Cosmology Problem Set #2

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The following exercises are due Friday, February 9.

1. The metric of the plane in polar coordinates r and ϕ is

$$ds^2 = dr^2 + r^2 d\phi^2$$

A curve between two points can be described parametrically by giving r and ϕ as functions of some parameter σ, which you can define to suit your own convenience. A curve is then described by two functions $r(\sigma)$ and $\phi(\sigma)$. Find the equations of a geodesic in this space. The equations are particularly simple is you use s as the parameter. Show that

$$\frac{d^2 r}{ds^2} = r\left(\frac{d\phi}{ds}\right)^2$$

$$\frac{d}{ds}\left(r^2 \frac{d\phi}{ds}\right) = 0$$

(These equations are hard to solve. You might say that this is getting straight lines the hard way.)

2. Consider the two-dimensional spacetime spanned by coordinates (v, x) with the line element

$$ds^2 = -x dv^2 + 2dv\, dx$$

This is a simple two-dimensional toy model for a black hole. It has the property that if you are trapped in the region $x < 0$, you can't get out!

(a) Calculate the light cone at a point (v, x).

(b) Draw a (v, t) spacetime diagram showing how light cones change with x.

(c) Show that a particle can cross from positive x to negative x but cannot cross from negative x to positive x.

3. Consider the three-dimensional space with the line element

$$ds^2 = \frac{dr^2}{(1 - 2M/r)} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

(a) Calculate the radial distance between the sphere $r = 2M$ and the sphere $r = 3M$.

(b) Calculate the spatial volume between the two spheres in part (a).

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1 $$ds = \sqrt{dr^2 + r^2 d\phi^2}$$

$$S = \int_A^B d\sigma \sqrt{\left(\frac{dr}{d\sigma}\right)^2 + r^2\left(\frac{d\phi}{d\sigma}\right)^2} = \int d\sigma\, L\left(\frac{dr}{d\sigma}, \frac{d\phi}{d\sigma}, r\right)$$

$$= \int_A^B d\sigma \sqrt{\dot{r}^2 + r^2\dot{\phi}^2}$$

$$\frac{\partial L}{\partial \dot{\phi}} = \frac{r^2}{L}\dot{\phi} \qquad \frac{\partial L}{\partial \dot{r}} = \frac{\dot{r}}{L} \qquad \frac{\partial L}{\partial r} = \frac{r}{L}\dot{\phi}^2$$

The Euler-Lagrange Eqn.'s ar

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = \frac{d}{d\sigma}\left(\frac{r^2}{L}\dot{\phi}\right) = 0$$

$$\frac{d}{d\sigma}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = \frac{d}{d\sigma}\left(\frac{\dot{r}}{L}\right) - \frac{r}{L}\dot{\phi}^2 = 0$$

But $L = ds/d\sigma$ so multiply by L and use S as the parameter rather than σ

$$\frac{d^2 r}{ds^2} = r\left(\frac{d\phi}{ds}\right)^2 \qquad \frac{d}{ds}\left(r^2\frac{d\phi}{ds}\right) = 0$$

2. This is a strange metric, partly because of the $dv\,dt$ term. The $\hat{e}_v$ and $\hat{e}_x$ axes are not orthogonal. I found it useful to look at a different coordinate system without the $dv\,dx$ term. Try $v' = v + f(x)$.

$$dv' = dv + \dot{f}\,dx \qquad \dot{f} \equiv df/dx$$
$$dv^2 = dv'^2 - 2\dot{f}\,dx\,dv - \dot{f}^2\,dx^2$$
$$ds^2 = -x\,[dv'^2 - 2\dot{f}\,\cancel{dx\,dv} - \dot{f}^2 dx^2] + \cancel{2\,dv\,dx}$$

We make the cross terms go away by setting

$$2x\dot{f} + 2 = 0 \qquad \dot{f} = -1/x$$
$$df = -dx/x$$
$$f = -\ln x + c$$

With this substitution

$$ds^2 = -x\,dv'^2 + x\dot{f}^2\,dx^2$$

The light cone is defined by $ds^2 = 0$

$$x\,dv'^2 = dx^2/x$$
$$x^2\,dv'^2 = dx^2$$

$$\boxed{dx = \pm x\,dv'}$$

This makes more sense. We can solve for x as a function of v

$$\ln x + c = \pm v'$$

$$x = x_0 e^{\pm v'}$$

what does this look like in terms of x & v?

$$v' = v + c - \ln x$$

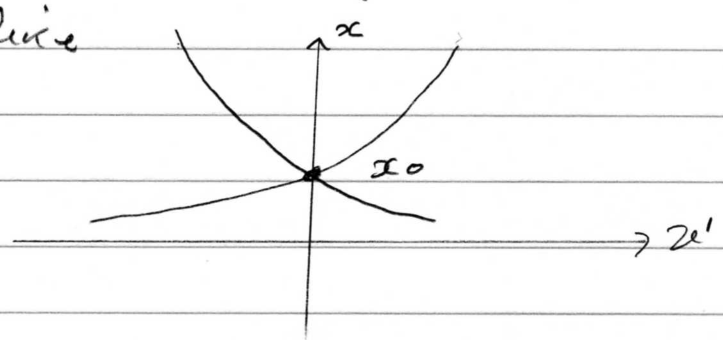
$$x = x_0 \exp\left(\pm v + c \mp \ln x\right)$$

$$x = x_0 e^{\pm v} x^{\mp 1}$$

upper sign $\Rightarrow$ $x^2 = x_0^2 e^{v}$ $\qquad x = x_0 e^{v/2}$

lower sign $\Rightarrow$ $1 - x_0 e^{-v}$ $\qquad v = \text{constant}$

It's hard to make a meaningful plot of this since x & v axes are not orthogonal. In the x, v' plane they look like



(c)

$$ds^2 = dv^2\left(-x + 2\frac{dx}{dv}\right)$$

If a particle can move from $A \to B$, A & B must be light-like separated i.e. negative ds^2 consequently $\frac{dx}{dv}$ must be negative if x is negative

3(a) Along the radius $d\theta = d\phi = 0$

$$R = \int_{r=2M}^{3M} \frac{dr}{\sqrt{1 - 2M/r}}$$

Maple had some trouble with this because of the singularity at $r = 2M$. The answer is well defined, however.

$$R = M\left(\sqrt{3} + \ln\left(2+\sqrt{3}\right)\right) = 3.05\,M$$

(b) This is a bit harder. The metric is

$$g_{11} = \frac{1}{1-2M/r} \qquad g_{22} = r^2 \qquad g_{33} = r^2 \sin^2\theta$$

So if you held θ & ϕ constant and varied r
$ds_r = \sqrt{g_{11}}\, dr$ similarly $ds_\theta = \sqrt{g_{22}}\, d\theta$ and
$ds_\phi = \sqrt{g_{33}}\, d\phi$. Differential volume is then

$$dV = \sqrt{g_{11}\, g_{22}\, g_{33}}\; dr\, d\theta\, d\phi$$

$$= \frac{r^2 \sin\theta \, d\phi\, d\theta\, dr}{\sqrt{1-2M/r}}$$

$$V = 4\pi \int_{r=2M}^{3M} \frac{r^2\, dr}{\sqrt{1-2M/r}} = \frac{1}{2} M^3 \left\{ 16\sqrt{3} + 5 \ln\left(2+\sqrt{3}\right) \right\} \times 4\pi$$

$$= 215.5\, M^3$$