The following exercises are due Monday, March 12.

1. In class I explained how to measure the mass in galaxies and galaxy clusters using the virial theorem. The basic formula is

\[ M = \frac{\langle v^2 \rangle r_h}{\alpha G}, \]

where \( r_h \) is the half-mass radius and \( \alpha \) is a parameter determined by data fitting. For our purposes \( \alpha \approx 0.4 \).

The Draco galaxy is a dwarf galaxy within the Local Group. Its luminosity is \( L = 1.8 \times 10^5 \) \( L_\odot \) and half its total luminosity is contained within a sphere of radius \( r_h = 120 \) pc. The red giant stars in the Draco galaxy are bright enough to have their line-of-sight velocities measured. The measured velocity is 31.5 \( \text{km s}^{-1} \). What is the mass of the Draco Galaxy? What is the mass-to-light ratio? Given the fact that typical stars have a mass-to-light ratio of \( 4M_\odot/L_\odot \), what fraction of the galaxy’s mass is dark matter?

2. One of the more recent speculations in cosmology is that the universe may contain a quantum field, called “quintessence,” which has a positive energy density and a negative value of the equation-of-state parameter \( w \). Assume, for the purposes of this problem, that the universe is spatially flat, and contains nothing but matter (\( w = 0 \)) and quintessence with \( w = -1/2 \). The current density parameter of matter is \( \Omega_{m,0} \leq 1 \), and the current density parameter of quintessence is \( \Omega_{Q,0} = 1 - \Omega_{m,0} \). At what scale factor \( a_{mQ} \) will the energy density of quintessence and matter be equal? Solve the Friedman equation to find \( a(t) \) of the universe. What is \( a(t) \) in the limit \( a \ll a_{mQ} \)? What is \( a(t) \) in the limit \( a \gg a_{mQ} \)? What is the current age of the universe, expressed in terms of \( H_0 \) and \( \Omega_{m,0} \)?
Problem Set #5 - Problem 1.

\[ v = 31.5 \text{ km/s} \quad \Gamma_n = 120 \text{ pc} \]

\[ v = 31.5 \frac{\text{km}}{\text{s}} \left( \frac{10^{16} \text{m}}{1 \text{ pc}} \right) = 3.15 \times 10^4 \text{ m} \]

\[ \Gamma_n = 120 \text{ pc} \left( \frac{3.09 \times 10^{16} \text{ m}}{1 \text{ pc}} \right) = 3.21 \times 10^{18} \text{ m} \]

\[ M = \left( \frac{3.15 \times 10^4 \text{ m/s}}{0.4 \times 6.67 \times 10^{-11} \text{ m}^3 \text{ kg}^{-1} \text{ s}^{-2}} \right)^2 \frac{3.21 \times 10^{18} \text{ m}}{1.19 \times 10^{38} \text{ kg}} = 6 \times 10^7 \text{ M}_\odot \]

\[ M = \frac{6 \times 10^7 \text{ M}_\odot}{1.8 \times 10^5 \text{ L}_\odot} = 3.3 \times 10^2 \text{ M}_\odot \text{ L}_\odot^{-1} \]

Compared with \( 4 \text{ M}_\odot \text{ L}_\odot^{-1} \), we would estimate that \( 1.8 \times 10^5 \text{ L}_\odot \left( \frac{4 \text{ M}_\odot}{\text{ L}_\odot} \right) = 7.2 \times 10^5 \text{ M}_\odot \) is luminous.

So the fraction of dark matter is

\[ \frac{6 \times 10^7 \text{ M}_\odot - 7.2 \times 10^5 \text{ M}_\odot}{6 \times 10^7} = 0.938 \]
Problem 2

\[ P_{\phi} = -\frac{1}{2} \rho \quad \omega = -\frac{1}{3} \quad \rho(t) = \rho_{0} a^{-3(1+\omega)} \quad (8.24) \]

So \[ \rho(t) = \rho_{0} \alpha(t)^{-3/2} \quad \rho_{\mu}(t) = \rho_{\mu 0} \alpha(t)^{-3} \]

1st Friedmann eqn.

\[ \left( \frac{\dot{a}}{a} \right)^{2} = \frac{8\pi G}{3} \left[ \rho_{\phi 0} a^{-3/2} + \rho_{\mu 0} a^{-3} \right] \]

\[ \left( \frac{\dot{a}}{a} \right)^{2} = H_{0}^{2} \left( \frac{8\pi G}{3 H_{0}^{2}} \right) \left[ \rho_{\phi 0} a^{+1/2} + \rho_{\mu 0} a^{-1} \right] \]

\[ a = H_{0} \left[ \sqrt{\rho_{\phi 0} a^{-1/2} + \rho_{\mu 0} a^{-1}} \right]^{-1/2} \]

\[ \int H_{0} \, dt = \int_{0}^{a} \frac{da}{\sqrt{\rho_{\phi 0} a^{-1/2} + \rho_{\mu 0} a^{-1}}} \]

\[ = \int_{0}^{a} \frac{da \, a^{-1/2}}{\sqrt{\rho_{\phi 0} a^{-3/2} + \rho_{\mu 0}}} \]

\[ H_{0} t = \frac{4}{3} \rho_{\phi 0} \left[ \sqrt{\rho_{\phi 0} a^{-3/2} + \rho_{\mu 0}} - \sqrt{\rho_{\phi 0}} \right] \]

This can be inverted, but it's not very illuminating!
The two will be equal when

\[ \Omega_{\text{m}, 0} \Omega_{\text{m}, 0} = \Omega_{\text{m}, 0} \]

or

\[ a = \left[ \frac{\Omega_{\text{m}, 0}}{\Omega_{\text{m}, 0}} \right]^{3/3} \]

\( a < a_{\text{cyc}} \) then \( \Omega_{\text{m}, 0} \Omega_{\text{m}, 0} < \Omega_{\text{m}, 0} \)

\[ H_0 = \frac{4}{3Q_{\text{m}, 0}} \left[ \sqrt{x + 1} - 1 \right] \]

where \( x = \frac{\Omega_{\text{m}, 0} a^{3/2}}{\Omega_{\text{m}, 0}} < 1 \)

\[ H_0 \propto \frac{4}{3Q_{\text{m}, 0}} \sqrt{\Omega_{\text{m}, 0}} + \frac{1}{2} \frac{2Q_{\text{m}, 0}}{\Omega_{\text{m}, 0}} a^{3/2} = \frac{2a^{3/2}}{3 \sqrt{Q_{\text{m}, 0}}} \]

\( a > > a_{\text{cyc}} \) then \( a_{\text{m}, 0} a^{3/2} >> \Omega_{\text{m}, 0} \)

\[ H_0 = \frac{4}{3Q_{\text{m}, 0}} a^{3/4} \]

To find the current age of the universe, set \( a = 1 \) \( \Omega_{\text{m}, 0} = 1 - \Omega_{\text{m}, 0} \)

\[ H_0 t_0 = \frac{4}{3Q_{\text{m}, 0}} \left[ 1 - \sqrt{\Omega_{\text{m}, 0}} \right] \]