A Summary of Useful Equations for Cosmology

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February 19, 2007

The starting point for mathematical cosmology is the Robertson-Walker metric

$$ds^2 = -dt^2 + a(t)^2 R^2 \left[d\chi^2 + \left\{ \begin{array}{l} \sin^2 \chi \\ \chi^2 \\ \sinh^2 \chi \end{array} \right\} (d\theta^2 + \sin^2 \theta \, d\phi^2) \right] \left\{ \begin{array}{c} \text{closed} \\ \text{flat} \\ \text{open} \end{array} \right\}$$

$$ds^2 = -dt^2 + a(t)^2 R^2 \left[\frac{dr^2}{1 - \kappa r^2} + r^2 (d\theta^2 + \sin^2 \theta \, d\phi^2) \right] \left(\begin{array}{l} \kappa = +1, \ \text{closed} \\ \kappa = 0, \text{flat} \\ \kappa = -1, \ \text{open} \end{array} \right)$$

These are equations for the invariant length ds. The parameter $a(t)$ is the cosmic scale factor. By convention it is normalized to unity at the present time, $a(t_0) = 1$. The time variable is "cosmic time." It is assumed that all observers can synchronize their watches to t. (In the language of Star Trek, it's the "stardate.") The other variables, χ, θ, ϕ and r are angles measured in a comoving coordinates. The distance R has many uses. Usually it cancels out somewhere in the calculation.

The Friedman equations:

$$\left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{\kappa c^2}{R_0^2 a^2}$$

$$\dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + P) = 0$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3P)$$

as a set predict the time dependence of a. G (sometimes called G_N) is Newton's gravitational constant. ρ is the density of matter and energy in the universe. It not only includes matter and radiation but also (in our present understanding) the dark energy

$$\rho_\Lambda = \frac{\Lambda}{8\pi G}$$

The equation of state gives pressure P in terms of ρ. $P = w\rho c^2$ where $w = 0$ for matter, $w = 1/3$ for radiation, and $w = -1$ for dark energy.

The critical density is

$$\rho_c = \frac{3H^2}{8\pi G}$$

where $H = \dot{a}/a$ is the Hubble parameter. It is the density required to make a flat universe. When densities are normalized by ρ_c they are given the symbol Ω

$$\Omega = \frac{\rho}{\rho_c}$$

Since most equations are linear in ρ it makes sense to break it up into its various constituents. $\Omega = \Omega_r + \Omega_m + \Omega_\Lambda$. The zero subscript always refers to the present time as in t_0, H_0, Ω_0, etc. By definition $a(t_0) = 1$. These quantities scale with a in a simple way.

$$\Omega_r(t) = \Omega_{r,0}a(t)^{-4} \qquad \Omega_m(t) = \Omega_{m,0}a(t)^{-3} \qquad \Omega_\Lambda(t) = \Omega_{\Lambda,0}$$

The proper distance d_p is the invariant distance ds measured along a straight line $d\theta = d\phi = 0$ from the observer to a distant object *at one value of the cosmic time.* The position of the distant object can be characterized in one of several ways: the time t_{em} at which it emitted some light we are just now observing, the value of the cosmic scale factor at this time, or the redshift of the light. These yield three equivalent expressions.

$$d_p(t_{em}) = \int_{t_{em}}^{t_0} \frac{cdt}{a(t)} \qquad d_p(a) = \int_{a_{em}}^{1} \frac{cda}{a^2(t)H(t)} \qquad d_p(z) = \int_0^z \frac{cdz'}{H(z')}$$