GENERALIZING EULER'S PENTAGONAL NUMBER THEOREM TO MULTIPARTITIONS

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Abstract. This paper examines the properties of pentagonal numbers and their respective Ferrers diagrams in an attempt to generalize Euler's Pentagonal Number Theorem to multipartitions. This endeavor resulted in numerous attempts at pairing partitions of a given n and an indepth investigation of properties surrounding Durfee squares for pentagonal numbers.

1. Introduction

A partition, λ, of a positive integer n is a nonincreasing sequence of positive integers called parts. We write $\lambda = (\lambda_1, \ldots, \lambda_k)$ to denote a partition of n such that $|\lambda_k| = n$.

Denote by $p(n)$ the number of partitions of an integer n. By convention we set $p(n) = 0$ when $n < 0$, and $p(0) = 1$.

Example 1.1. *The partitions of* 5*:*

(5)
(4,1)
(3,2)
(3,1,1)
(2,2,1)
(2,1,1,1)
(1,1,1,1,1)

We can also represent partitions by means of a Ferrers diagram.

Definition 1.2. *A Ferrers diagram of a partition $\lambda = (\lambda_1, \cdots, \lambda_r)$ is an array of left justified boxes, where the first row has λ_1 boxes, the second λ_2 boxes, so on until the last row has λ_r boxes.*

It is common also to study restricted partitions of n. For instance, considering the set of partitions of n into odd parts, and the set of partitions of n into distinct parts, Euler proved the following.

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Theorem 1.3 (Euler)**.** *For all $n \geq 0$, the number of partitions of n into distinct parts equals the number of partitions n into odd parts.*

Example 1.4. *There are three partitions of 5 into odd parts : (5) (3,1,1) (1,1,1,1,1), and three partitions of 5 into distinct parts are (5) (4,1) (3,2).*

1.1. **Generating Functions.** The generating function for $p(n)$, attributed to Euler, is:

$$\sum_{n\geq 0} p(n)q^n = \prod_{n\geq 1} \frac{1}{1-q^n}.$$

That is, the coefficient of q^n in the right hand side is the number of partitions of n. We can see this by looking at how we get our q^n's. Say for example, that we want to look at the partitions of 3 : (3) (2,1) (1,1,1). We can choose three 1s : q^{1+1+1}, one 2 and one 1 : q^1q^2, and finally one three q^3. Adding these together we get $q^{1+1+1} + q^{1+1}q^2 + q^3 = 3q^3$, where $p(3) = 3$.

By raising the generating function to the k^{th} power, we have the generating function for multipartitions.

Definition 1.5. *A $k-$component multipartition (also called a $k-$colored partition) is a partition where each part is assigned one of k colors. Denote the number of $k-$component multipartitions of n by $p_k(n)$. We have the generating function for $p_k(n)$*

$$\sum_{n\geq 0} p_k(n)q^n = \prod_{n\geq 1} \frac{1}{(1-q^n)^k}.$$

We can also look at partitions into distinct parts. If we let $p(n, \mathcal{D})$ count the number of partitions of n into distinct parts, then

$$\sum_{n\geq 0} p(n, \mathcal{D})q^n = \prod_{n\geq 1} (1+q^n).$$

Here we see that we can have at most one part of a given size.

For this paper, we are interested in the following generating function for the difference between the number of partitions of n into an even number of parts colored distinctly by k colors , and the number of partitions of n into an odd number of parts colored distinctly into k colors. Namely, let $p_k(n, \mathcal{E})$ (resp. $p_k(n, \mathcal{O})$ is the number of partitions of n into an even (resp. odd) number of parts with colored parts distinct. And we let $d_k(n) = p_k(n, \mathcal{E}) - p_k(n, \mathcal{O})$ denote their difference.

$$\begin{aligned}
\sum_{n\geq 0} d_k(n)q^n &= \prod_{n\geq 1} (1-q^n)^k \\
&= (1-q)^k(1-q^2)^k(1-q^3)^k \cdots \\
&= \sum_{n\geq 0} (p_k(n, \mathcal{E}) - p_k(n, \mathcal{O}))q^n \\
&= \sum_{n\geq 0} d_k(n)q^n
\end{aligned}$$

If we let $k = 1$, this tells us the difference between the number of partitions of n into an even (resp. odd) number of distinct parts.

Example 1.6. *The partitions of* 5 *into distinct parts are : (5) (4,1) (3,2), so we have that*

$$p_1(5, \mathcal{E}) = 2,$$

and

$$p_1(5, \mathcal{O}) = 1.$$

Thus

$$d_1(5)q^5 = (2-1)q^5 = q^5.$$

1.2. **Euler's Pentagonal Number Theorem.** Euler came up with an identity for the generating function for $d_1(n)$.

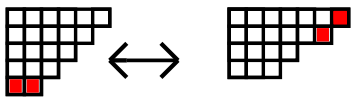
Theorem 1.7 (Euler's Pentagonal Number Theorem)**.** *For $|q| > 1$ we have*

$$\prod_{n\geq 1}(1-q^n) = \sum_{r\in\mathbb{Z}}(-1)^r q^{\frac{r(3r-1)}{2}}.$$

This theorem shows the connection between pentagonal numbers and the number of partitions into even and odd distinct parts. Given $n \in \mathbb{Z}$, $d_1(n)$ is ± 1 if n is a pentagonal number and 0 otherwise. By representing these two subsets of partitions of n with their Ferrers' diagrams, it is easy to combinatorically show the existence of this bijection.[An76]

Franklin's involution acts on the set of partitions into distinct parts which reverses the parity of the number of parts. Let λ be a partition of n into distinct parts and let a_λ denote the smallest part, $b = b_\lambda$ denote the largest b such that $\lambda_b = \lambda_1 + 1 - b$ (so that $\lambda_k = \lambda_1 + 1 - k$ if and only if $1 \leq k \leq b$). If a partition into distinct parts, λ is not a misfit (defined below), the we define a new partition λ' as follows. If $a_\lambda \leq b_\lambda$, we obtain λ' by removing the smallest part from λ and then adding 1 to the largest a_λ parts of this new partition. If $a_\lambda > b_\lambda$ we obtain λ' by subtracting 1 from the b_λ largest parts of λ and then appending a new part b_λ to this new partition.

Example 1.8. *Here is an example of the map of Franklin on Ferrers diagrams that provides a one-to-one correspondence between partitions of n into even distinct and partitions of n odd distinct parts, with some exceptions.*



But for some n, this is not a bijection. This happens when n is a pentagonal number and of a specific shape.

Definition 1.9. *A pentagonal number is a number of the form $p_r = \frac{r(3r-1)}{2}$ where $r \in \mathbb{Z}$.*

Definition 1.10. *A misfit, μ_r, is a partition of a pentagonal number into r parts, where :*

$$\mu_r^+ = (2r, 2r-1, \ldots, r+2, r+1)$$

$$\mu_r^- = (2r-1, 2r-2, \ldots, r+1, r)$$

For these two expressions, we will say that μ_r^- is a *minus misfit*. And similarly, μ_r^+ is a *plus misfit*. We also notice that $|\mu_r^+| = p_r$ and $|\mu_r^-| = p_{-r}$

Misfits occur when its smallest part equals the number of parts, r, or if the smallest part equals the number of parts plus one.

1.3. **Lacunarity.** We recall[On04] the Dedekind η function :

$$\eta^k(z) = q^{k/24}\prod_{n\geq 1}(1-q^n)^k = q^{k/24}\sum_{n\geq 0} d_k(n)q^n, \text{ where } q = e^{2\pi iz}.$$

We say that a series is *lacunary* if the set of non-zero terms has zero density. Serre proved that for even k, $\eta^k(z)$ is lacunary if and only if k is one of $2, 4, 6, 8, 10, 14$ or 26.

When $k = 24$, the coefficients of $\eta^{24}(z)$ are given by Ramanujan's tau function. We write

$$\eta^{24}(z) = \Delta(z) = \sum_{n\geq 0}\tau(n)q^n.$$

Lehmer conjectured that none of the coefficients $\tau(n)$ are zero. That is, for all n, $\tau(n) \neq 0$.

2. A two color identity

2.1. **Squaring Euler's Pentagonal Identity.** Our motivating question is to study the coefficients of q^n in the product $\prod(1-q^n)^2$. We begin by squaring the following identity:

$$\prod_{n\geq 1}(1-q^n)^2 = \left[\sum_{n\in\mathbb{Z}}(-1)^n q^{n(3n-1)/2}\right]^2 = \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}(-1)^{r+s}q^{\frac{r(3r-1)}{2}+\frac{s(3s-1)}{2}}, \tag{1}$$

which follows from Euler's pentagonal number theorem. We prove this identity combinatorially in Section 2.3

This shows that the only n for which the coefficient of q^n is not zero, are those n that are the sums of two pentagonal numbers. For a given pair of pentagonal numbers $p_r + p_s = n$, we will see the term $(-1)^{r+s}q^{\frac{r(3r-1)}{2}+\frac{s(3s-1)}{2}}$ twice where $r \neq s$, as both indices are free to range over the integers. Because of this, we see that even though 7 can be expressed as sums of pentagonal numbers, $p_{-2} + p_0 = 7 + 0$ and $p_2 + p_{-1} = 5 + 2$, the coefficient of q^7 is :

$$[2(-1)^{-2+0} + 2(-1)^{2+(-1)}]q^7 = (2-2)q^7 = 0q^7.$$

However, we can say for certain one instance when the coefficient of q^n will never be zero.

Lemma 2.1. *Suppose we have $n = 2p_d$, for some $d \in \mathbb{Z}$. Then $d_2(n) \neq 0$).*

Proof. Whereas for distinct r, s where $p_r + p_s = n$, the coefficient $(-1)^{r+s}$ appears twice in the sum, the coefficient $(-1)^{d+d} = 1$ only appears once. No number of 2's and -2's and a single 1 can ever add to 0. □

2.2. **Compositions of Partitions and two colored partitions.** We can interpret the set of partitions where each part is repeated at most k times, as the set of partitions where each part is assigned one of k colors, where the colored parts are distinct.

Definition 2.2. *Let* $C = \{$*Partitions in 2 colors with colored parts distinct*$\}$ *Let*

$$\Lambda = \{\lambda = (\lambda_1, ...\lambda_k)\} = \{\textit{Partitions in color 1 with distinct parts}\}$$

Let

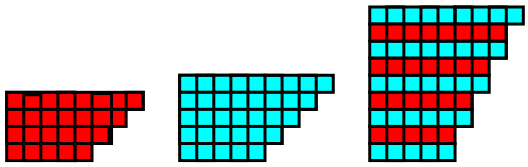
$$M = \{\mu = (\mu_1, ..., \mu_k)\} = \{\textit{Partitions in color 2 with distinct parts}\}$$

Define the composition of two partitions to be the map

$$* : \Lambda \times M \to C$$

defined by arranging the parts of $\lambda \in \Lambda$ *and* $\mu \in M$ *in increasing order with color 1 always before color 2.*

Example 2.3. *Here is the composition of* $\mu_4^+ * \lambda_5^-$ *and its resulting Ferrers diagram.*

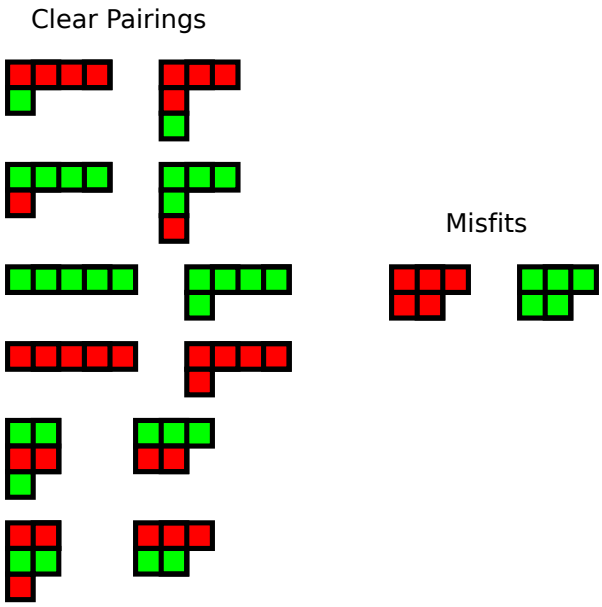


2.3. **Generalizing Franklin's Involution.** For our project, our first course of action was to exted Franklin's involution to give a combinatorial proof of the identity 1.

Definition 2.4. *We define an involution on the set of partitions with k colored parts distinct, to be the regular Franklin's involution but applied to a single color. We start with the color that partitions the largest number. If there is more than one partition that partitions the largest number, choose the color with the lower index. Apply Franklin's involution to the color chosen so if possible, if not, move to the next largest color.*

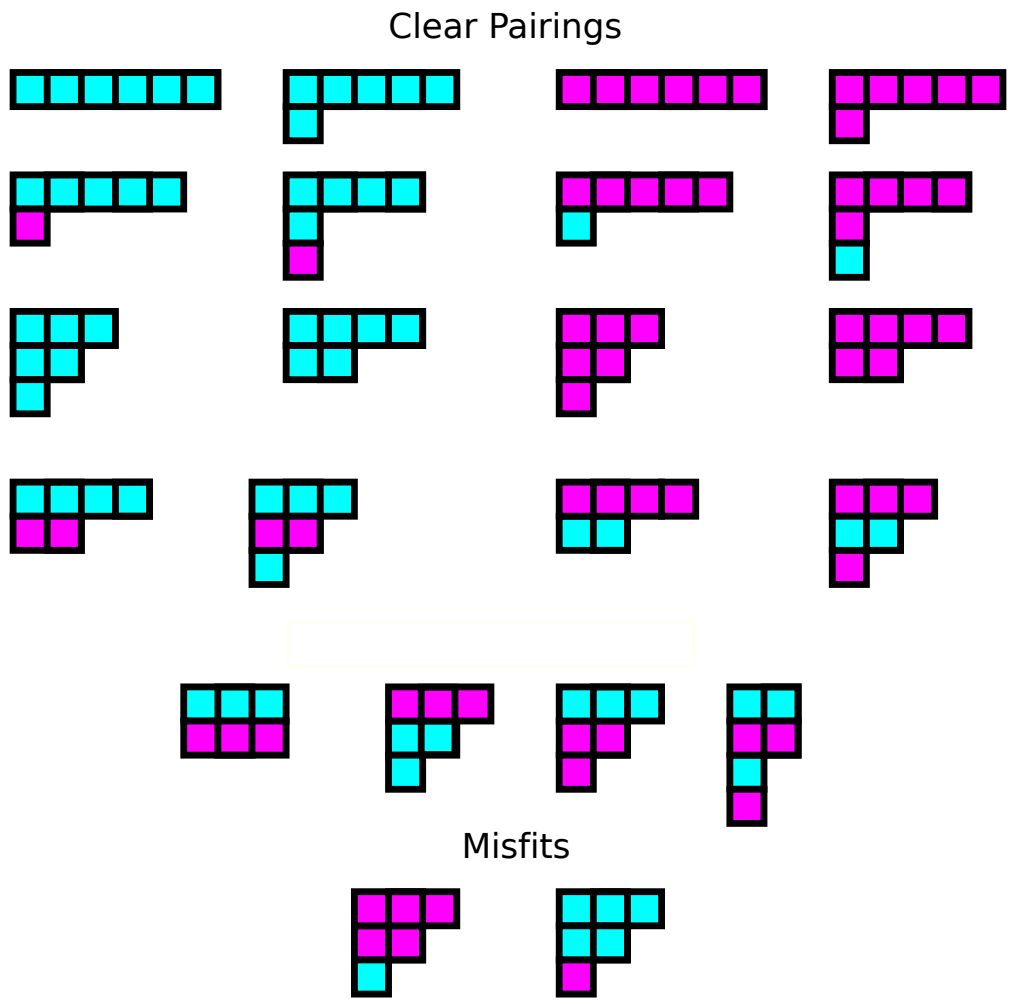
If we can apply this involution to our set of colored parts distinct partitions of n, we see that we change the parity and so come up with a bijection between the set of partitions of n into an even number parts colored distinctly and the set of partitions of n into an odd number of parts colored distinctly. However, when we have a composition of two misfits, the involution breaks down, corresponding exactly to the non-zero terms in th eright hand side of the identity 1.

Example 2.5. *Here is an example of pairing partitions for* 5.



The "Clear Pairings" represent the pairings that arise from our involution. Each partition is paired with a partition of a different parity and, more specifically, is matched with a partition that represents the end result of Franklin's involution for the pairing partition. But as you can see, some partitions are unable to undergo this involution, i.e. when said partition is a misfit.

Example 2.6. *Here is another example of pairing but for the partitions of* 6.



Much like the example of 5, there are "Clear Pairings" for many of the partitions. There are some unique properties surrounding misfits and the shape of their Ferrers diagrams. Given r, μ_r^- will have r parts with largest part of length $2r-1$ and smallest part of length r. For a given r, μ_r^+ will have r parts with largest part of length $2r$ and smallest part of length $r+1$.

2.4. **Durfee Squares of $2-$misfits.** For a given $r \in \mathbb{N}$, r dictates the parity of a misfit because it dictates the number of parts. This being the case, we thought a natural next step in our project would be to begin analyzing properties of r and other attributes of misfits that r dictates. In particular, we began to study Durfee squares.

Definition 2.7. *The* Durfee square *of a partition λ is the largest square that fits in the Ferrers diagram of λ. The size of the Durfee square, $d(\lambda)$ is given by :*

$$d(\lambda) = \max_i\{i : \lambda_i \geq i\}.$$

We are interested in $d(\mu_r^+)$ and $d(\mu_r^-)$ for the misfits, μ_r^- and μ_r^+.

Proposition 2.8. *For a given $r \in \mathbb{N}$, $d(\mu_r) = r$.*

Proof. Let $r \in \mathbb{N}$ be a positive integer. Consider

$$\mu_r^+ = (2r, 2r-1, 2r-2, ..., r+1).$$

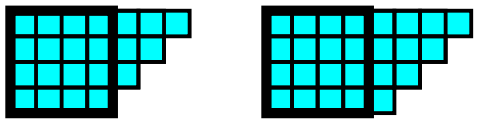
We can see that μ_r^+ has r parts. This means that $\lambda_r = r+1 \geq r$ and therefore $d(\mu_r^+) = r$. Consider

$$\mu_r^- = (2r-1, 2r-2, ..., r+1, r).$$

We can see that μ_r^- has r parts. This means that $\lambda_r = r \geq r$ and therefore $d(\mu_r^-) = r$

□

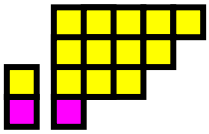
Example 2.9. *Here are two examples of the Ferrers diagrams for μ_4^- and μ_4^+. Each has a Durfee square of size 4.*



Theorem 2.10. *Let s and r be integers such that $s \leq r$, $d(\mu_r * \mu_s) \geq r$.*

Proof. When composing two misfits, we are interested in the size of the new Durfee square. First, we find a lower and upper bound. By fixing positive integers r and s such that $s \leq r$ the Durfee square of $\mu_s * \mu_r$ can have sides no smaller than r. Proposition 2.8 states that $d(\mu_r) = r$. This is because $d(\mu_s)>0$ so the composition of μ_s and μ_r cannot have Durfee square smaller than $d(\mu_r)$. □

Example 2.11. *It should be clear from these examples and Theorem 2.10 that the Durfee square of the composition of 2-misfits cannot decrease. The first example is the composition of two misfits of size 1 ($r = 1$ and $s = 1$ and both are minus misfits). Clearly, the 1×1 Durfee square cannot be decreased to 0×0. Similarly, $d(\mu_3 * \mu_1)$ is dictated by the larger r and is, thus, a 3×3 Durfee square that cannot be decreased with the composition of another misfit.*



Definition 2.12. *Define a lonely misfit to be the composition of a misfits with itself. Let*

$$L_r^+ = \mu_r^+ * \mu_r^+$$

and

$$L_r^- = \mu_r^- * \mu_r^-.$$

Theorem 2.13. *Let $\mu_{r,s}^{\pm,\pm} = \mu_r^\pm * \mu_s^\pm$ be a $2-$misfit. Then the size of the durfee square $d(\mu_{r,s}^{\pm,\pm})$ is given by :*

$$d(\mu_{r,s}^{\pm,\pm}) = \begin{cases} \left\lfloor \frac{3r+c+3}{3} \right\rfloor & \text{if } c \geq 0 \text{ with } \mu_r^+ \\ \left\lfloor \frac{3r+c+1}{1} \right\rfloor & \text{if } c \geq 0 \text{ with } \mu_r^- \\ r & : c < 0 \end{cases}$$

Where c is the difference between the largest part of μ_s and the smallest part of μ_r.

Proof. Suppose we have a 2−misfit $\mu^{+,+}_{r,s} = (2r, \cdots, r+1) * (\mathbf{2s}, \cdots, \mathbf{s+1})$. Then

$$\begin{aligned}\mu^{+,+}_{r,s} &= (\overbrace{2r, \cdots, r+c+2}^{r-c-1 \text{ parts of } \mu_r}, \overbrace{r+c+1, \mathbf{2s}, \cdots, r+1, \mathbf{2s-c}}^{2c+2 \text{ parts of } \mu_r \text{ and } \mu_s}, \overbrace{\mathbf{2s-c-1}, \cdots, \mathbf{s+1}}^{s-c-1 \text{ parts of } \mu_s}) \\ &= (\underbrace{\lambda_1, \cdots, \lambda_{r-c-1}}, \underbrace{\lambda_{r-c}, \lambda_{\mathbf{r-c+1}}, \cdots, \lambda_{r+c}, \lambda_{\mathbf{r+c+1}}}, \underbrace{\lambda_{\mathbf{r+c+2}}, \cdots, \lambda_{\mathbf{r+s}}}),\end{aligned}$$

where $2s = r + 1 + c$.

The Durfee square has size at least $r - c - 1$, since $\lambda_{r-c-1} = r + c + 2 \geq r - c - 1$, and has size less than $r + c + 2$, since $\lambda_{r+c+2} = 2s - c - 1 = r + c + 1 - c - 1 = r < r + c + 2$.

So we need only to consider the $2c + 2$ parts in the middle in our calculation of the Durfee square size. We determine the maximum i such that

$$\lambda_{r-c+i} \geq r - c + 1$$

for $2c + 1 \geq i \geq 0$.

We see that for i even,

$$\lambda_{r-c+i} = r + (c - \frac{i}{2}) + 1,$$

and for i odd,

$$\lambda_{r-c+i} = 2s - \frac{i-1}{2}.$$

These in turn imply that

$$\frac{4c+2}{3} \geq i,$$

when i is even, and

$$\frac{4c+3}{3} \geq i,$$

when i is odd.

Case 2.14 ($c \equiv 0 \pmod 3$)**.** *We have that $c = 3k$, and wehave two possible bounds on i :*

$$i \leq \frac{12k+2}{3} = 4k + \frac{2}{3}$$

and

$$i \leq \frac{12k+3}{3} = 4k + 1$$

So we take $4k + 1$ as our maximum i, and we have

$$d(\mu^{+,+}_{r,s}) = r - c + i = r - c + 4k + 1 = r + k + 1 = r + \frac{c}{3} + 1 = \frac{3r + c + 3}{3}$$

Case 2.15 ($c \equiv 1 \pmod 3$)**.** *We havec $= 3k + 1$, so finding i we get:*

$$i \leq \frac{12k+6}{3} = 4k + 2$$

and

$$i \leq \frac{12k+7}{3} = 4k + 2\frac{1}{3}$$

So we take $4k+2$ as our maximum i, and we have

$$d(\mu^{+,+}_{r,s}) = r - c + i = r - c + 4k + 2 = r + k + 1 = r + \frac{c-1}{3} + 1 = \frac{3r + c + 2}{3}$$

Case 2.16 ($c \equiv 2 \pmod 3$)**.** *We have $c = 3k+2$, finding i we get:*

$$i \leq \frac{12k + 10}{3} = 4k + 3\frac{1}{3}$$

and

$$i \leq \frac{12k + 11}{3} = 4k + 3\frac{2}{3}$$

So we take $4k+3$ as our maximum i, and we have

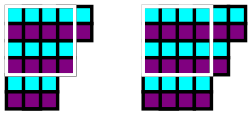
$$d(\mu^{+,+}_{r,s}) = r - c + i = r - c + 4k + 3 = r + k + 1 = r + \frac{c-2}{3} + 1 = \frac{3r + c + 1}{3}$$

So we can say now that $d(\mu^{+,+}_{r,s}) = \left\lfloor \frac{3r+c+3}{3} \right\rfloor$. □

Lemma 2.17. *Given a lonley misfits L^-_r and L^+_r, we have that $d(L^-_r) = r + \lfloor \frac{r}{3} \rfloor$ and $d(L^+_r) = r + \lfloor \frac{r+2}{3} \rfloor$.*

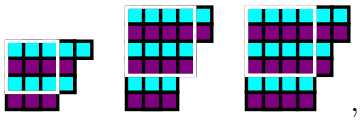
Lemma 2.18. *For a lonely misfit, if $d(L_r)$ is odd then it has the shape:*

where the $d(L_r)^{th}$ part is of color 1 and has size $d(\lambda)+1$. If $d(L_r)$ is even then it has one of the two shapes:



where the $d(L_r)^{th}$ part is always of color 2 and has size $d(\lambda)$ or $d(\lambda)+1$.

Proof. Given a lonely misfit of two colors, there are three possible woven Durfee squares


,

for reasons we will now describe.

The bottom row is color 2 with lengths $d(\lambda)$ or $d(\lambda)+1$, or the bottom row is the color 1 of length d.
These are the only possibilities because of how lonely misfits grow in relation to their Durfee squares. When $d(\lambda)$ is odd it will always end with color one because the rows are alternating. More specifically, odd $d(\lambda)$ must have last row of size $d(\lambda)$ because to have a last row of size of $d(\lambda)+1$ will increase the Durfee square by one by virtue of its alternating rows. When

$d(\lambda)$ is even the last row will always be color 2l because it has rows occurring in pairs. The last row can have size $d(\lambda)$ or $d(\lambda)+1$ because they preserve the size of the Durfee square, but having last row of $d(\lambda)+2$ will increase the Durfee square by one.
A lonely misfit is a 2-misfit where every row is alternating. If the Durfee square has an even side length, then it has a bottom row of color 2, so by above there are two lonely misfit possibilities with an even Durfee square of size d. Namely,

$$(2d-\frac{d}{2}-1),(2d-\frac{d}{2}-2),...,\begin{cases}(\frac{2d-\frac{d}{2}-1}{2}+1), & \text{if } \frac{d}{2} \text{ is odd,}\\ (\frac{2d-\frac{d}{2}}{2}), & \text{if } \frac{d}{2} \text{ is even}\end{cases}$$

and

$$(2d-\frac{d}{2}),(2d-\frac{d}{2}-1),...,\begin{cases}(\frac{2d-\frac{d}{2}-1}{2}), & \text{if } \frac{d}{2} \text{ is odd}\\ (\frac{2d-\frac{d}{2}}{2}), & \text{if } \frac{d}{2} \text{ is even.}\end{cases}$$

Similarly there is only one lonely misfit with an odd Durfee square of size d. Namely,

$$(2d-\frac{d-1}{2}-1),(2d-\frac{d-1}{2}-2),...,(\frac{2d-\frac{d-1}{2}}{2})$$

□

Corollary 2.19. *Let d bea positive integer, and ϵ_d be defined so $d \equiv \epsilon_d \mod 2$. Then there are $2-\epsilon_d$ lonely misfits that have Durfee square size d.*

We are also interested in the number of misfits that have Durfee square size d. By looking at what forms the Durfee squares can take, we can calculate the number of $2-$misfits that have the same size Durfee square.

Theorem 2.20. *Given a positive number d, there are $7d$ $2-$misfits that have a Durfee square of size d.*

Proof. We can break the $2-$misfits into two forms :

$$\mu_{r,s}=\begin{cases}({}_1\mu_r,\cdots,{}_r\mu_r,{}_1\mu_s,\cdots,{}_s\mu_s) & \text{if } {}_1\mu_s \le r \le {}_r\mu_r\\ ({}_1\mu_r,\cdots,{}_{r-c-1}\mu_r,\overbrace{{}_{r-c}\mu_r,{}_1\mu_s,\cdots,{}_r\mu_r,{}_{c+1}\mu_s}^{\text{doubling}},{}_{c+2}\mu_s,\cdots,{}_s\mu_s) & \text{if } {}_1\mu_s={}_r\mu_r+c \text{ and } c\ge 1\end{cases}$$

Suppose $d(\mu_{r,s})=d$. For the first type of $2-$misfit, we have that

$$\mu_{r,s}=({}_1\mu_r,\cdots,{}_r\mu_r,{}_1\mu_s,\cdots,{}_s\mu_s)$$

where $r=d$, $0\le {}_1\mu_s\le d$, and ${}_r\mu_r=d$ or $d+1$. So we have $d+1$ possible misfits μ_s that satisfy this relation, $2(d+1)$ ways to choose μ_s for the two different values of ${}_r\mu_r$ and $4(d+1)$ possibilities by switching colors.

For the second type of $2-$misfit, we have

$$\mu_{r,s} = (\underbrace{{}_1\mu_r, \cdots, {}_{r-c-1}\mu_r, \overbrace{{}_{r-c}\mu_r, {}_1\mu_s, \cdots, \lambda_d}^{\text{doubling}}}_{d \text{ elements of the Durfee square}}, \cdots, {}_r\mu_r, {}_{c+1}\mu_s, {}_{c+2}\mu_s, \cdots, {}_s\mu_s)$$

Where if λ_d is color 2, it can have size d or $d+1$. We can fill in the rest of the square in $\lfloor \frac{d}{2} \rfloor$ ways, giving us a total of $4\lfloor \frac{d}{2} \rfloor$ Durfee squares of this type with λ_d having color 2 by switching colors.

If λ_d is color 1, it can only have size d, and we can fill in therestof the square in $\lfloor \frac{d-1}{2} \rfloor$ways. In total, we get $2\lfloor \frac{d-1}{2} \rfloor$ ways by switching colors.

However, we need to account for the lonely misfts. When d is even we have 2 lonely misfits by the above theorem, and when d is odd we have 1 lonely misft.

□

2.5. **Ranges for r.** Next we want to know, given a fixed partition, n, what are the bounds on r such that there exist r and s such that

$$|\mu_r^{+/-} * \mu_s^{+/-}| = n$$

with $r \geq s$. In other words, for any partitioned number what is the smallest and largest pentagonal number that bounds n.

For a given positive integer N, the largest possible pentagonal number will be if N was composed entirely of a pentagonal number, so the least upper bound on r will be a pentagonal number such that

$$N \geq \frac{r(3r-1)}{2}.$$

For the lower bound, we want the greatest lower bound, this occurs when N is composed of the same two misfits i.e.,$N \leq |\mu_r * \mu_r|$ Thus, the greatest lower bound will arise when

$$\frac{r(3r+1)}{2} \geq \frac{N}{2}.$$

This is the smallest r can possibly be because n can be defined as the composition of two pentagonal numbers, r and s, so for $s \leq r$ the smallest r can be and still sum to n is if $r = s$.

Theorem 2.21. *For a fixed positive integer N, if there exists positive integers $r \geq s$ such that $|\mu_{r,s}| = N$, then $\frac{-1+\sqrt{1+12N}}{6} \leq r \leq \frac{1+\sqrt{1+24N}}{6}$.*

Proof. Fix N to be a positive integer. We want there to be a bound that will satisfy a least upper bound for N. This implies that $N \geq \frac{r(3r\pm1)}{2}$ because if $\frac{r(3r\pm1)}{2} > N$ then r is too big. Thisidea is expressed by the inequality

$$\frac{r(3r+1)}{2} \leq N$$

which in turn implies

$$3r^2 + r - 2N \leq N.$$

Similarly we can see that

$$\frac{r(3r-1)}{2} \leq N$$

implies

$$3r^2 - r - 2N \leq 0.$$

Solving the quadratic equations for r yields

$$\frac{-1 \pm \sqrt{1+24N}}{6} \text{ and } \frac{1 \pm \sqrt{1+24N}}{6}$$

respectively.

Plotting these limits on a graph makes the choice of least upper bound more obvious. Clearly $\frac{1+\sqrt{1+24N}}{6}$ is the least upper bound for r. For the lower bound we want to find r such that it is the greatest lower bound. This occurs when N is composed entirely of two misfits of equal size i.e.

$$\text{If } \frac{r(3r \pm 1)}{2} \geq \frac{N}{2} \text{ then } r(3r \pm 1) \geq N.$$

$$\text{If } r(3r+1) \geq N \text{ then } 3r^2 + r - N \geq 0$$

and

$$\text{If } r(3r-1) \geq N \text{ then } 3r^2 - r - N \geq 0.$$

Solving for the quadratic equation for r yields

$$\frac{-1 \pm \sqrt{1+12N}}{6} \text{ and } \frac{1 \pm \sqrt{1+12N}}{6}$$

respectively.

Again, plotting these points makes the choice of greatest lower bound apparent. Clearly $\frac{1+\sqrt{1+12N}}{6}$ is the greatest lower bound for r.

□

Bounds on r are very important, as it is imperative to our project to know how a given n can be expressed as a sum of two pentagonal numbers, and thus, these bounds on r gives us a range for pairs of pentagonal numbers that could possibly sum to n. Hence giving us more information regarding the coefficient of q^n in $\prod(1-q^n)^2$.

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