Cosmology Problem Set #1

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The following exercises are due Friday, February 2.

1. An observer on earth watches a spaceship traveling with a speed of $\beta = 0.9$. He uses a telescope to watch a clock ticking on board the ship. How fast does he *see* the clock tick relative to the proper time. (Hint: Make allowances for the travel time of the light signal.)

2. Here is an exercise in using tensor notation. The scalar product between the four-vectors x and y in special relativity is

$$x \cdot y = x^\mu \eta_{\mu\nu} y^\nu$$

Lorentz transformations, which we write

$$x'^\mu = [\boldsymbol{L}]^\mu{}_\nu x^\nu,$$

leave this product invariant, *and do not change* η. Use this to prove that

$$\eta_{\gamma\delta} = [\boldsymbol{L}]^\alpha{}_\gamma \eta_{\alpha\beta} [\boldsymbol{L}]^\beta{}_\delta$$

We sometimes use this as a definition of the Lorentz transformation.

3. Exercise (2.3) in the text. I realize that a brief solution is given in the back of the book. I would like you to write it out carefully so that you understand it.

4. Here is another way to derive the formula for the Doppler shift. It make use of the fact that the $k^\alpha = (\omega/c, \boldsymbol{k})$ is a four-vector. Apply the Lorentz transformation to this quantity to show that the frequency observed in the laboratory for a source moving in the $\hat{\boldsymbol{x}}$ direction with velocity $v = \beta c$ for arbitrary $\boldsymbol{k}$ is given by the formula

$$\frac{\omega}{\omega_0} = \frac{1}{\gamma(1 - \beta\cos\theta)}$$

where ω_0 is the frequency in the rest frame of the source, and θ is the angle of $\boldsymbol{k}$ relative to $\hat{\boldsymbol{x}}$ in the laboratory frame of reference.

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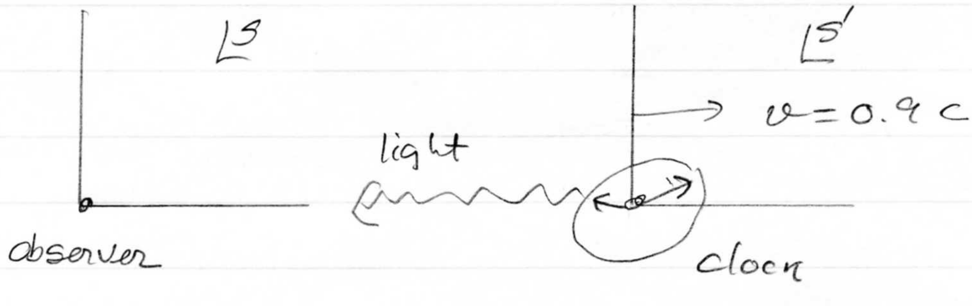
1.



To make it easy assume the first tick of the clock occurs at $t = t' = 0$ when the origins. We can then identify 3 point events:

#1 the clock ticks $x_1 = x_1' = t_1 = t_1' = 0$

2 the clock ticks a second time

$$t_2' = \Delta t \quad x_2' = 0 \quad x_2 = v t_2$$

3 the light from # 2 reaches the observer

$$t_3 = ? \quad x_3 = 0$$

Because of the travel time of light $t_3 = t_2 + x_2/c$

$$t_3 = t_2 (1 + v/c)$$

The times are related by $t = \gamma (t' + \beta x'/c)$

In this case

$$t_2 = \gamma (t_2' + \beta x_2'/c) = \gamma \Delta t$$

$$t_3 = \gamma \Delta t (1 + v/c)$$

$$\frac{t_3}{\Delta t} = \frac{(1+\beta)}{\sqrt{1-\beta^2}} = 4.36$$

So if the clock ticks once a second, the observer will see it click every 4.36 sec.

2. $x' \cdot y' = x'^{\mu} \eta_{\mu\nu} y'^{\nu}$

$$= L^{\mu}{}_{\alpha} x^{\alpha} \eta_{\mu\nu} L^{\nu}{}_{\beta} y^{\beta} = x \cdot y = x^{\alpha} \eta_{\alpha\beta} y^{\beta}$$

this must be true for all x^{α} and y^{β}, so

$$L^{\mu}{}_{\alpha} \eta_{\mu\nu} L^{\nu}{}_{\beta} = \eta_{\alpha\beta}$$

3. (a) chain rule:

$$\frac{\partial}{\partial x'^{\mu}} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} \frac{\partial}{\partial x^{\nu}}$$

but $x^{\nu} = [L^{-1}]^{\nu}{}_{\mu} x'^{\mu}$

or $\dfrac{\partial x^{\nu}}{\partial x'^{\mu}} = [L^{-1}]^{\nu}{}_{\mu}$

and $\dfrac{\partial}{\partial x'^{\mu}} = [L^{-1}]^{\nu}{}_{\mu} \dfrac{\partial}{\partial x^{\nu}}$

or if $\partial_{\mu} \equiv \dfrac{\partial}{\partial x^{\mu}}$ $\quad \partial'_{\mu} = [L^{-1}]^{\nu}{}_{\mu} \partial_{\nu}$

(b) assume $\partial_{\mu}' = [I]_{\mu}{}^{\nu} \partial_{\nu}$ where I is some unknown matrix. Then

$$\frac{\partial x'^{\nu}}{\partial x'^{\mu}} = \partial'_{\mu} x'^{\nu} = [I]_{\mu}{}^{\alpha} \partial_{\alpha} [L]^{\nu}{}_{\beta} x^{\beta}$$

$$= [I]_{\mu}{}^{\alpha} [L]^{\nu}{}_{\beta} \delta^{\beta}_{\alpha}$$

$$= [\bar{L}]_\mu{}^\alpha [L]^\nu{}_\alpha = \delta^\nu_\mu$$

so $\bar{L} = L^{-1}$ as promised.

4. $$k^\alpha = (\omega/c, \vec{k})$$

$$k'^\alpha = [L]^\alpha{}_\beta k^\beta$$

$$\begin{bmatrix} \omega'/c \\ k'_x \\ k'_y \\ k'_z \end{bmatrix} = \begin{bmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \omega/c \\ k_x \\ k_y \\ k_z \end{bmatrix}$$

$$\frac{\omega'}{c} = \gamma \left(\frac{\omega}{c} - \beta k_x \right)$$

but $k_x = k \cos\theta = \frac{\omega}{c} \cos\theta$

so $\omega' = \gamma \omega (1 - \beta \cos\theta)$