Final Exam Solutions – Grad and Undergrad
PH 405/505, Cosmology

March 22, 2007

1. Answer the following questions in terms of the currently accepted picture of the universe. (10 points each)

   (a) Is the universe finite or infinite? How do you know?

   The universe is flat. (See next question.) Flat universes are infinite.

   (b) What is the curvature of the universe? How do you know?

   We know it's flat from the angular size of the microwave anisotropies.

   (c) What fraction of the energy density of the universe is made up of dark energy? How do you know?

   Roughly 70%. We know that the expansion of the universe is accelerating from the redshift-luminosity curves of type Ia supernovas. This together with the flatness of the universe constrains the dark energy fraction.

2. Given that the cosmic helium synthesis took place when the average thermal energy of particles was of the order of MeV, how would you go about estimating the number density ratio of neutron to proton n_m/n_p at that epoch? (30 points)

   At sufficiently high temperatures, neutrons and protons are in thermal equilibrium because of the weak interactions that transform one into the other. So long as this is true, their number density is governed by the Maxwell-Boltzman distribution. The ratio is then

   $$\frac{n_n}{n_p} = \exp\left(\frac{-Q}{kT}\right)$$

   As the universe expands, however, the interaction rate falls rapidly. At some temperature called the "freeze out" temperature, the transformations stop and the ratio is frozen at $\exp(-Q/kT_{freez})$ until the neutrons decay or are incorporated into nuclei.

3. What causes the anisotropies in the cosmic microwave background?

The anisotropies are controlled by the distribution of dark matter at the time of photon decoupling. The anisotropies in the dark matter density are the result of quantum fluctuations enormously magnified by inflation. They in turn affect the temperature distribution vis several mechanisms. One is the Sachs-Wolfe effect in which photons are red shifted or blue shifted as they climb in or out of gravitational potential wells. Another source of anisotropies is acoustic waves resulting from the compression and rarefaction of the electron-photon "fluid" inside potential wells.

4. What is the horizon problem, and how does the inflation postulate explain it? Give as much theoretical detail as possible. (30 points)

   The cosmic microwave background comes to us from a thin layer of space-time called the sphere of last scattering roughly 350,000 years from the origin of the universe. The Hubble distance at this time was a small fraction of the radius of the sphere, so that from our perspective it subtends an angular size of roughly 1°. Regions separated by more than this presumably have no causal connection with one another, and yet the entire sphere is uniform to one part in 10^5. We believe this is because the entire sphere was in causal contact before inflation. Inflation simply magnifies a very small smooth region of space.

5. Why should the accelerating universe lead us to observe the galaxies, at a given redshift, to be dimmer than expected (in an empty or decelerating universe)? Give as much theoretical detail as possible. (30 points)

   Luminosity distance is related to proper distance by means of an integral over $H(z)$, which means that this relationship depends on the past history of the universe back to the time the light was emitted. In an empty universe, for example, this would be a simple relation, but if the expansion is accelerating, then the expansion rate was smaller in the past. Thus to have a given redshift (i.e. recession speed) a star must be located farther away than expected, and it will appear dimmer than expected.

6. Our universe is spatially flat with the dominant component being matter and positive dark energy. Its fate is an unending exponential expansion. Now consider the same flat universe by with a negative dark energy $\Omega_\Lambda = 1 - \Omega_{M,0} < 0$, which provides a gravitational attraction. Show that this will slow the expansion down to a standstill when the scale factor reaches $a_{max} = (-\Omega_\Lambda/\Omega_{M,0})^{1/3}$. The universe will then start to contract. Show how you would calculate the age of the universe when $a(t)$ reaches zero again. (At some point you will encounter a difficult integral. Stop there!) (40 points)

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} = H_0^2\frac{\rho}{\rho_{c,0}} = H_0^2\left(\frac{\Omega_{M,0}}{a^3} + \Omega_\Lambda\right)$$

   If $\dot{a} = 0$

$$a = (-\Omega_{M,0}/\Omega_\Lambda)^{1/3} \equiv a_{max}$$

The universe oscillates with a sinusoidal motion taking same time to get from $0 \rightarrow a_{max}$ as from $a_{max} \rightarrow 0$.

$$t_{total} = \frac{2}{H_0} \int_0^{a_{max}} \frac{da}{\sqrt{\frac{\Omega_{M,0}}{a} + a^2 \Omega_\Lambda}}$$

7. A low-density universe may be approximated by setting the density function in the Friedmann equation to zero. Beside the uninteresting possibility of $\dot{a} = k = 0$ there is a non-trivial solution with curvature. Find $a(t)$. Find the Hubble relation between the proper distance and redshift in such a model universe. (40 points)

   Take $\rho = 0$ and $\kappa = -1$.

$$\frac{\dot{a}^2}{a^2} = \frac{c^2}{R_0^2 a^2}$$

$$a(t) = ct/R_0 \qquad H = \frac{1}{t} = \frac{c}{R_0 a} = \frac{c}{R_0}(1+z)$$

$$d_p(z) = \int_0^z \frac{c dz'}{H(z')} = \int_0^z \frac{c dz'}{c(1+z')/R_0} = R_0 \ln(1+z)$$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{\kappa c^2}{R_0^2 a^2} \qquad \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P) \qquad 1 + z = 1/a \qquad d_L(z) = (1+z)d_p(z)$$

$$d_p(t_{em}) = \int_{t_{em}}^{t_0} \frac{cdt}{a(t)} \qquad d_p(a_{em}) = \int_{a_{em}}^{1} \frac{cda}{a^2 H(a)} \qquad d_p(z) = \int_0^z \frac{cdz'}{H(z')}$$

$$\rho_c = \frac{3H^2}{8\pi G} \qquad \rho_M = \frac{\rho_{M,0}}{a^3} \qquad \rho_R = \frac{\rho_{R,0}}{a^4}$$

$$n = g\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2} \exp\left(-\frac{mc^2}{kT}\right)$$