PH421 F12 Exam solution

1. SOLUTION:

(a) $V(t)$ has period T. $V(t)$ is zero for half a period, and a sinusoid (with period T) for half a period.

With this choice of origin, $V(t)=\begin{cases} V_0\cos(2\pi t/T) & -T/4<t<T/4 \\ 0 & T/4<t<3T/4 \end{cases}$

(b) The function is even, so a cosine expansion is appropriate using fundamental period T and fundamental frequency $\omega=2\pi/T$. **All sine coefficients b_n are zero**.

$$V(t)=\frac{a_0}{2}+\sum_{n=1}^{\infty}\left[a_n\cos(n\omega t)\right] \text{ with } a_n=\frac{2}{T}\int_{-T/4}^{3T/4}V(t)\cos(n\omega t)\,dt$$

$$a_n=\frac{2}{T}\left[\int_{-T/4}^{T/4}V_0\cos(2\pi t/T)\cos(n2\pi t/T)\,dt+\int_{T/4}^{3T/4}0\cos(n2\pi t/T)\,dt\right]$$

$$=\frac{2V_0}{T}\int_{-T/4}^{T/4}\cos(2\pi t/T)\cos(n2\pi t/T)\,dt$$

$$=\frac{V_0}{\pi}\int_{-\pi/2}^{\pi/2}\cos u\cos(nu)\,du \qquad \text{with } u=\frac{2\pi t}{T}$$

$n=1$

$$a_1=\frac{V_0}{\pi}\int_{-\pi/2}^{\pi/2}\cos^2 u\,du=\frac{V_0}{\pi}\int_{-\pi/2}^{\pi/2}\left[\tfrac{1}{2}+\tfrac{1}{2}\cos 2u\right]du=\frac{V_0}{\pi}\left[\tfrac{u}{2}-\tfrac{1}{4}\cancel{\sin(2u)}\right]_{-\pi/2}^{\pi/2} \qquad \sin\pi=-\sin\pi=0$$

$$=\frac{V_0}{2}$$

$n>1$

$$a_n=\frac{V_0}{\pi}\int_{-\pi/2}^{\pi/2}\cos u\cos nu\,du$$

$$=\frac{V_0}{4\pi}\int_{-\pi/2}^{\pi/2}\left(e^{iu}+e^{-iu}\right)\left(e^{inu}+e^{-inu}\right)du=\frac{V_0}{4\pi}\int_{-\pi/2}^{\pi/2}\left(e^{i(n+1)u}+e^{-i(n+1)u}+e^{i(n-1)u}+e^{-i(n-1)u}\right)du$$

$$=\frac{V_0}{4\pi}\int_{-\pi/2}^{\pi/2}\left[2\cos((n+1)u)+2\cos((n-1)u)\right]du$$

$$=\frac{V_0}{2\pi}\left[\frac{\sin((n+1)u)}{(n+1)}+\frac{\sin((n-1)u)}{(n-1)}\right]_{-\pi/2}^{\pi/2}=\frac{V_0}{\pi}\left[\frac{\sin((n+1)\frac{\pi}{2})}{(n+1)}+\frac{\sin((n-1)\frac{\pi}{2})}{(n-1)}\right] \qquad \sin(x)-\sin(-x)=2\sin(x)$$

$$=\begin{cases}\frac{V_0}{\pi}\left(-\frac{1}{3}+1\right)=\frac{2V_0}{3\pi} & n=2\\ 0 & n=3\end{cases}$$

SOLUTION 2:

(a) Newton: $F_{net} = ma = m\ddot{x}(t)$

The net force on the mass comes from 2 sources:

(i) restoring force of the spring directed opposite to the displacement from equilibrium $x(t)$, $F_{spring} = -kx(t)$

(ii) the resistive damping force, directed opposite to the velocity $v(t) = x(dot)(t)$, $F_{damping} = -b\dot{x}(t)$ b characterizes the damping force.

Newton reads: $F_{spring} + F_{damp} = m\ddot{x}(t) \Rightarrow -kx(t) - b\dot{x}(t) = m\ddot{x}(t)$.

Divide by m and rearrange, and define

$$2\beta = b/m; \omega_0^2 = k/m \Rightarrow \ddot{x}(t) + 2\beta\dot{x}(t) + \omega_0^2 x(t) = 0$$

b) Solution to DE is:

$x(t) = Ae^{-\beta t}\cos(\omega_1 t + \theta)$ which oscillates with frequency $\omega_1 = \omega_0\sqrt{1 - \frac{\beta^2}{\omega_0^2}}$ with β = damping constant (1/β = time to decay to 1/e).

If β is small,

$$\beta^2 = \omega_0^2\left(1 - \frac{\omega_1^2}{\omega_0^2}\right) = \frac{4\pi^2}{T_0^2}\left(1 - \frac{T_0^2}{T_1^2}\right) = 4\pi^2\left(1 - \frac{1}{\left(1 + 2\times10^{-4}\right)^2}\right)s^{-2}$$

$$\beta \approx 2\pi\sqrt{1 - \left[1 - 2\times2\times10^{-4}\right]}s^{-1} = 2\pi\times2\times10^{-2}s^{-1} = 0.125s^{-1}$$

c) $1/\beta$ = time to decay to e^{-1} => $2/\beta$ = time to decay to e^{-2}

$$\frac{2}{\beta} \approx \frac{2}{0.125s^{-1}} = \frac{4}{0.25}s = 16s$$

SOLUTION 3:

(a) Establish A and B form relations

$$x(t) = A\cos(\omega t + \phi) = A\cos\phi\cos\omega t - A\sin\phi\sin\omega t$$

$$x(t) = B_p \cos\omega t + B_q \sin\omega t$$

$$\Rightarrow B_p = A\cos\phi; \quad B_q = -A\sin\phi;$$

$$A^2 = B_p^2 + B_q^2$$

$$\tan\phi = \frac{-B_q}{B_p}$$

(b) We are told $x(0) = x_0$ so $x(t) = B_p \cos\omega t + B_q \sin\omega t \Rightarrow x_0 = B_p$

We aren't told v(0) directly, but call it v_0, and we know it is opposite in direction to x_0, and it is given by $v_0 = -\omega B_p \sin\omega 0 + \omega B_q \cos\omega 0 \Rightarrow v_0 = B_q \omega$.

We are told $A = 2x_0$ so

$$A^2 = B_p^2 + B_q^2 \Rightarrow 4x_0^2 = x_0^2 + \left(\frac{v_0}{\omega}\right)^2$$

$$\Rightarrow v_0^2 = 3x_0^2\omega^2$$

$$\Rightarrow v_0 = \pm\sqrt{3}x_0\omega$$

We choose the –ve sign because v is directed in the negative direction at t=0, and x_0 is positive.

So now in B form:

$$x(t) = B_p \cos\omega t + B_q \sin\omega t$$

$$x(t) = x_0 \cos\omega t - \sqrt{3}x_0 \sin\omega t$$

In A form

$$x(t) = A\cos(\omega t + \phi) = 2A\cos(\omega t + \phi)$$

$$\tan\phi = \frac{-B_q}{B_p} = \frac{-\left(-\sqrt{3}x_0\right)}{x_0} = \sqrt{3} \Rightarrow \phi = 60°$$

$$x(t) = 2A\cos(\omega t + 60°)$$

SOLUTION 4

(a) Resonance widths are all the same, so $\beta_1 = \beta_2 = \beta_3$. Resonance freqs are different:

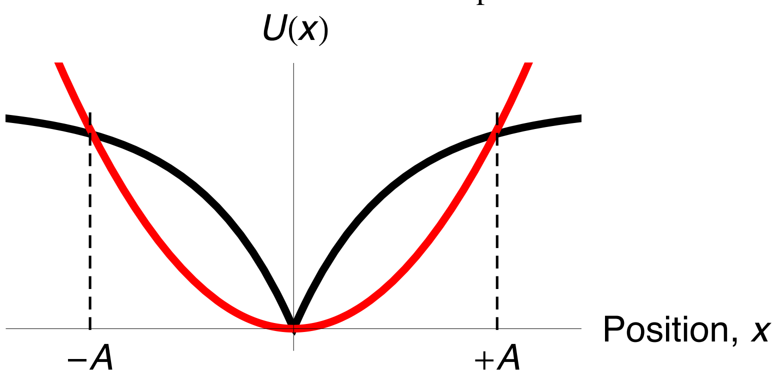
$\omega_{0,3} > \omega_{0,2} > \omega_{0,1}$. Q-values is $Q = \dfrac{\omega_0}{2\beta} \Rightarrow Q_3 > Q_2 > Q_1$ (even though shapes look the same, the Q is larger for larger ω_0 (shorter period) because there are more decay cycles in the same damping time.

(b) For our LRC circuit, L ≈ 0.14 H. R ≈ 280 Ω. Damping time is

$$\frac{1}{\beta} = \frac{2L}{R} = \frac{2 \times 0.14H}{270\Omega} \approx \frac{0.28H}{280\Omega} = 0.001s$$

(c) At frequencies well above resonance, the phase of the acceleration $\ddot{x}(t)$ is in phase with the driving force. We know that the phasor diagram for the electric system has the phase of I-dot 90-deg ahead of I which is in turn 90-deg ahead of *q*: $\phi_{\dot{I}} = \phi_I + \dfrac{\pi}{2} = \phi_q + \pi$

(d) If the <u>same</u> amplitude *A*, then the <u>same</u> energy total *E*, because both have zero KE at ± *A*. At any position, the KE in the harmonic oscillator (red) potential is larger (PE is smaller), so the mass travels faster at every position. It travels the same distance in both cases, so it traverses the distance faster in the case of the HO potential. Period is therefore shorter in the HO potential.



SOLUTION 5

$$V(t)=V_0\cos(\Omega t)\cos(3\Omega t)=\frac{V_0}{4}\left(e^{i\Omega t}+e^{-i\Omega t}\right)\left(e^{i3\Omega t}+e^{-i3\Omega t}\right)$$

(a) $$=\frac{V_0}{4}\left(e^{i4\Omega t}+e^{-i4\Omega t}+e^{i2\Omega t}+e^{-i2\Omega t}\right)=\frac{V_0}{4}\left(2\cos(4\Omega t)+2\cos(2\Omega t)\right)$$

$$V(t)=\frac{V_0}{2}\cos(4\Omega t)+\frac{V_0}{2}\cos(2\Omega t)$$

two sinusoidal voltages of same magnitude $V_0/2$ applied simultaneously.

EACH sinusoidal component of the voltage induces a sinusoidal current component in the circuit $I_\omega(t)=Y(\omega)V_{app,\omega}(t)$

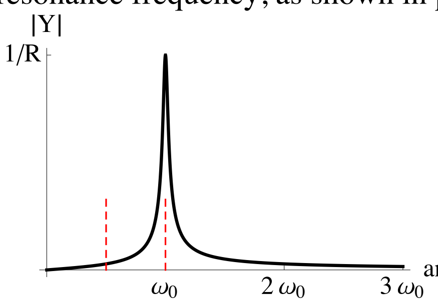
To get the total current in the circuit, sum up the component currents. $I_{tot}(t)=I_{4\Omega}(t)+I_{2\Omega}(t)$

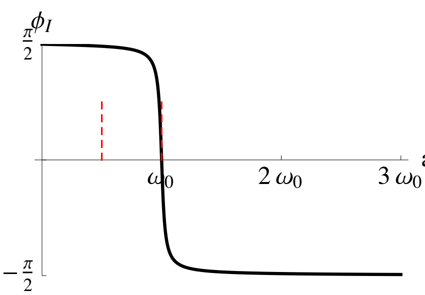
Hence the voltage across the resistor is given by

$V_R(t)=\left(I_{4\Omega}+I_{2\Omega}\right)R=\left(Y(4\Omega)V_{app,4\Omega}+Y(2\Omega)V_{app,2\Omega}\right)R$.

(b)(i) $\Omega=\frac{\omega_0}{4}\Rightarrow 4\Omega=\omega_0, 2\Omega=\frac{\omega_0}{2}\Rightarrow V(t)=\frac{V_0}{2}\cos(\omega_0 t)+\frac{V_0}{2}\cos\left(\frac{\omega_0 t}{2}\right)$

Two sinusoidal voltages of same magnitude applied simultaneously at resonance and half resonance frequency, as shown in plot.





For calculation write $V_{app}(t)=\frac{V_0}{2}e^{i\omega_0 t}+\frac{V_0}{2}e^{i\frac{\omega_0}{2}t}$

$$Y(\omega_0)=\left|Y(\omega_0)\right|e^{i\phi(\omega_0)}=\frac{1}{R}e^{i0}$$

<u>Estimate</u> admittance from the plot

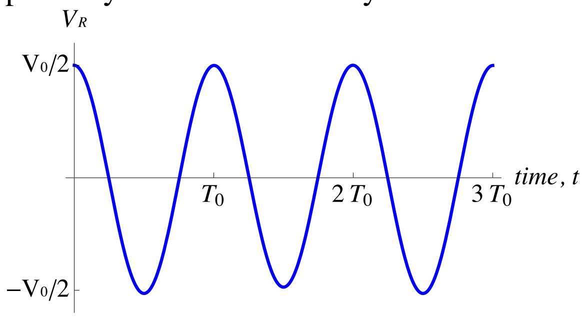
$$Y(\omega_0/2)=\left|Y(\omega_0/2)\right|e^{i\phi(\omega_0/2)}\approx\frac{1}{30R}e^{i\pi/2}$$

$$V_R(t)=\left(I_{\omega_0}+I_{\omega_0/2}\right)R=\left(Y(\omega_0)V_{app,\omega_0}+Y(\omega_0/2)V_{app,\omega_0/2}\right)R$$

$$=\left(\frac{1}{R}e^{i0}\frac{V_0}{2}e^{i\omega_0 t}+\frac{1}{30R}e^{i\pi/2}\frac{V_0}{2}e^{i\frac{\omega_0}{2}t}\right)R$$

$$V_R(t)=\frac{V_0}{2}e^{i\omega_0 t}+\frac{V_0}{30}e^{i\left(\frac{\omega_0}{2}t+\frac{\pi}{2}\right)}$$

On the scope, we see the real part $V_{app}(t)=\frac{V_0}{2}\cos(\omega_0 t)+\frac{V_0}{30}\cos\left(\frac{\omega_0}{2}t+\frac{\pi}{2}\right)$. Second term is probably too small to modify the dominant sinusoidal term at the resonance frequency.

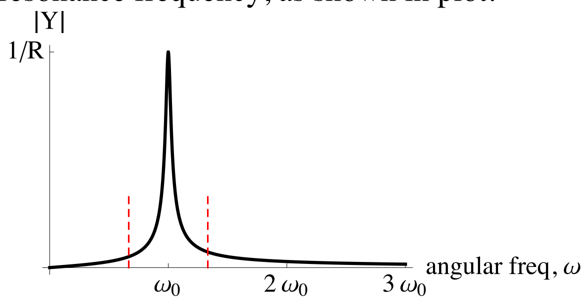


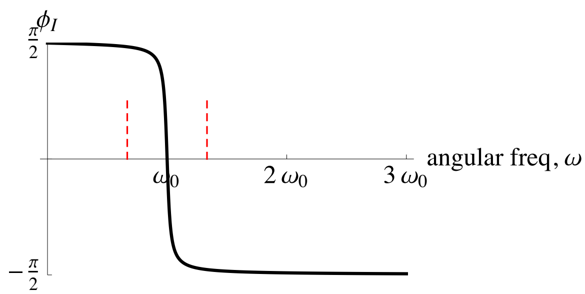
(b)(ii)

$$\Omega=\frac{\omega_0}{3}\Rightarrow 4\Omega=\frac{4\omega_0}{3},2\Omega=\frac{2\omega_0}{3}$$

$$\Rightarrow V(t)=\frac{V_0}{2}\cos\left(\frac{4}{3}\omega_0 t\right)+\frac{V_0}{2}\cos\left(\frac{2}{3}\omega_0 t\right)$$

Two sinusoidal voltages of same magnitude applied simultaneously equidistant from the resonance frequency, as shown in plot.





For calculation write $V_{app}(t)=\frac{V_0}{2}e^{i\frac{4}{3}\omega_0 t}+\frac{V_0}{2}e^{i\frac{2}{3}\omega_0 t}$

Estimate admittance from the plot – about same (small) magnitude, and π out of phase with each other.

$$Y\left(\frac{4}{3}\omega_0\right)=\left|Y\left(\frac{4}{3}\omega_0\right)\right|e^{i\phi\left(\frac{4}{3}\omega_0\right)}\approx\frac{1}{20R}e^{-i\pi/2}$$

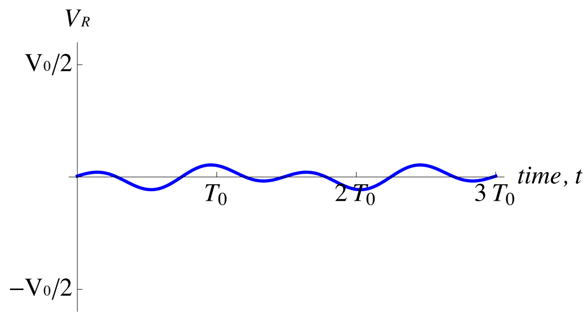
$$Y\left(\frac{2}{3}\omega_0\right)=\left|Y\left(\frac{2}{3}\omega_0\right)\right|e^{i\phi\left(\frac{2}{3}\omega_0\right)}\approx\frac{1}{20R}e^{i\pi/2}$$

$$V_R(t)=\left(I_{4\omega_0/3}+I_{2\omega_0/3}\right)R=\left(Y(4\omega_0/3)V_{app,4\omega_0/3}+Y(2\omega_0/3)V_{app,2\omega_0/3}\right)R$$

$$=\left(\frac{1}{20R}e^{-i\pi/2}\frac{V_0}{2}e^{i\frac{4}{3}\omega_0 t}+\frac{1}{20R}e^{i\pi/2}\frac{V_0}{2}e^{i\frac{2}{3}\omega_0 t}\right)R$$

$$V_R(t)=\frac{V_0}{20}e^{i\left(\frac{\omega_0}{2}t-\frac{\pi}{2}\right)}+\frac{V_0}{20}e^{i\left(\frac{\omega_0}{2}t+\frac{\pi}{2}\right)}$$

On the scope, we see the real part $V_{app}(t) = \frac{V_0}{20}\cos\left(\frac{4}{3}\omega_0 t - \frac{\pi}{2}\right) + \frac{V_0}{20}\cos\left(\frac{2}{3}\omega_0 t + \frac{\pi}{2}\right)$. Both terms of equal (small) magnitude, but π out of phase with each other. So something like this:



This is a hard one to draw – make your best effort. Here are the components (on an expanded scale)