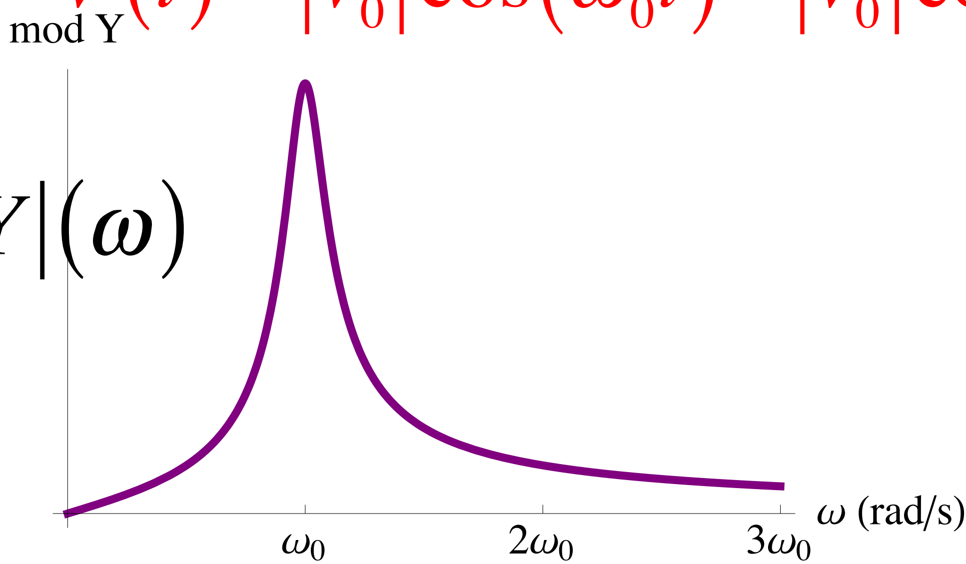




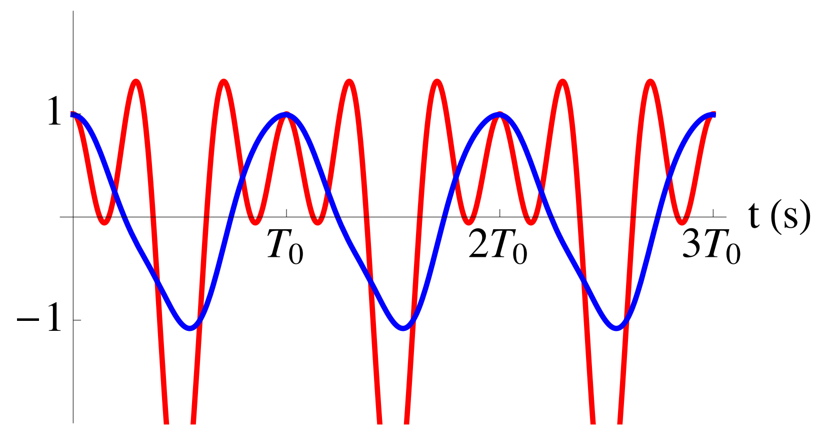


$$V(t) = |V_0|\cos(\omega_0 t) - |V_0|\cos(2\omega_0 t) + |V_0|\cos(3\omega_0 t)$$

$|Y|(\omega)$

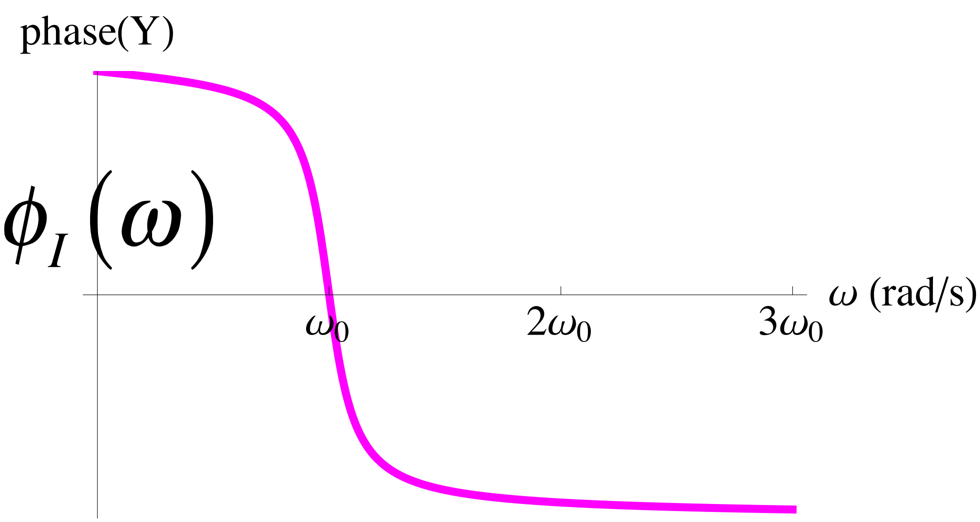


$$I \neq Y\, V_{app}$$



$t \longrightarrow$

$\phi_I(\omega)$



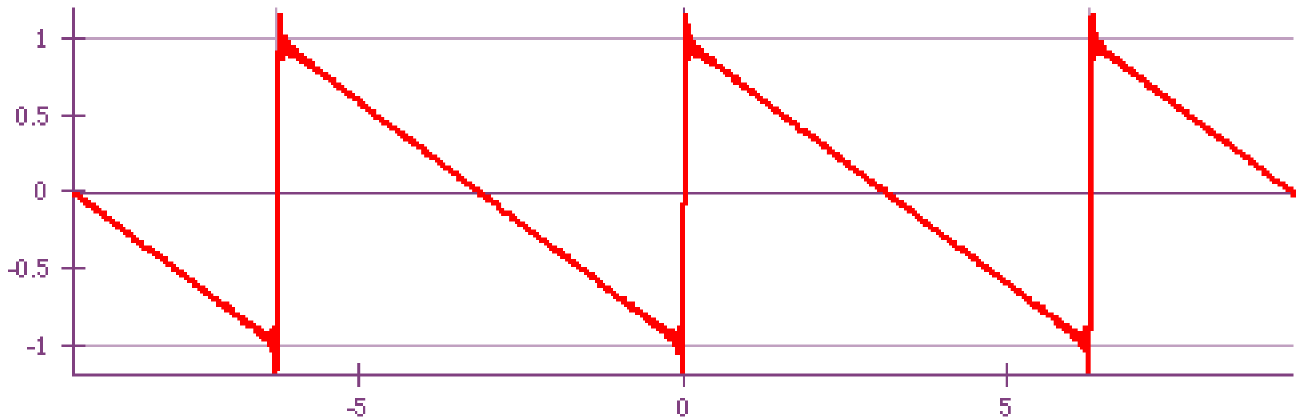
$$I(t) = |V_0||Y|_{\omega_0}\cos\left(\omega_0 t + \phi_{\omega_0}\right)$$
$$-|V_0||Y|_{2\omega_0}\cos\left(2\omega_0 t + \phi_{2\omega_0}\right) + |V_0||Y|_{3\omega_0}\cos\left(3\omega_0 t + \phi_{3\omega_0}\right)$$

Response of a damped oscillator to a periodic driving force of arbitrary shape:

So now you can see how useful the admittance function is, and why it is important to know how its magnitude and phase shift vary with frequency: if there is a driving force that can be expressed as the sum of sinusoids, we simply use the admittance function to find the response at each of the driving frequencies, and add the responses to get the net response.

It turns out that *any* periodic driving force can be decomposed into such a sum of sinusoidal components, and obviously it must be our job to learn how to break down such a periodic function into its component sinusoids. This technique is called **Fourier analysis**.

What if V(t) is this?



$$t \longrightarrow$$

$$V(t) = 1 - \frac{t}{\pi}$$

$$0 < t < 2\pi$$

And periodic repetitions

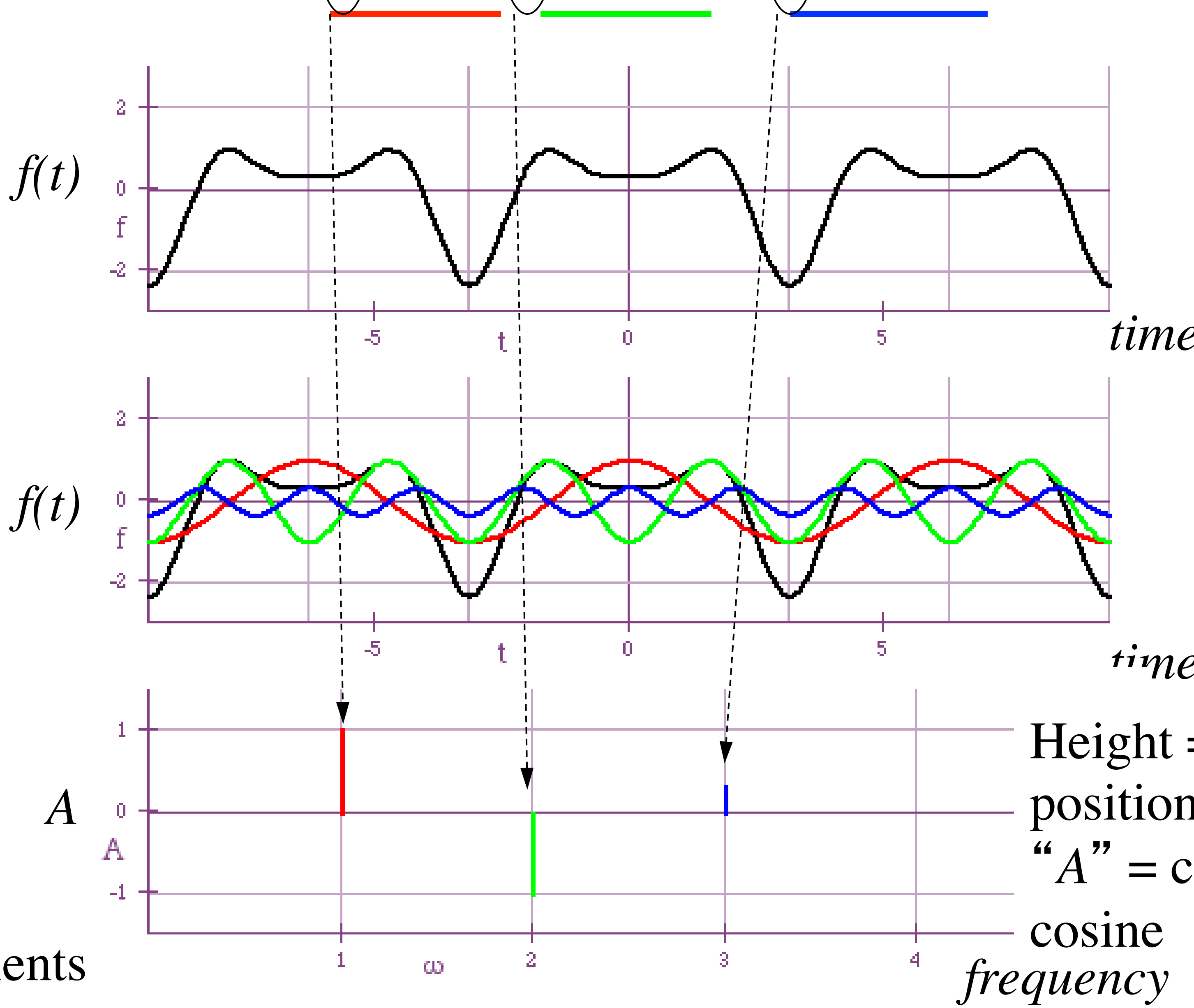
Use Fourier Analysis!

Fourier Analysis

- is easy
- is a sensible thing to do
- has a bad reputation (unjustly)
- does not involve impossible integrals

Before we tackle the problem we just posed, backtrack a bit to review some terminology involving Fourier series

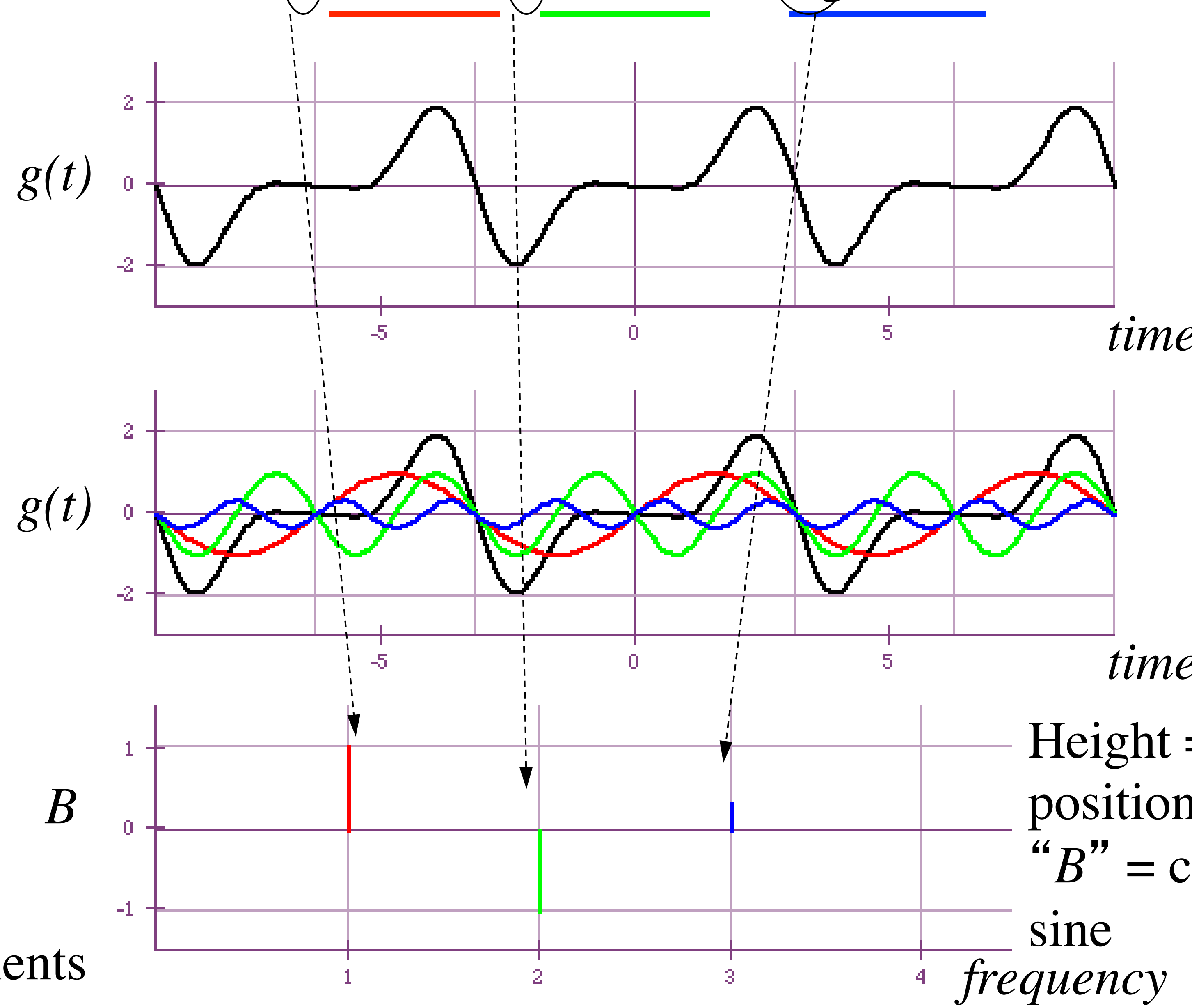
$$f(t) = 1\cos(t) - \cos(2t) + \frac{1}{3}\cos(3t)$$



Height = amp
position = freq
"A" = code for cosine

Fourier Coefficients

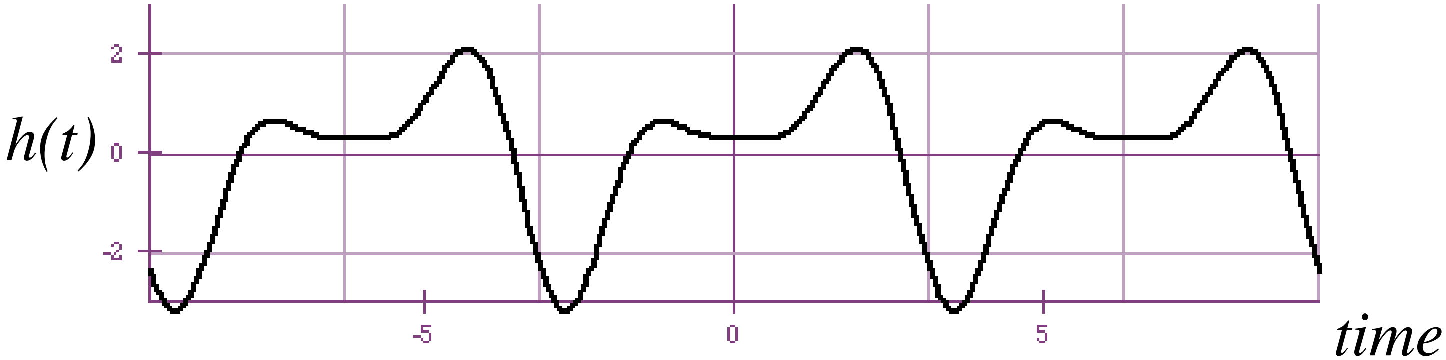
$$g(t) = 1\sin(t) - \sin(2t) + \frac{1}{3}\sin(3t)$$



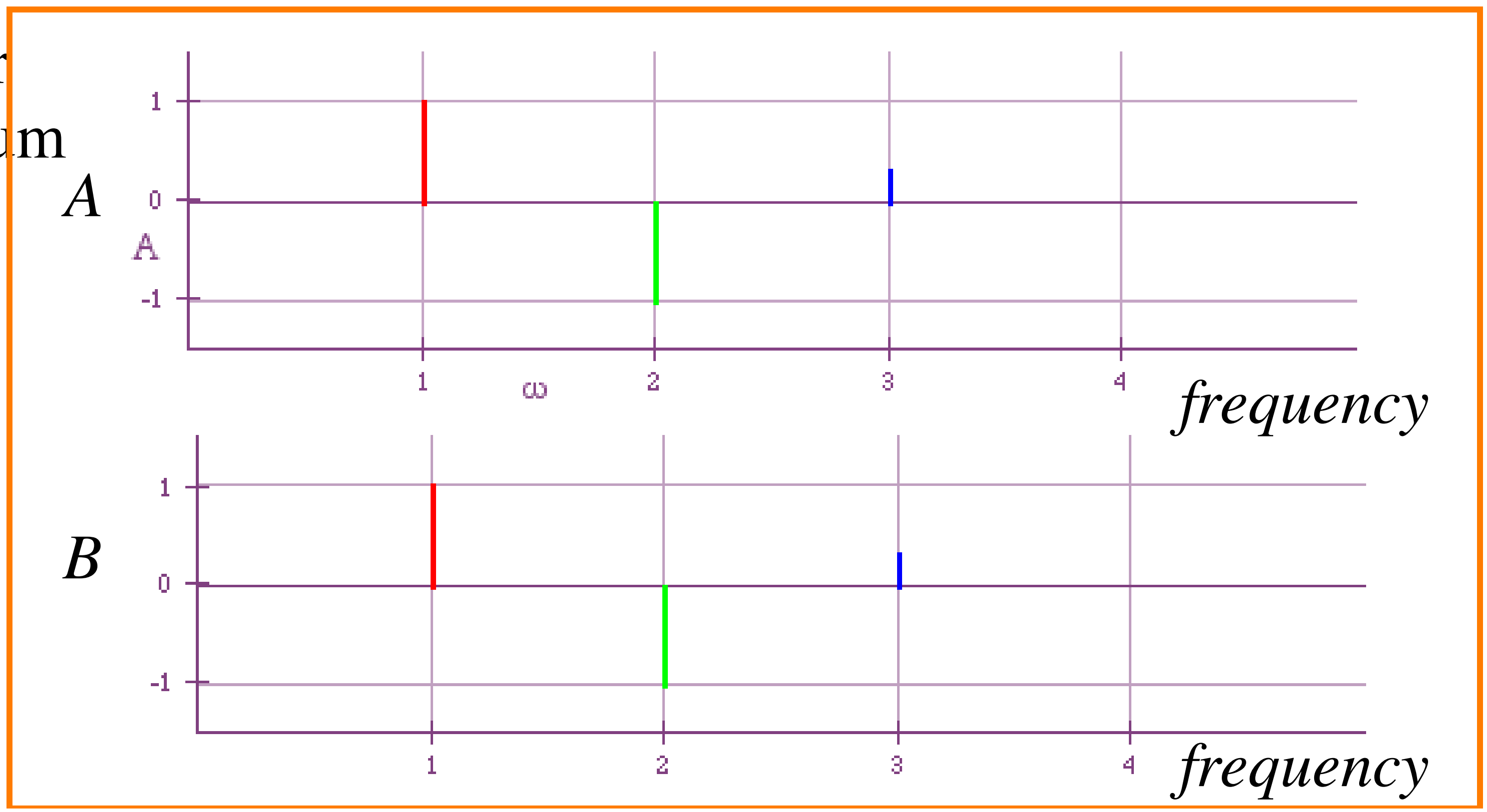
Fourier Coefficients

Height = amp
position = freq
"*B*" = code for sine

$$h(t) = f(t) + g(t)$$



Fourier Spectrum

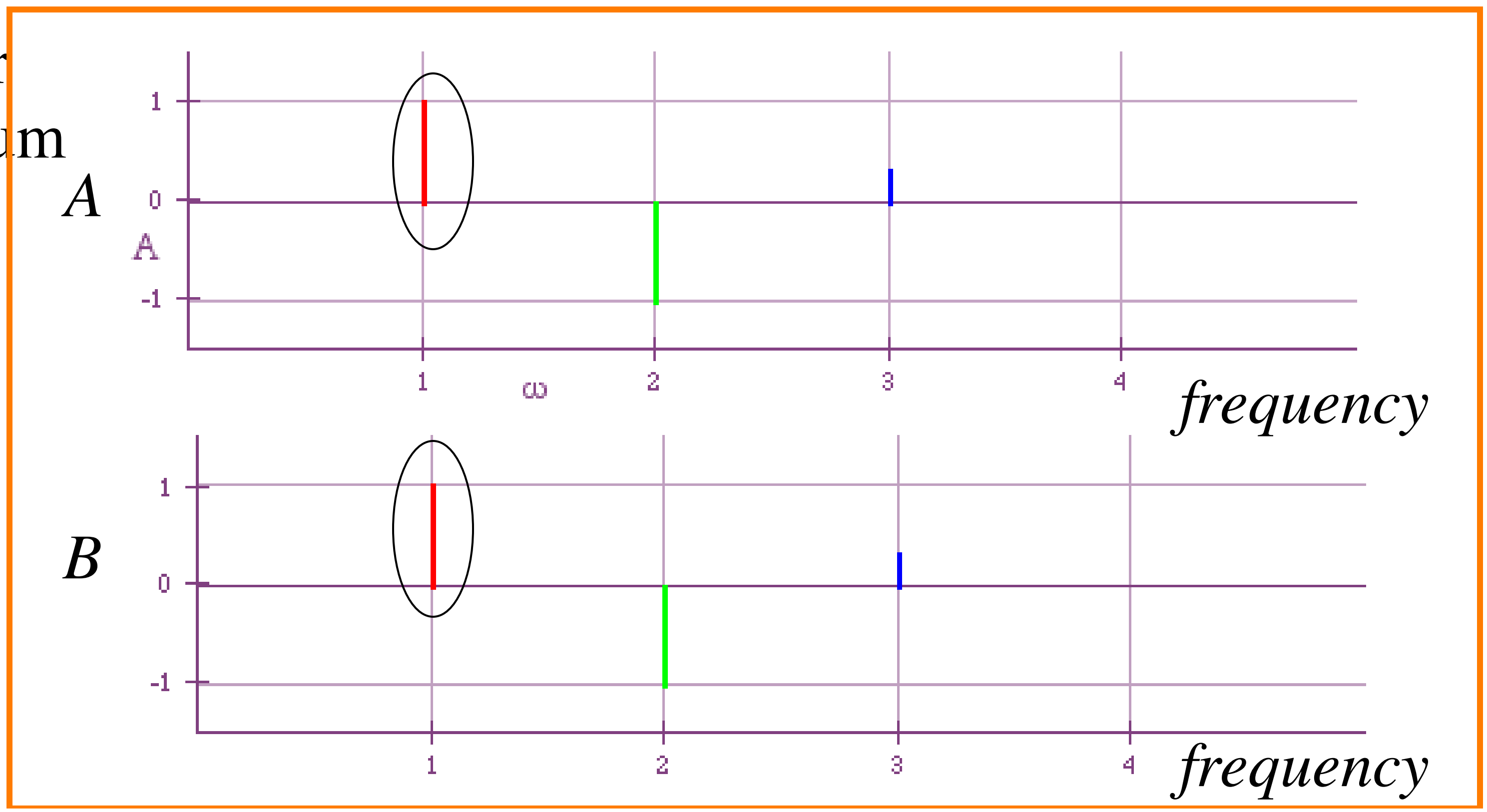


Look at fundamental: $1\cos(\omega t) + 1\sin(\omega t)$

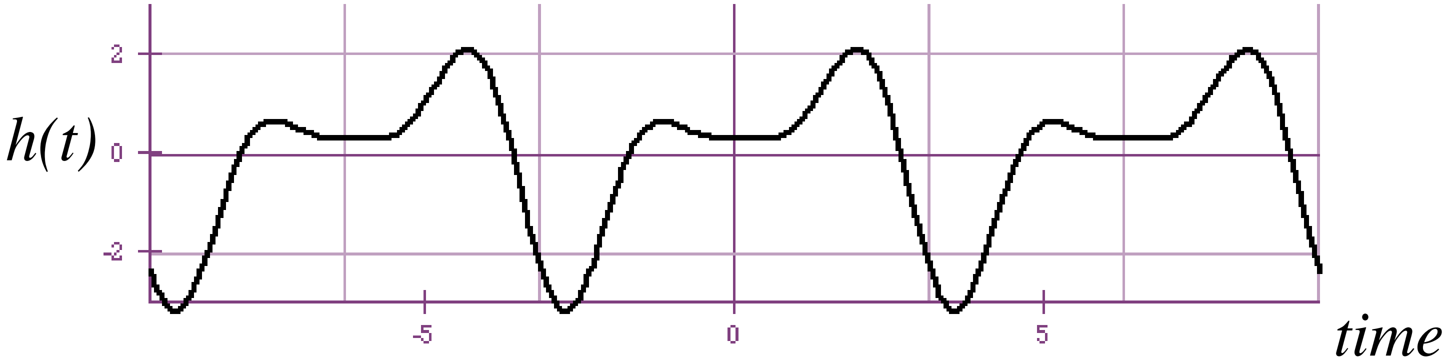
Compute: $\sqrt{1^2 + 1^2} = \sqrt{2}; \arctan\left(\frac{-1}{1}\right) = -\frac{\pi}{4}$

Aha! Fundamental also $\mathrm{Re}\left[\sqrt{2}e^{i(\omega t - \pi/4)}\right]$

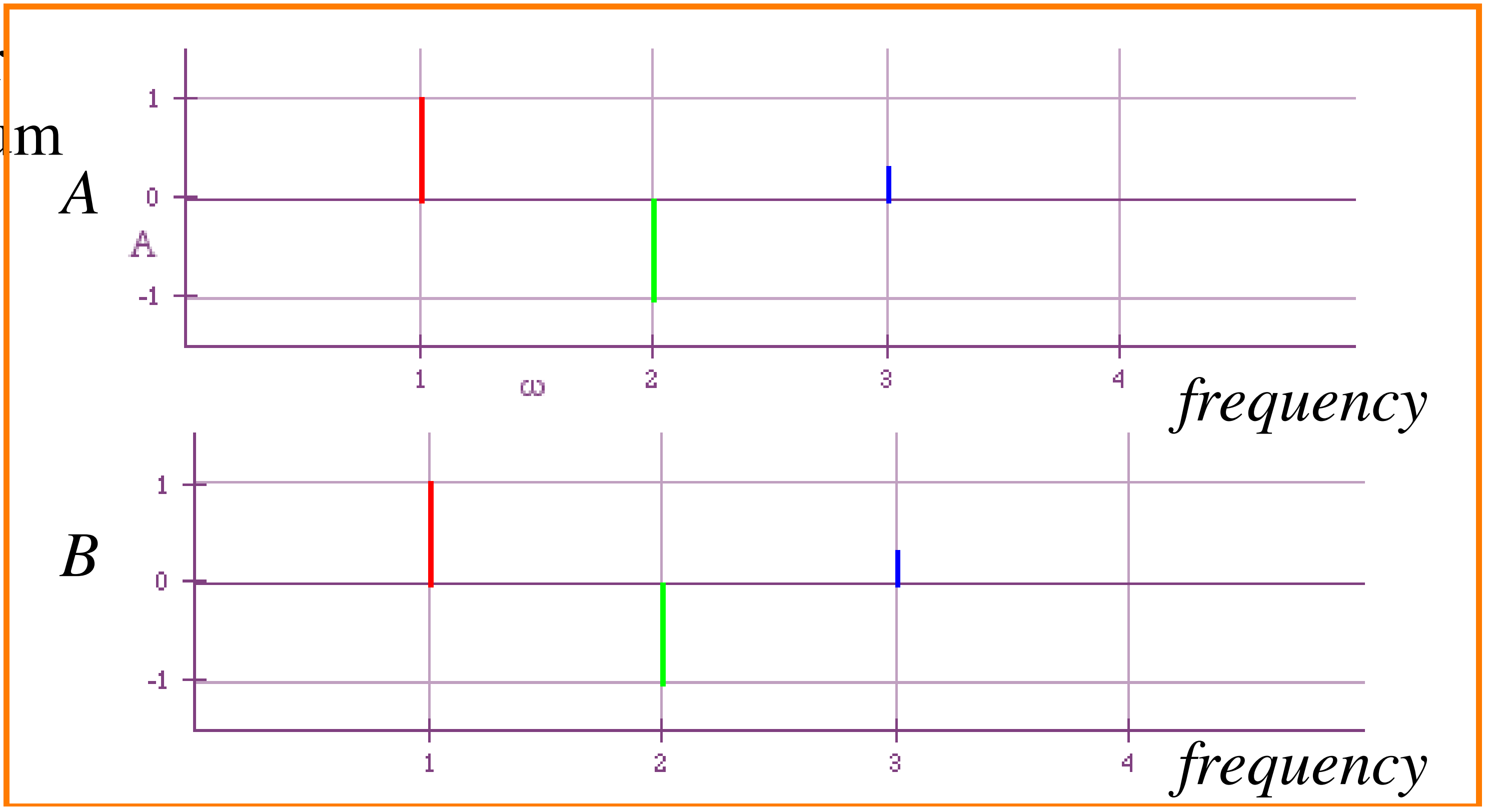
Fourier Spectrum



$$h(t) = f(t) + g(t) \quad = \mathrm{Re}\left[\sqrt{2} e^{-i\pi/4} e^{i\omega t} + ?\, e^{i2\omega t} + ?\, e^{i3\omega t} \right]$$



Fourier Spectrum



What have we learned so far?

- Odd periodic functions with period $T=2\pi/\omega$ can be represented by a series of **sin**$(n\omega t)$ functions
- Even periodic functions with period $T=2\pi/\omega$ can be represented by a series of **cos**$(n\omega t)$ functions
- If these functions represent physical motion, then we think of the motion as the superposition of the motions of SHOs with increasing frequencies (each a multiple of the fundamental one)
- In these special odd/even cases the coefficient of the **sin**$(n\omega t)$ (or cos) term represents the amplitude of that particular SHO
- In these special odd/even cases, we can plot the size of a coefficient of the $n\omega t$ term at the $n\omega t$ frequency to give an alternative representation of the function

What have we learned so far?

- Periodic functions with period $T=2\pi/\omega$ that are neither even nor odd can be represented by a series of **sin(*n*ω*t*)** and **cos(*n*ω*t*)** functions, all with different coefficients
- If these functions represent physical motion, then we think of the motion as the superposition of the motions of SHOs with increasing frequencies (each a multiple of the fundamental one)
- In these general cases the coefficients of the **sin(*n*ω*t*)** **cos(*n*ω*t*)** terms must be combined to represents the **amplitude and phase** of that particular SHO
- In these general cases, we must make **two plots** to give an alternative representation of the function. We can chose to plot sin coeff and cos coeff, or magnitude and phase

What have not learned yet?

• How to find the coefficients if the function is not explicitly written in terms of sines and cosines (we will)
• How to deal with exponential functions
• Lots of practice needed, of course ….