1. The metric of the plane in polar coordinates $r$ and $\phi$ is

$$ds^2 = dr^2 + r^2 d\phi^2$$

A curve between two points can be described parametrically by giving $r$ and $\phi$ as functions of some parameter $\sigma$, which you can define to suit your own convenience. A curve is then described by two functions $r(\sigma)$ and $\phi(\sigma)$. Find the equations of a geodesic in this space. The equations are particularly simple is you use $s$ as the parameter. Show that

$$\frac{d^2 r}{ds^2} = r \left( \frac{d\phi}{ds} \right)^2$$

$$\frac{d}{ds} \left( r^2 \frac{d\phi}{ds} \right) = 0$$

(These equations are hard to solve. You might say that this is getting straight lines the hard way.)

2. Consider the two-dimensional spacetime spanned by coordinates $(v, x)$ with the line element

$$ds^2 = -xdv^2 + 2dv \, dx$$

This is a simple two-dimensional toy model for a black hole. It has the property that if you are trapped in the region $x < 0$, you can’t get out!

(a) Calculate the light cone at a point $(v, x)$.

(b) Draw a $(v, t)$ spacetime diagram showing how light cones change with $x$.

(c) Show that a particle can cross from positive $x$ to negative $x$ but cannot cross from negative $x$ to positive $x$. 
3. Consider the three-dimensional space with the line element

\[ ds^2 = \frac{dr^2}{(1 - 2M/r)} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \]

(a) Calculate the radial distance between the sphere \( r = 2M \) and the sphere \( r = 3M \).

(b) Calculate the spatial volume between the two spheres in part (a).
Cosmology Problem Set #2

1. \[ ds = \sqrt{dr^2 + r^2 d\phi^2} \]

\[ S = \int_A^B \sqrt{\left(\frac{dr}{ds}\right)^2 + r^2 \left(\frac{d\phi}{ds}\right)^2} \, ds = \int_A^B L \left(\frac{dr}{ds}, \frac{d\phi}{ds}, r\right) \, ds \]

\[ = \int_A^B \sqrt{\frac{1}{r^2} \frac{dr^2}{ds} + r^2 \left(\frac{d\phi}{ds}\right)^2} \]

\[ \frac{\partial L}{\partial \phi} = r^2 \frac{d\phi}{ds} \quad \frac{\partial L}{\partial r} = \frac{r}{L} \quad \frac{\partial L}{\partial \theta r} = \frac{r}{L} \frac{d\phi}{ds} \]

The Euler–Lagrange Eqn.'s ar

\[ \frac{d}{ds} \left( \frac{\partial L}{\partial \phi} \right) - \frac{\partial L}{\partial \phi} = \frac{d}{ds} \left( \frac{r^2 \phi}{L} \right) = 0 \]

\[ \frac{d}{ds} \left( \frac{\partial L}{\partial r} \right) - \frac{\partial L}{\partial r} = \frac{d}{ds} \left( \frac{r}{L} \right) - \frac{r}{L} \left(\frac{d\phi}{ds}\right)^2 = 0 \]

But \[ L = ds/d\phi \] so multiply by \[ L \] and use \[ S \] as the parameter rather than \[ 0 \]

\[ \frac{d^2r}{ds^2} = r \left(\frac{d\phi}{ds}\right)^2 \quad \frac{d}{ds} \left( \frac{v^2 d\phi}{ds} \right) = 0 \]
2. This is a strange metric, partly because of the $dx^2 dt$ term. The $e_2$ and $e_3$ axes are not orthogonal. I found it useful to look at a different coordinate system without the $dx dt$ term. Try $x' = x + f(x)$.

$$\begin{align*}
dx' &= dx + f \, dx & f &= df/dx \\
d^2 = dxe'^2 &+ x \int dx \, dx' - \int f^2 dx^2 \\
ds^2 &= -x [dxe'^2 + \int dx \, dx' - \int f^2 dx^2] + 2dxdx'
\end{align*}$$

We make the cross terms go away by setting

$$\begin{align*}
x' f + 2 &= 0 & f &= -1/x \\
df &= -dx/x & f &= -\ln x + c
\end{align*}$$

With this substitution

$$ds^2 = -x \, dx' \, dx' + x f^2 \, dx^2$$

The light cone is defined by $ds^2 = 0$. Thus

$$\begin{align*}
x' dx' &= dx^2/x \\
x^2 \, dx' \, dx' &= dx^2
\end{align*}$$

This makes more sense. We can solve for $x$ as a function of $x'$

$$\ln x + c = \pm x'$$
\[ x = x_0 \, e^{\pm \nu} \]

What does this lozenge line in terms of \( x \) & \( \nu \)?

\[ \nu' = \nu^2 + C - \ln x \]
\[ x = x_0 \exp \left( \pm \nu + C + \ln x \right) \]

\[ x = x_0 \, C^{\pm \nu} \, x^1 \]

Upper sign \( \Rightarrow x^2 = x_0^2 \, e^{2\nu} \quad x = x_0 \, e^{\nu^2/2} \)

Lower sign \( \Rightarrow 1 - x_0 \, e^{-2\nu} \quad \nu = \text{constant} \)

It's hard to make a meaningful plot of this since \( x & \nu \) axes are not orthogonal. In the \( x, \nu \) plane, this lozenge line:

\[ ds^2 = d\nu^2 \left( -x + 2 \frac{dx}{d\nu} \right) \]

If a particle can move from \( A \to B \), \( A & B \) must be light-like separated i.e. negative \( ds^2 \)

Consequently, \( dx \) must be negative if \( x \) is negative.
3(a) Along the radius, \( d\phi = 0 \)

\[
R = \int_{r=2M}^{3M} \frac{dr}{\sqrt{1 - \frac{2M}{r}}}
\]

Maple had some trouble with this, because of the singularity at \( r = 2M \). The answer is well defined, however:

\[
R = M \left( \sqrt{3} + \ln \left( 2 + \sqrt{3} \right) \right) = 3.05M
\]

(b) This is a bit harder. The metric is:

\[
\begin{align*}
\not{g}_{11} &= \frac{1}{1 - \frac{2M}{r}} \quad \not{g}_{22} = r^2 \quad \not{g}_{33} = r^2 \sin^2 \theta
\end{align*}
\]

So if you hold \( \theta + \phi \) constant and varied \( r \)

\[
ds^2 = \sqrt{\not{g}_{11}} \, dr
\]

Similarly, \( ds_\theta = \sqrt{\not{g}_{22}} \, d\theta \) and \( ds_\phi = \sqrt{\not{g}_{33}} \, d\phi \). Differential volume is then:

\[
\begin{align*}
\text{d}V &= \sqrt{\not{g}_{11}} \, \not{g}_{22} \, \not{g}_{33} \, \text{d}r \, \text{d}\theta \, \text{d}\phi \\
&= \frac{r^2 \sin \theta \, d\phi \, dr}{\sqrt{1 - \frac{2M}{r}}}
\end{align*}
\]

\[
V = 4\pi \int_{r=2M}^{3M} \frac{r^2 \, dr}{\sqrt{1 - \frac{2M}{r}}} = \frac{1}{2} M^3 \left( \frac{1}{16\sqrt{3}} + 5 \ln \left( 2 + \sqrt{3} \right) \right)^{3/4} 4\pi = 315.5 \, M^3
\]