# Cosmology Problem Set #5

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The following exercises are due Monday, March 12.

1. In class I explained how to measure the mass in galaxies and galaxy clusters using the virial theorem. The basic formula is

   $$M = \frac{\langle v^2 \rangle r_h}{\alpha G},$$

   where $r_h$ is the half-mass radius and $\alpha$ is a parameter determined by data fitting. For our purposes $\alpha \approx 0.4$.

   The Draco galaxy is a dwarf galaxy within the Local Group. Its luminosity is $L = 1.8 \times 10^5 \ L_\odot$ and half its total luminosity is contained within a sphere of radius $r_h = 120$ pc. The red giant stars in the Draco galaxy are bright enough to have their line -of-sight velocities measured. The measured velocity is 31.5 km s$^{-1}$. What is the mass of the Draco Galaxy? What is the mass-to-light ratio? Given the fact that typical stars have a mass-to-light ratio of $4M_\odot/L_\odot$, what fraction of the galaxy's mass is dark matter?

2. One of the more recent speculations in cosmology is that the universe may contain a quantum field, called "quintessence," which has a positive energy density and a negative value of the equation-of-state parameter $w$. Assume, for the purposes of this problem, that the universe is spatially flat, and contains nothing but matter ($w = 0$) and quintessence with $w = -1/2$. The current density parameter of matter is $\Omega_{m,0} \leq 1$, and the current density parameter of quintessence is $\Omega_{Q,0} = 1 - \Omega_{m,0}$. At what scale factor $a_{mQ}$ will the energy density of quintessence and matter be equal? Solve the Friedman equation to find $a(t)$ of the universe. What is $a(t)$ in the limit $a \ll a_{mQ}$? What is $a(t)$ in the limit $a \gg a_{mQ}$? What is the current age of the universe, expressed in terms of $H_0$ and $\Omega_{m,0}$?

Problem Set # 5 - Problem 1.

$v = 31.5$ km/s  $r_h = 120$ pc

$$v = 31.5 \frac{km}{s} \left(\frac{10^3 m}{1 km}\right) = 3.15 \times 10^4 m$$

$$r_h = 120 \text{ pc} \left(\frac{3.09 \times 10^{16} m}{1 pc}\right) = 3.21 \times 10^{18} m$$

$$M = \frac{(3.15 \times 10^4 m/s)^2 \; 3.21 \times 10^{18} m}{0.4 \times 6.67 \times 10^{-11} m^3 kg^{-1} s^{-2}} = 1.19 \times 10^{38} kg$$

$$1.19 \times 10^{38} kg \left(\frac{1 M_\odot}{1.99 \times 10^{30} kg}\right) = 6 \times 10^7 M_\odot$$

$$\frac{M}{L} = \frac{6.0 \times 10^7 M_\odot}{1.8 \times 10^5 L_\odot} = 3.33 \times 10^2 M_\odot / L_\odot$$

compared with $4 M_\odot / L_\odot$ we would estimate

that $1.8 \times 10^5 L_\odot \left(\frac{4 M_\odot}{L_\odot}\right) = 7.20 \times 10^5 M_\odot$ is luminous

so the fraction of dark matter is

$$\frac{6. \times 10^7 M_\odot - 7.20 \times 10^5 M_\odot}{6. \times 10^7} = 0.988$$

Problem 2

$$P_Q = -\tfrac{1}{2}\rho_Q \qquad w = -1/3 \qquad \rho(t) = \rho_0\, a^{-3(1+w)} \qquad (8.24)$$

so $\rho_Q(t) = \rho_{Q,0}\, a(t)^{-3/2}$ $\qquad \rho_M(t) = \rho_{M,0}\, a(t)^{-3}$

1st Friedman eqn. $\left(\dfrac{\dot a}{a}\right)^2 = \dfrac{8\pi G}{3}\left[\rho_{Q,0}\, a^{-3/2} + \rho_{M,0}\, a^{-3}\right]$

$$(\dot a)^2 = H_0^2\left(\frac{8\pi G}{3H_0^2}\right)\left[\rho_{Q,0}\, a^{+1/2} + \rho_{M,0}\, a^{-1}\right]$$

$$\dot a = H_0^2\left[\Omega_{Q,0}\, a^{+1/2} + \Omega_{M,0}\, a^{-1}\right]^{1/2}$$

$$\int H_0\, dt = \int_0^a \frac{da}{\sqrt{\Omega_{Q,0}\, a^{+1/2} + \Omega_{M,0}\, a^{-1}}}$$

$$= \int_0^a \frac{da\ a^{1/2}}{\sqrt{\Omega_{Q,0}\, a^{3/2} + \Omega_{M,0}}}$$

$$H_0 t = \frac{4}{3\Omega_Q}\left[\sqrt{\Omega_{Q,0}\, a^{3/2} + \Omega_{M,0}} - \sqrt{\Omega_{M,0}}\right]$$

This can be inverted, but it's not very illuminating!

The two will be equal when

$$\Omega_{Q,0}\, a_{MQ}^{3/2} = \Omega_{M,0}$$

$$\text{or} \quad a_{MQ} = \left[ \Omega_{M,0} / \Omega_{Q,0} \right]^{2/3}$$

$a \ll a_{MQ}$ Then $\Omega_{Q,0}\, a^{3/2} \ll \Omega_{M,0}$

$$H_0 t = \frac{4}{3\Omega_{Q0}} \sqrt{\Omega_{M0}} \left[ \sqrt{x+1} - 1 \right]$$

$$\text{where} \quad x \equiv \frac{\Omega_{Q,0}\, a^{3/2}}{\Omega_{M,0}} \ll 1$$

$$H_0 t \approx \frac{4}{3\cancel{\Omega_{Q0}}} \sqrt{\Omega_{M,0}}\ \frac{1}{2} \frac{\cancel{\Omega_{Q,0}}\, a^{3/2}}{\Omega_{M,0}} = \frac{2 a^{3/2}}{3\sqrt{\Omega_{M,0}}}$$

$a \gg a_{MQ}$ then $\Omega_{Q0}\, a^{3/2} \gg \Omega_{M0}$

$$H_0 t = \frac{4}{3\sqrt{\Omega_{Q0}}}\, a^{3/4}$$

To find the current age of the universe, just set $a = 1$ $\quad \Omega_{Q0} = 1 - \Omega_{M0}$

$$H_0 t_0 = \frac{4}{3\Omega_Q} \left[ 1 - \sqrt{\Omega_{M0}} \right]$$