Cosmology Problem Set #2

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The following exercises are due Friday, February 9.

1. The metric of the plane in polar coordinates r and ϕ is

$$ds^2 = dr^2 + r^2 d\phi^2$$

A curve between two points can be described parametrically by giving r and ϕ as functions of some parameter σ, which you can define to suit your own convenience. A curve is then described by two functions $r(\sigma)$ and $\phi(\sigma)$. Find the equations of a geodesic in this space. The equations are particularly simple is you use s as the parameter. Show that

$$\frac{d^2 r}{ds^2} = r\left(\frac{d\phi}{ds}\right)^2$$

$$\frac{d}{ds}\left(r^2 \frac{d\phi}{ds}\right) = 0$$

(These equations are hard to solve. You might say that this is getting straight lines the hard way.)

2. Consider the two-dimensional spacetime spanned by coordinates (v, x) with the line element

$$ds^2 = -x dv^2 + 2dv\, dx$$

This is a simple two-dimensional toy model for a black hole. It has the property that if you are trapped in the region $x < 0$, you can't get out!

   (a) Calculate the light cone at a point (v, x).

   (b) Draw a (v, t) spacetime diagram showing how light cones change with x.

   (c) Show that a particle can cross from positive x to negative x but cannot cross from negative x to positive x.

3. Consider the three-dimensional space with the line element

$$ds^2 = \frac{dr^2}{(1 - 2M/r)} + r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$

(a) Calculate the radial distance between the sphere $r = 2M$ and the sphere $r = 3M$.

(b) Calculate the spatial volume between the two spheres in part (a).