IV. The Wave Equation

1. Introduction.

We shall consider well-posed problems for the wave equation in two and three variables. Recall that the well-posed initial value problem for the second order ordinary differential equation

$$y''(t) = f(t, y, y')$$

results (under various hypothesis on f) from asking for a solution in some interval, $|t - t_0| < a$, for which the function and its first derivative are specified at the point, t_0. For example, if f is analytic at (t_0, a_0, a_1), the solution is obtained as the sum of a power series, $y(t) = \sum_{n=0}^{\infty} a_n (t - t_0)^n$. The first two coefficients are determined by the initial conditions, $a_0 = y(t_0)$, $a_1 = y'(t_0)$, and all the remaining ones are then determined by the equation, e.g.,

$$2!a_2 = y''(t_0) = f(t_0, a_0, a_1) \ .$$

A direct generalization of the initial-value problem to the second order semi-linear partial differential equation

$$a(x, y)u_{xx} + 2b(x, y)u_{xy} + c(x, y)u_{yy} = f(x, y, u, u_x, u_y)$$

is to ask for a solution for which the function and its first order derivatives are specified along a curve. This is called a *Cauchy problem*. Hence, suppose we are given a curve in the parametric form $x = x(t)$, $y = y(t)$ and we want to find a solution u of the equation which satisfies the "initial conditions"

$$u(x(t), y(t)) = u_0(t) \ , \quad u_x(x(t), y(t)) = p(t) \ , \quad u_y(x(t), y(t)) = q(t) \ .$$

In order to attempt the procedure used above for the ordinary differential equation, we shall try to evaluate the second order derivatives of u from the information above. Differentiate the second and third equations in the initial conditions to obtain

$$u_{xx}(x(t), y(t))x'(t) + u_{xy}(x(t), y(t))y'(t) = p'(t) \ ,$$
$$u_{xy}(x(t), y(t))x'(t) + u_{yy}(x(t), y(t))y'(t) = q'(t) \ .$$

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These two identities together with the partial differential equation above give us, for each fixed t, three equations from which to determine the three second order derivatives, u_{xx}, u_{xy}, u_{yy}, along the curve. This system can be solved uniquely only if the determinant of its coefficients is non-zero. This determinant is just

$$ay'(t)^2 - 2bx'(t)y'(t) + cx'(t)^2 \ ,$$

so we have shown that the above initial value problem for the partial differential equation determines the second order derivatives of a solution along the curve if the curve is non-characteristic. (Cf. I, §5.) One can show similarly that all the derivatives of u are uniquely determined at any point on the curve where the tangent does not possess the characteristic direction. We shall later show that the Cauchy problem is well-posed for hyperbolic equations if the "initial curve" is nowhere characteristic.

For comparison, consider the elliptic equation of Laplace. There are no characteristics, so we consider the Cauchy problem

$$u_{xx} + u_{yy} = 0 \ , \quad u(0,y) = 0 \ , \quad u_x(0,y) = f(y) \ .$$

First, we note that there is a solution only if f is analytic, for u_x is harmonic in a neighborhood of the y-axis. But taking $f_n(y) = (\frac{1}{n})\sin(ny)$, we obtain the solutions $u_n(x,y) = \frac{1}{n^2}\sin h(nx)\sin(ny)$. Suppose $\varepsilon > 0$, $x_0 > 0$, and $M > 0$ are given. Choose n with $1/n < \varepsilon$ and $\sin h(nx_0)/n^2 > M$. For each integer k, define $y_k = (k+\frac{1}{2})\pi/n$. Then we have $|f_n(y_k)| < \varepsilon$ but $|u_n(x_0,y_k)| > M$. Thus, we can find a solution of Laplace's equation with initial data on $x = 0$ arbitrarily small for which the values can be made arbitrarily large at points arbitrarily close to $x = 0$. This shows that the Cauchy problem for the Laplace equation is not well-posed even if we restrict the initial data to be analytic.

Exercises.

(1.1) Show there is at most one solution of the Cauchy problem above for the Laplace equation.

(1.2) Discuss the well-posedness of the Cauchy problem

$$u_{xx} + u_{yy} = 0 \ , \quad u(0,y) = f(y) \ , \quad u_x(0,y) = 0 \ .$$

2. The Cauchy Problem.

The Cauchy problem for the hyperbolic semi-linear equation

$$u_{xy} = f(x, y, u, u_x, u_y) \tag{1}$$

is the following:

Let a curve $C : x = x(t),\ y = y(t)$ in the plane and functions $u_0(t),\ p(t),\ q(t)$ be given. Find a function twice continuously differentiable in a neighborhood of C satisfying (1) in that neighborhood and the initial conditions

$$u(x(t), y(t)) = u_0(t)\ , \quad u_x(x(t), y(t)) = p(t)\ , \quad u_y(x(t), y(t)) = q(t)\ . \tag{2}$$

Note that the three functions in the Cauchy data cannot be prescribed independently. The chain rule gives the necessary compatibility condition

$$u_0'(t) = p(t)x'(t) + q(t)y'(t) \tag{3}$$

so only two of the three functions need be prescribed, the remaining one then coming from (3). Essentially, we can prescribe the solution along the curve (and thereby the derivative tangential to the curve) and any other non-tangential derivative along the curve. Note, further, that along a characteristic curve, $x = x_0$, the equation (1) is an ordinary differential equation for the function $p(\cdot)$,

$$p'(y) = f(x_0, y, u_0, p, q)$$

and, hence, the Cauchy data is even more restricted along the characteristics. In particular, we need only to specify $u_0(y)$ in the case above, for then $q(y) = u_0'(y)$ and p is determined from the equation above. The preceding remarks apply at any point on the curve C at which the tangent has a characteristic direction, so we shall assume hereafter that the curve C is nowhere-characteristic. Thus we may represent it in the form

$$C\colon y = y_0(x)\ . \tag{4}$$

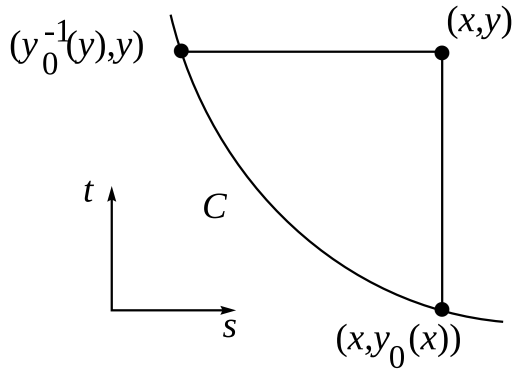
where y_0 is strictly monotone. The Cauchy data (2) is thus given with $x(t) = t = x$ and $y(t) = y_0(x)$, and so (2) takes the form

$$u(x, y_0(x)) = u_0(x)\ , \quad u_x(x, y_0(x)) = p(x)\ , \quad u_y(x, y_0(x)) = q(x)\ , \tag{2'}$$

while the compatibility condition becomes

$$(3') \qquad u_0'(x) = p(x) + q(x)y_0'(x) \ .$$

Let u be a solution of the homogeneous equation, $u_{xy} = 0$, in some open connected set G which contains a portion of the curve (4), and let u satisfy the Cauchy condition $(2')$. From the equation we have $u(x,y) = \varphi(x) + \psi(y)$ in G for some pair of functions φ, ψ. Hence, $u_x(x,y) = \varphi'(x)$ and $u_y(x,y) = \psi'(y)$ for $(x,y) \in G$. Let $(x,y) \in G$ be such that the closure of the region in the plane bounded by the characteristics through (x,y) and the curve C is contained in G.



Then we have $p(x) = \varphi'(x)$ and $q(x) = \psi'(y)$ in this region and hence follows

$$\int_{y_0^{-1}(y)}^{x} p(s)\,ds = \varphi(x) - \varphi(y_0^{-1}(y)) = u(x,y) - u(y_0^{-1}(y), y) \ .$$

Using $(2')$ again, we obtain

$$u(x,y) = u_0(y_0^{-1}(y)) + \int_{y_0^{-1}(y)}^{x} p(s)\,ds \ .$$

A similar calculation gives

$$u(x,y) = u_0(x) + \int_{y_0(x)}^{y} q(y_0^{-1}(t))\,dt \ .$$

So we have two representations for the value of the solution u at the point (x,y) in terms of the Cauchy data. However, using the change-of-variable theorem and then the condition $(3')$ we have

$$\int_{y_0(x)}^{y} q(y_0^{-1}(t))\,dt = \int_{x}^{y_0^{-1}(y)} q(s)y_0'(s)\,ds = \int_{y_0^{-1}(y)}^{x} p(s)\,ds + u_0(y_0^{-1}(y)) - u_0(x) \ ,$$

so these two representations are the same. By averaging these two, we obtain the symmetric representation

$$(5)\quad u(x,y) = \frac{1}{2}(u_0(x) + u_0(y_0^{-1}(y))) + \frac{1}{2}\int_{y_0^{-1}(y)}^{x} p(s)\,ds + \frac{1}{2}\int_{y_0(x)}^{y} q(y_0^{-1}(t))\,dt\ .$$

We note that the integrals in (5) can be written as line integrals along C and we obtain, finally, the representation

$$(5')\qquad u(x,y) = \frac{1}{2}(u_0(x) + u_0(y_0^{-1}(y))) + \frac{1}{2}\int_{(y_0^{-1}(y),y)}^{(x,y_0(x))} (p(s)\,ds - q(y_0^{-1}(t))\,dt)$$

giving the value of the solution u of the homogeneous equation at (x,y) in terms of the Cauchy data along a portion of C depending on (x,y).

Consider now the non-homogeneous linear equation

$$(6)\qquad u_{xy} = F(x,y)\ .$$

Let u be a solution of (6) in an open connected set G containing a part of the curve C, and suppose u and its partial derivatives vanish along C. Choose a point $(x,y) \in G$ as in the preceding discussion. Then we have

$$\int_{y_0(x)}^{y}\int_{y_0^{-1}(t)}^{x} F(s,t)\,ds\,dt = \int_{y_0(x)}^{y} u_y(x,t)\,dt = u(x,y)\ ,$$

and this is a representation for solutions to (6) with homogeneous Cauchy data. Since the equation (6) is linear we have obtained a representation for its solutions which satisfy general Cauchy data (2′) along non-characteristic curves. But we have also settled some existence and uniqueness questions.

Theorem 1. *Let C be the curve* (4) *where y_0 is continuously differentiable on the interval $[a,b]$ with $y'(x) \neq 0$ for $x \in [a,b]$, and twice continuously differentiable in (a,b). Let G be the rectangle with corners at the points $(a, y_0(a))$ and $(b, y_0(b))$. Let F be continuous on G, p and q continuously differentiable on (a,b), and u_0 twice continuously differentiable on (a,b). Then there is exactly one solution of the Cauchy problem* (6), (2′) *and it is given by*

$$(7)\qquad u(x,y) = \frac{u_0(x) + u_0(y_0^{-1}(y))}{2} + \frac{1}{2}\int_{(y_0^{-1}(y),y)}^{(x,y_0(x))} p(s)\,ds - q(y_0^{-1}(t))\,dt$$
$$+ \int_{y_0(x)}^{y}\int_{y_0^{-1}(t)}^{x} F(s,t)\,ds\,dt\ .$$

Corollaries. *The Cauchy problem* $(2')$ *is well-posed for the linear hyperbolic equation* (6)*. The solution at a point* (x,y) *depends on its value at the two points on* C *intersected by the characteristics through* (x,y)*, its derivatives along that portion of* C *between these two points, and the function* F *in the region bounded by these two characteristics and* C*.*

Exercises.

(2.1) Show that for solutions of the heat equation $u_{xx} = u_y$, the functions p and q in (2) are determined by u_0 when the data is assigned along the curve $y =$ constant. Show, also that all second-order derivatives of a solution of the heat equation are determined by u_0 along such a curve.

(2.2) Verify that the functions $u_n(x,y) = \sin(nx)/ne^{n^2 y}$ satisfy the heat equation and the initial condition $u_n(x,0) = \sin(nx)/n$. Show that solutions of the heat equation in the half-plane $y < 0$ can *not* be expected to depend continuously on the data given along the x-axis.

(2.3) Use Green's Theorem on the identity $u_{xy} = \frac{1}{2}\{(u_y)_x + (u_x)_y\}$ to obtain the representation (7).

(2.4) The *Characteristic Cauchy Problem* is to find a $u \in C^2(G)$, G a given open square centered at (a,b), such that

$$u_{xy} = F(x,y) \ , \quad (x,y) \in G \ , \quad u(a,y) = \varphi_1(y) \ , \quad u(x,b) = \varphi_2(x) \ .$$

Show there is at most one solution to this problem. A necessary and sufficient condition for the existence of a solution is that $F \in C(G)$, φ_1 and φ_2 are twice continuously differentiable at each y and x, respectively, for which $(x,y) \in G$, and $\varphi_1(b) = \varphi_2(a)$. Find a representation for the solution similar to (7).

(2.5) Show that if u is sufficiently smooth and satisfies $u_{xy} = 0$ in an open set G, then $u(A) - u(B) + u(C) - u(D) = 0$, where A, B, C, D are the consecutive corners of any square in G with sides parallel to the coordinate axes..

(2.6) Theorem 1 gives sufficient conditions for existence and uniqueness of a solution. Give sufficient conditions for uniqueness of a solution.

3. Successive Approximations.

Our objective in this section is to show that the Cauchy problem for the hyperbolic semi-linear equation (1) is well-posed. First, we show that this problem is equivalent to an integro-differential equation. Let the curve (4), the rectangle G, and Cauchy data (2′) be as in Theorem 1. Let $f\colon G \times H \to \mathbb{R}$ be continuous where H is an open subset of $\mathbb{R}^3$, and let u be a continuously differentiable solution of (1), (2′) on G with $(u, u_x, u_y) \in H$ for $(x, y) \in G$. Define $F(\xi, \eta) = f(\xi, \eta, u(\xi, \eta), u_x(\xi, \eta), u_y(\xi, \eta))$, $(\xi, \eta) \in G$. Then u satisfies (6) and hence we obtain from Theorem 1

(8)
$$u(x,y) = \frac{u_0(x) + u_0(y_0^{-1}(y))}{2} + \frac{1}{2}\int_{(y_0^{-1}(y),x)}^{(x,y_0(x))} p(s)\,ds - q(y_0^{-1}(t))\,dt$$
$$+ \int_{y_0(x)}^{y}\int_{y_0^{-1}(t)}^{x} f(s,t,u(s,t),u_x(s,t),u_y(s,t))\,ds\,dt\,, \qquad (x,y) \in G\,.$$

Conversely, a continuously differentiable solution of (8) is a solution of the Cauchy problem (1), (2′).

Before continuing further, we simplify the problem (1), (2′) in two ways. First, we assume that $u_0 \equiv p \equiv q \equiv 0$ in (2′). This causes no loss in generality since otherwise we may subtract from our desired function u the expression v defined by (5). This difference satisfies the null Cauchy data (2′) and an equation like (1) but with $f(x, y, r, s, t)$ replaced by $f(x, y, r + v(x, y), s + p(x), t + q(y))$, and this new function will satisfy the hypotheses we place below on f. Second, we assume that the curve C is given by the straight line $x + y = 0$. Otherwise, we introduce the change of variable $\xi = y_0(x)$, $\eta = -y$. In the variable (ξ, η), the curve C is transformed to the straight line $\xi + \eta = 0$ and the transformed equation is like (1) with f modified but satisfying the hypotheses we place below on f. After these two simplifications, the Cauchy problem (1), (2′) is equivalent to the equation

(9)
$$u(x,y) = \int_{-x}^{y}\int_{-t}^{x} f(s,t,u,u_x,u_y)\,ds\,dt$$
$$= \int_{-y}^{x}\int_{-s}^{y} f(s,t,u,u_x,u_y)\,dt\,ds\,.$$

Let $x_0 + y_0 = 0$ and f be a real-valued function which is continuous in some neighborhood of $(x_0, y_0, 0, 0, 0)$. Hence, we can choose $a > 0$ and $b > 0$ so small that this neighborhood contains $\overline{G}_a \times \overline{H}_b$, where $G_a = \{(x,y)\colon \max(|x-x_0|, |y-y_0|) < a\}$ and $H_b = \{(r,s,t)\colon \max(|r|,|s|,|t|) < b\}$. Let M be the maximum value of f on $\overline{G}_a \times \overline{H}_b$, and for each $c > 0$ let $J_c = \{(x,y)\colon |x+y| < c\}$.

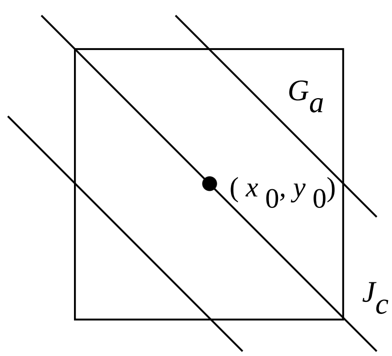
Let $u\colon G_a \to \mathbb{R}$ be continuously differentiable and suppose $(u, u_x, u_y) \in H_b$ for all $(x,y) \in G_a$. Then we can define a continuously differentiable function on G_a by

$$(10) \qquad v(x,y) = \int_{-x}^{y} \int_{-t}^{x} f(s,t,u,u_x,u_y)\, ds\, dt\ , \qquad (x,y) \in G_a\ .$$

For those $(x,y) \in G_a \cap J_c$, we have the estimates

$$|v(x,y)| \le Mc^2/2\,, \quad |v_x(x,y)| \le Mc\,, \quad \text{and } |v_y(x,y)| \le Mc\,.$$

If c is chosen with $Mc^2/2 \le b$ and $Mc \le b$, it follows that $(v, v_x, v_y) \in H_b$ whenever $(x,y) \in G_a \cap J_c$. Hence, with c as above, if u is continuously differentiable on $G_a \cap J_c$ with $(u, u_x, u_y) \in H_b$, then the same is true of the function v defined by (10).



Given the function f and the constants $a, b, c > 0$ as above, we can define a sequence of functions u^n by

$$(11.a) \qquad \begin{cases} u^0(x,y) = 0\ , \\ u^{n+1}(x,y) = \displaystyle\int_{-x}^{y} \int_{-t}^{x} f(x,t,u^n,u^n_x,u^n_y)\, ds\, dt\ , \end{cases} \qquad (x,y) \in G_a \cap J_c\ .$$

Note that the sequences of partial derivatives are given by

$$(11.b) \qquad u^{n+1}_x(x,y) = \int_{-x}^{y} f(x,t,u^n,u^n_x,u^n_y)\, dt\ ,$$

$$(11.c) \qquad u^{n+1}_y(x,y) = \int_{-y}^{x} f(s,y,u^n,u^n_x,u^n_y)\, ds\ .$$

Assume now that f satisfies the Lipschitz condition

$$(12) \quad |f(x,y,r,s,t) - f(x,y,r',s',t')| \le K(|r-r'| + |s-s'| + |t-t'|)$$

$$(x,y) \in G_a\ ;\ (r,s,t), (r',s',t') \in H_b\ ,$$

and define for each $n \geq 0$

$$M_n = \sup_{G_a \cap J_c} \{|u^{n+1} - u^n| + |u_x^{n+1} - u_x^n| + |u_y^{n+1} - u_y^n|\} .$$

From the Lipschitz condition (12) it follows that on $G_a \cap J_c$, $|u^{n+2} - u^{n+1}| \leq KM_n c^2/2$, $|u_x^{n+2} - u_x^{n+1}| \leq KM_n c$, and $|u_y^{n+2} - u_y^{n+1}| \leq KM_n c$, so we obtain the fundamental estimate

$$M_{n+1} \leq Kc(2 + c/2)M_n \ , \qquad n \geq 0 \ .$$

A trivial induction gives

$$M_n \leq [Kc(2 + c/2)]^n M_0 \ ,$$

and hence the series $\sum M_n$ is dominated by the geometric series $\sum [Kc(2 + c/2)]^n$. Therefore, these series converge if c is chosen so small that the quantity in brackets is less than one. Hence, if $Kc(2 + c/2) < 1$, we have shown that each of the series

$$\sum (u^{n+1} - u^n) \ , \quad \sum (u_n^{n+1} - u_x^n) \ , \quad \sum (u_y^{n+1} - u^n)$$

converges uniformly on $G_a \cap J_c$. Since the n^{th} partial sums of the series are just u^n, u_x^n and u_y^n, respectively, the sequences $\{u^n\}$, $\{u_x^n\}$ and $\{u_y^n\}$ converge uniformly on $G_a \cap J_c$ to continuous functions u, v, w. The Lipschitz condition and the uniform convergence permit us to take the limits in (11) to obtain

(13.a) $$u(x, y) = \int_{-x}^{y} \int_{-t}^{x} f(s, t, u, v, w) \, ds \, dt \ ,$$

(13.b) $$v(x, y) = \int_{-x}^{y} f(x, t, u, v, w) \, dt$$

(13.c) $$w(x, y) = \int_{-y}^{x} f(s, y, u, v, w) \, ds \ .$$

Since these functions are continuous, we can differentiate (13.a) and thereby find that $u_x = v$ and $u_y = w$. Substitution of these quantities back in (13.a) shows that u is a solution of (9) on $G_a \cap J_c$, thus establishing the *existence* of a solution to the Cauchy problem.

The Lipschitz condition also implies the uniqueness of the solution. To see this, suppose U is another solution of (9) on $G_a \cap J_c$, and let $V = U_x$, $W = U_y$. Then U,

V and W satisfy (13.a), (b) and (c), respectively so we can use these six equations and (12) to obtain the estimate $m \leq Kc(2 + c/2)m$, where

$$m = \sup_{G_a \cap J_c} \{|u - U| + |v - V| + |w - W|\} \ .$$

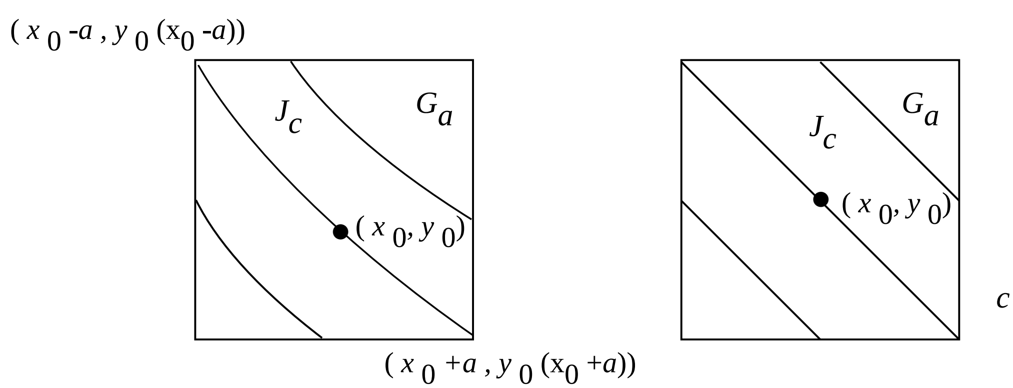
But the choice of c then implies $m = 0$, so $u = U$ on $G_a \cap J_c$. The above has provided a proof of the following result.

Theorem 2. *Let C be the curve (4) where y_0 is twice continuously differentiable on the interval $I_a = [x_0 - a, x_0 + a]$ and $a > 0$. Let G_a be the rectangle with corners at the points $(x_0 - a, y_0(x_0 - a))$ and $(x_0 + a, y_0(x_0 + a))$ and sides parallel to the axes. Let p and q be continuously differentiable and u_0 twice continuously differentiable on I_a. Let $b > 0$ and assume that the real-valued function f is defined and continuous on the set*

$$S = \{(x, y, r, s, t) : (x, y) \in G_a \ , \ |r - u_0(x)| < b \ , \ |s - p(x)| < b \ , \ |t - q(x)| < b\} \ ,$$

and that it satisfies the Lipschitz condition (12) on S. Then there is a positive number $c > 0$ such that, on the set of those $(x, y) \in G_a$ for which $|y - y_0(x)| < c$, there exists exactly one continuously differentiable solution of (8).

This is precisely the result we obtained for the special case given by (9). The formulation given here merely corresponds to the situation of the more general case of (8). The figures below indicate the geometry involved in (8) and (9), respectively.



Exercises.

(3.1) Verify all the details in the proof of Theorem 2.

(3.2) State and prove a result like Theorem 2 for the characteristic Cauchy problem for the nonlinear equation (1). (See Exercise (2.4).)

(3.3) The *Goursat Problem* is to find a function u that satisfies (1) in the region where $0 < y < x < a$, and the boundary conditions

$$u(x,x) = \varphi_1(x)\ , \quad u(x,0) = \varphi_2(x)\ , \qquad 0 < x < a\ .$$

(a) State and prove a result like Theorem 1 for the Goursat Problem with the linear equation (6).

(b) State and prove a result like Theorem 2 for the Goursat Problem with equation (1).

(3.4) Discuss existence, uniqueness and representation of solutions of the mixed Goursat problem

$$u_x(x,x) = p(x)\ , \quad u_y(x,x) = q(x)\ , \quad u(x,0) = u_0(x)\ , \qquad 0 < x < a$$

for the linear equation (6).

4. The Effect of Data.

We shall make some observations on the Cauchy problem (1), (2′) concerning the manner in which the solution is influenced by the Cauchy data. These remarks all follows from the integro-differential equation (8) and the method of proof of Theorem 2.

First we discuss the continuous dependence of the solution on the Cauchy data (2′) and the function f in the equation (1). Since in the previous section we simplified the problem by incorporating the Cauchy data in the function f, it follows that we need only to consider the case of homogeneous data where the integrand in (9) is permitted to vary. So suppose that for each $\lambda \in [0,1]$ we are given a function $f(\lambda;\cdot)$ on $\overline{G}_a \times \overline{H}_b$ as in the proof of Theorem 2, that the function $(\lambda,x,y,r,s,t) \mapsto f(\lambda;x,y,r,s,t)$ is continuous, hence, uniformly continuous and bounded on $[0,1] \times \overline{G}_a \times \overline{H}_b$, and that we have a Lipschitz condition

$$|f(\lambda;x,y,r,s,t) - f(\lambda;x,y,r',s',t')| \le K(|r-r'| + |s-s'| + |t-t'|)$$
$$\lambda \in [0,1]\ , \quad (x,y) \in G_a\ , \quad (r,s,t),(r',s',t') \in H_b\ .$$

For each λ we obtain a solution $u(\lambda;x,y)$ of (9) as the uniform limit of the sequence (11). In the identity like (11) for $u^n(\lambda;x,y)$, each term of the sequence is expressed as a continuous function λ in terms of the preceding one. (This follows by induction on n and the uniform continuity of the integrands in the identities like (11).) Our Lipschitz condition shows that the estimates of the majorants M_n are independent of λ, so the convergence of the sequence of approximations to $u(\lambda; x, y)$ is uniform on $[0,1] \times G_a$. But uniform convergence of the sequence and continuity of each term of the sequence imply that the limit of the sequence is continuous. That is, u is a continuous function of the variable λ, as well as x and y, and this is the desired result.

The solution obtained in Theorem 2 was "local" in the sense that it was obtained on some (possibly small) region containing the initial curve. This constraint arose from the necessity of choosing the number c sufficiently small to obtain two objectives: (1) to make the definition of the sequence possible and (2) to make the sequence converge. We shall show that a "global" solution on G_a can be obtained in certain cases that include the linear equation

$$u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y) = F(x,y) \ . \tag{14}$$

Suppose the nonlinear function f of (1) is defined on $G_a \times \mathbb{R}^3$ and satisfies there the Lipschitz estimate (12). Then for every k, there is a unique solution of (1) in $G_a \cap J_c(k)$ with Cauchy data given on the curve $y = y_0(x) + k$, where the number c is chosen so that $[Kc(2 + c/2)] < 1$ and is independent of k. Hence, we have the solution on $J_c(0)$; this gives Cauchy data along $y = y_0(x) \pm c/2$ for which we obtain solutions on $J_c(\pm c/2)$ which by uniqueness agree with the original solution on $J_c(0)$. Using the solutions on $J_c(\pm c/2)$ to give Cauchy data on $y = y_0(x) \pm c$, we obtain the respective extensions to $J_c(\pm c)$. In a finite number of steps, we have obtained a solution of the Cauchy problem on all of G_a. This technique applies to the linear equation (14) in which the coefficients and function F are assumed bounded and continuous on G_a. Then the function

$$f(x,y,r,s,t) \equiv F(x,y) - c(x,y)r - a(x,y)s - b(x,y)t$$

satisfies a Lipschitz condition (12) in which the constant K depends only on the bounds on the coefficients.

Our final remarks follow from the equation (8) for which we assume we have a solution on the set G_a. For a point (x, y) in G_a, define the domain of dependence of (x, y) as the set $D_{(x,y)}$ of points in G_a bounded by the two characteristics through (x, y) and the curve C. From (8) it is clear that the value of the solution at (x, y) depends on the Cauchy data and f in $D_{(x,y)}$.

Let A be a connected subset of the curve C. The *domain of influence* of A defined to be the set $I(A)$ of all (x, y) in G_a for which $D_{(x,y)} \cap A$ is non-empty. Then the data given along A *influences* the solution in the region $I(A)$. The *domain of determinacy* of A is the set $D(A)$ of points (x, y) in G_a such that $D_{(x,y)} \subset A$. At such points, the value of the solution depends at most on points of A, hence, data along A determines the solution in $D(A)$.

Exercises.

(4.1) Verify the identities $D(A \cap B) = D(A) \cap D(B)$ and $I(A \cup B) = I(A) \cup I(B)$, where A and B are subsets of the curve C.

(4.2) For the Goursat problem (3.3), discuss the domain of dependence at a point and domains of influence and determinancy of the data.

(4.3) Repeat (4.2) for the characteristic Cauchy problem (2.4).

(4.4) Explain why the characteristic Cauchy problem (2.4) and the Goursat problem (3.3) for the equation (14) are well-posed on the first quadrant when the functions a, b, c and F are continuous on $[0, \infty) \times [0, \infty)$.

(4.5) Find the solution of $u_{xy} = 0$ on the region $\{(x, y) : x + y > 0,\ 0 < x - y < 2\}$ such that $u(x, -x) = u_0(x)$, $0 \le x \le 1$, $u(x, x - 2) = u_0(x)$, $1 \le x$, $u_x(x, x) = 0$, $x > 0$ and $u_x(x, -x) = 0$ for $0 < x < 1$. Find the domains of dependence of the solution at points of this region.

(4.6) (a) Show that there is at most one solution u of (14) on the set $\{(x, y) : x+y > 0,\ 0 < x - y < 2\}$ such that

$$u,\ u_x,\ u_y \text{ are given on } x + y = 0\ ,$$

$$u_x,\ u_y \text{ are given on } x - y = 0\ , \text{ and}$$

$$u \text{ is given on } x - y = 2\ .$$

(b) Find necessary and sufficient compatibility conditions on the data in (a) for the existence of a solution.

5. Riemann's Representation.

In the preceding section we showed that the Cauchy problem for the linear equation (14) is well-posed and, in fact, has a global solution when the data is given on a large region. The result is satisfying to some extent, but not as sharp as having a representation like that obtained in Theorem 1 for the special case of equation (b). We shall obtain an integral representation for solutions of (14). The kernel of this representation can be obtained by solutions of a family of characteristic Cauchy problems for an equation adjoint to (14).

Let the linear part of (14) be denoted by

$$L[u] = u_{xy} + a(x,y)u_x + b(x,y)u_y + c(x,y)u \tag{15}$$

and assume the coefficients a and b are continuously differentiable while c is continuous on the domain of determinancy $D(C)$ of the non-characteristic curve C. Suppose we are given Cauchy data (2′) for the solution u of $L[u] = F(x,y)$ in $D(C)$. We introduce the adjoint L^* of L by the requirement that the quantity $vL[u] - uL^*[v]$ be a divergence expression. By direct calculation we find

$$vL[u] - uL^*[v] = (P)_x + (Q)_y \tag{16}$$

where $P = (1/2)vu_y - (1/2)v_y u + avu$, $Q = (1/2)vu_x - (1/2)v_x u + bvu$, and the adjoint is given by

$$L^*[v] = v_{xy} - (av)_x - (bv)_y + cv \ . \tag{17}$$

Let (ξ,η) be a point in $D(C)$ and denote by $G(\xi,\eta)$ the domain of dependence of (ξ,η). Let $M(\eta)$ and $N(\xi)$ be the points of intersection of C with the characteristics $y = \eta$ and $x = \xi$, respectively. We use the divergence theorem to integrate (16) over $G(\xi,\eta)$ and obtain

$$\iint_{G(\xi,\eta)} (vL[u] - uL^*[v])\,dx\,dy = \int_{\partial G(\xi,\eta)} P\,dy - Q\,dx \ . \tag{18}$$

On those portions of the line integral along the characteristic segments, we integrate by parts to obtain

$$\int_{N(\xi)}^{(\xi,\eta)} vu_y\,dy = uv\Big|_{N(\xi)}^{(\xi,\eta)} - \int_{N(\xi)}^{(\xi,\eta)} v_y u\,dy \ ,$$
$$\int_{M(\eta)}^{(\xi,\eta)} vu_x\,dx = uv\Big|_{M(\eta)}^{(\xi,\eta)} - \int_{M(\eta)}^{(\xi,\eta)} v_x u\,dx \ .$$

Substituting these in (18) gives the identity

$$(19)\qquad u(\xi,\eta)v(\xi,\eta) = (1/2)\,[u(N(\xi)v(N(\xi) + u(M(\eta))v(M(\eta))] + \int_{M(\eta)}^{(\xi,\eta)} u(v_x - bv)\,dx + \int_{N(\xi)}^{(\xi,\eta)} u(v_y - av)\,dy - \int_{M(\eta)}^{N(\xi)} P\,dy - Q\,dx + \iint\limits_{G(\xi,\eta)} (vL[u] - uL^*[v])\,dx\,dy\ .$$

The above will provide a representation for $u(\xi,\eta)$ in terms of Cauchy data along C and the non-homogeneous term F in (14) if v can be chosen so as to satisfy

$$(20.a)\qquad L^*[v] = 0 \ \text{ in } \ G(\xi,\eta)\ ,$$

$$(20.b)\qquad \begin{cases} v_x = bv \ \text{ on } \ y = \eta\ , \\ v_y = av \ \text{ on } \ x = \xi\ , \\ v(\xi,\eta) = 1\ . \end{cases}$$

The conditions in (20.b) are equivalent to

$$(20.b')\qquad \begin{cases} v(x,\eta) = \exp\left\{\int_\xi^x b(s,\eta)\,ds\right\} \\ v(\xi,y) = \exp\left\{\int_\eta^y a(\xi,t)\,dt\right\} \end{cases}$$

so (20) is the usual characteristic Cauchy problem with data on the characteristics $x = \xi$, $y = \eta$.

For each point (ξ,η) in $D(C)$, there is a unique solution of (20) which we denote by $R(x,y;\xi,\eta)$ (see 4.4). Substitute $v(x,y) = R(x,y;\xi,\eta)$ in (19) to obtain the identity

$$(21)\qquad u(\xi,\eta) = (1/2)[u(N)R(N;\xi,\eta) + u(M)R(M;\xi,\eta)] + \int_M^N ((1/2)Ru_x - (1/2)R_x u + bRu)\,dx - ((1/2)Ru_y - (1/2)R_y u + aRu)\,dy + \iint\limits_{G(\xi,\eta)} R(x,y;\xi,\eta)F(x,y)\,dx\,dy\ .$$

The equation (21) is known as *Riemann's representation* for the solution of the Cauchy problem for (6). Such properties of the solution as its continuous dependence on the data are immediate from the explicit formula (21). When the Riemann function R is known, this formula establishes existence of a solution of the Cauchy problem provided, of course, the data satisfies the compatibility condition (3′).

Exercises.

(5.1) Show that (7) is a special case of (21).

(5.2) Let $L[u] = u_{xy} + au$, where $a > 0$ is constant.

(a) Show L is self-adjoint: $L = L^*$

(b) Show that the Riemann function is given by $R(x, y; \xi, \eta) = f((x-\xi)(y-\eta))$, where f satisfies $xf''(x) + f'(x) + af(x) = 0$, $f(0) = 1$, $f'(0) = 0$.

(c) Show that the above equations become $tf'' + f' + tf = 0$ under the change of variables $t = 2\sqrt{as}$. This is Bessel's equation of order zero, so that $f(s) = J_0(2\sqrt{as})$.

6. The Wave Equation.

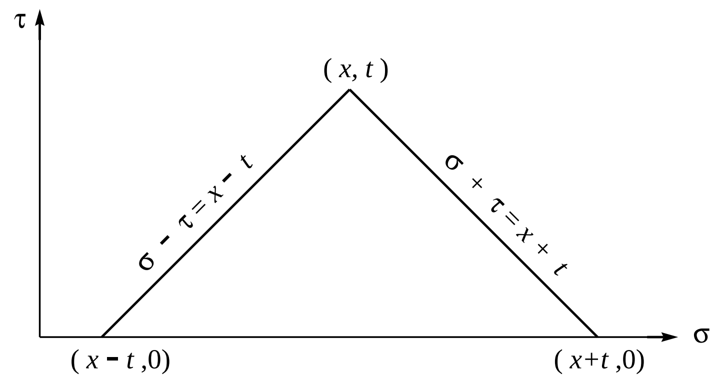
In this and the next two sections we shall be concerned with the wave equation. First, we obtain the D'Alembert formula for the one dimensional wave equation. This follows from corresponding results of Theorem 1 by the usual change of variable, but we prefer to present the straightforward computations so as to make them independent of our preceding work. (The importance of the wave equation certainly justifies this inefficiency in our discussion.) Then we discuss the wave equation in space of dimension ≥ 2.

Let (x, t) be a point with $t > 0$ and suppose u is a solution of the one-dimensional *wave equation*

$$u_{tt} - u_{xx} = F(x, t) \tag{22}$$

in the region bounded by the two characteristics through (x, t) and x-axis. We may use the divergence theorem to integrate (22) over this region and obtain

$$\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} F(\sigma, \tau)\, d\sigma\, d\tau = -\oint u_x\, d\tau + u_t\, d\sigma$$



The indicated line integral can be evaluated over the three line segments $\Sigma_1(\tau = 0)$, $\Sigma_2(\sigma + \tau = x + t)$ and $\Sigma_3(\sigma - \tau = x - t)$ with their induced orientation

$$\int_{\Sigma_1} u_x\, d\tau + u_t\, d\sigma = \int_{x-t}^{x+t} u_t(\sigma, 0)\, d\sigma$$

$$\int_{\Sigma_2} u_x(-d\sigma) + u_t(-d\tau) = -\int_{\Sigma_2} du = u(x+t, 0) - u(x,t) \ ,$$

$$\int_{\Sigma_3} u_x\, d\sigma + u_t\, d\tau = \int_{\Sigma_3} du = u(x-t, 0) - u(x,t) \ .$$

This gives us the *D'Alembert representation*

$$(23) \quad u(x,t) = (1/2)(u(x+t,0) + u(x-t,0)) + (1/2)\int_{x-t}^{x+t} u_t(\sigma,0)\, d\sigma + (1/2)\int_0^t \int_{x-(t-\tau)}^{x+(t-\tau)} F(\sigma,\tau)\, d\sigma\, d\tau$$

for a solution of (22) in terms of Cauchy data along the x-axis. Similar formulae can be obtained which represent solutions of (22) in terms of u (or u_x) along nowhere characteristic curves $\sigma = g(\tau)$. (See Exercises.)

The homogeneous analogue of (22) in three space dimensions is the wave equation

$$(24.a) \qquad u_{tt} - (u_{xx} + u_{yy} + u_{zz}) = 0 \ ,$$

and the Cauchy problem asks for a solution with

$$(24.b) \qquad u(x,y,z,0) = \psi(x,y,z) \quad , \quad u_t(x,y,z,0) = \varphi(x,y,z)$$

given in some region. If we could solve the Cauchy problem (24) with $\psi \equiv 0$ and obtain a solution $u^{(\varphi)}$ then we could likewise solve the general problem, for the function $u_t^{(\psi)}$ satisfies (24) with $\varphi \equiv 0$. Thus a general solution of (24) could be given in the form

$$u \equiv u^{(\varphi)} + u_t^{(\psi)} \ .$$

This shows that we may consider the special case of (24) with $\psi \equiv 0$.

Suppose we seek a representation for solutions u of (24) with $\psi \equiv 0$. First we consider the special case in which u depends only on r and t, where r is the distance from some point $Q = (x_0, y_0, z_0)$ in $\mathbb{R}^3$. Then we have the computations

$$\frac{\partial r}{\partial x} = \frac{x - x_0}{r} \ , \quad \frac{\partial^2 r}{\partial x^2} = \frac{1}{r} - \frac{(x-x_0)^2}{r^3}$$

which give the Chain rule

$$\frac{\partial^2 u}{\partial x^2} = \frac{(x-x_0)^2}{r^2}\,\frac{\partial^2 u}{\partial r^2} + \left(\frac{1}{r} - \frac{(x-x_0)^2}{r^3}\right)\frac{\partial u}{\partial r} \ .$$

This and similar computations for $\partial^2/\partial y^2$ and $\partial^2/\partial z^2$ give the identity

$$u_{xx} + u_{yy} + u_{zz} = u_{rr} + \left(\frac{2}{r}\right) u_r = \frac{1}{r}(ru)_{rr} \ .$$

Hence, the solution of the special type we are considering satisfies the equation

$$(ru)_{tt} - (ru)_{rr} = 0 \ , \quad u = u(r,t) \ ,$$

and our three-dimensional Cauchy problem (24) is equivalent to the one-dimensional problem

(25) $$v_{tt} - v_{rr} = 0 \ , \quad v(r,0) = 0 \ , \quad v_t(r,0) = r\varphi(r) \ ,$$

where $v(r,t) = ru(r,t)$. We know that (25) has a unique solution in the region where $0 < |t| < r$, and it is given by the D'Alembert formula as

$$v(r,t) = (1/2)\int_{r-t}^{r+t} \sigma\,\varphi(\sigma)\,d\sigma \ .$$

Finally, if we extend φ for negative values of the argument as an even function ($\varphi(-\sigma) = \varphi(\sigma)$) the integrand above is odd ($\varphi(-\sigma)(-\sigma) = -\varphi(\sigma)\sigma$) so we obtain the representation

$$u(r,t) = \frac{1}{2r}\int_{|r-t|}^{r+t} \sigma\varphi(\sigma)\,d\sigma$$

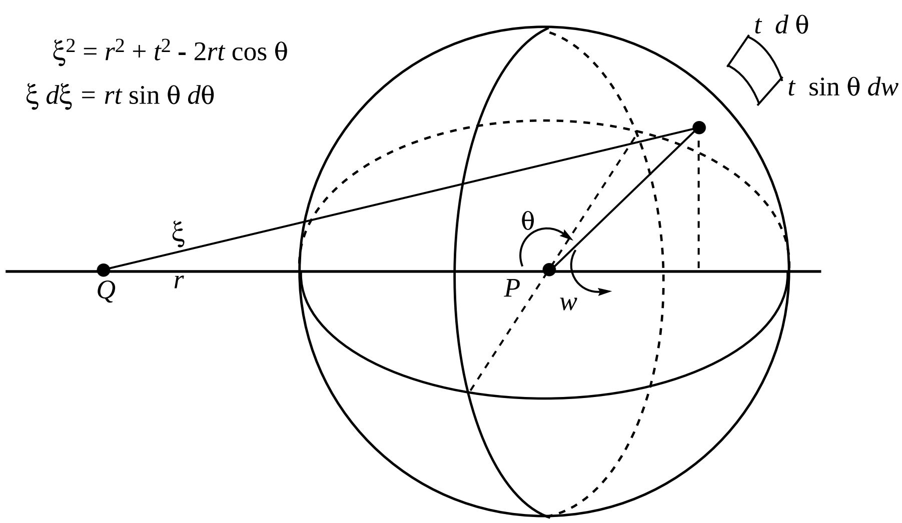
for a solution of (24) (with $\psi \equiv 0$) which depends only on the distance r from some point and the variable t.

Let S_t denote the sphere of radius $t > 0$ centered at $P \equiv (x, y, z) \in \mathbb{R}^3$. The *spherical mean* or average value of the function φ on S_t is given by

$$\overline{\varphi}(S_t) = \frac{1}{4\pi t^2} \iint\limits_{S_t} \varphi \, dS \ .$$

Since φ was assumed to depend only on the distance from Q, this spherical mean can be evaluated by the computation

$$\begin{aligned} \overline{\varphi}(S_t) &= \frac{1}{4\pi t^2} \int_0^{2\pi} \int_0^{\pi} \varphi t^2 \sin\theta \, d\theta \, dw \\ &= \frac{1}{2} \int_0^{\pi} \varphi \sin\theta \, d\theta = \frac{1}{2} \int_{|r-t|}^{r+t} \varphi \frac{\xi \, d\xi}{rt} = \frac{1}{t} u(r, t) \end{aligned}$$



Hence, we obtain the representation

$$u(P, t) = t\overline{\varphi}(S_t) \tag{26}$$

for solutions of (24) which depend only on the distance from Q. The formula (26) states that the value of the solution at (P, t) is the *mean value* of the initial value, φ, over the sphere of radius t at P multiplied by t. This statement is independent of Q and suggests we look for a solution of (24) in the form

$$u(x, y, z, t) \equiv \frac{1}{4\pi t} \iint\limits_{S_t(x,y,z)} \varphi \, dS \tag{27}$$

where $S_t(x, y, z)$ is the sphere of radius t and center at (x, y, z). In fact, we have the

Theorem 3. *If φ is twice continuously differentiable in the sphere $x^2+y^2+z^2 < r_0^2$, then the function defined by* (26) *is a solution of the Cauchy problem* (24) *(with $\psi \equiv 0$) in the cone $t^2 < x^2+y^2+z^2 < r_0^2$.*

Proof. First note that (27) may be expressed by either of

$$u(x,y,z,t) = \frac{1}{4\pi t} \iint\limits_{S_1(0)} \varphi(x+n_1 t, y+n_2 t, z+n_3 t)\, dS$$
$$= \frac{t}{4\pi} \iint \varphi(x+n_1 t, y+n_2 t, z+n_3 t)\, d\Omega$$

where dS denotes surface area, $d\Omega$ the polar angle and (n_1,n_2,n_3) a point on the unit sphere. It follows from the smoothness of φ that we may differentiate the above and obtain

$$\Delta_3 u = \frac{t}{4\pi} \iint \Delta_3 \varphi \, d\Omega \ , \tag{28}$$

where Δ_3 denotes the Laplacian operator $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. Similarly, we have

$$\frac{\partial u(x,y,z,t)}{\partial t} = \frac{1}{4\pi} \iint \varphi(x+n_1 t, y+n_2 t, z+n_3 t)\, d\Omega + \frac{t}{4\pi} \iint \nabla\varphi \cdot n \, d\Omega$$

where $n = (n_1,n_2,n_3)$ and $\nabla\varphi = (\varphi_x, \varphi_y, \varphi_z)$ is the gradient of φ. The second term can be expressed as a surface integral over the sphere and then by the divergence theorem as a volume integral,

$$\frac{t}{4\pi} \iint \nabla\varphi \cdot n \, d\Omega = \frac{1}{4\pi t} \iint \nabla\varphi \cdot n \, dS \tag{29}$$
$$= \frac{1}{4\pi t} \iiint \Delta_3\varphi \cdot dV \ .$$

By expressing this volume integral as an iterated surface integration over the radius, we obtain

$$\frac{\partial u(x,y,z,t)}{\partial t} = \frac{1}{4\pi} \iint \varphi(x+n_1 t, y+n_2 t, z+n_3 t)\, d\Omega$$
$$+ \frac{1}{4\pi t} \int_0^t \left\{ \iint \Delta_3\varphi(x+n_1 r, y+n_2 r, z+n_3 r) r^2 \, d\Omega \right\} dr \ .$$

We differentiate this identity to obtain

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{4\pi} \iint \nabla\varphi \cdot n\, d\Omega - \frac{1}{4\pi t^2} \int_0^t \left\{ \iint \Delta_3 \varphi r^2 \, d\Omega \right\} dr$$
$$+ \frac{t^2}{4\pi t} \iint \Delta_3\varphi(x+n_1 t, y+n_2 t, z+n_3 t)\, d\Omega \ .$$

But (29) shows that the difference of the first two terms is zero and the third is by (28) just $\Delta_3 u$, so u satisfies the equation (24.a).

To check the Cauchy data, remember that φ is continuous, $\lim_{t\downarrow 0} \overline{\varphi}(S_t) = \varphi(x,y,z)$, so (24.b) follows easily from (26).

Corollary. *If φ is twice continuously differentiable and ψ is three times continuously differentiable in the sphere $x^2+y^2+z^2<r_0^2$, then the function*

$$u(x,y,z,t) = \frac{1}{4\pi t}\iint\limits_{S_t(x,y,z)} \varphi\, dS + \frac{\partial}{\partial t}\left\{\frac{1}{4\pi t}\iint\limits_{S_t(x,y,z)} \psi\, dS\right\} \tag{30}$$

is a solution of the Cauchy problem (24) *in the cone $t^2 < x^2+y^2+z^2 < r_0^2$.*

Note that we have demonstrated only the existence of a solution to (24). We shall show in Section 8 that this is the only problem.

Exercises.

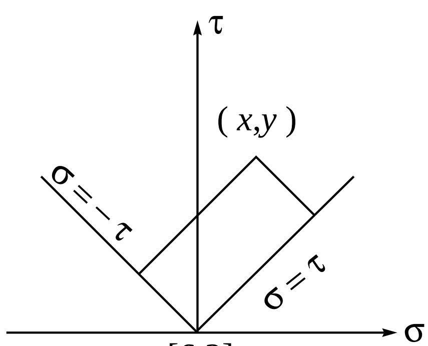
(6.1) Show that (23) is equivalent to (7) under an appropriate change of variable.

(6.2) The characteristic Cauchy Problem for (22) is to find a solution whose values are specified along an interesting pair of characteristics (cf. (2.4)). Use the divergence theorem to obtain the representation

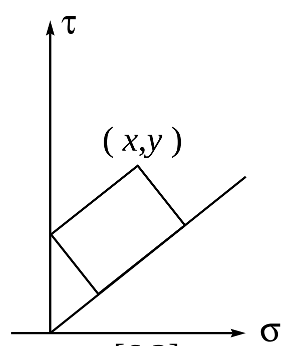
$$u(x,y) = u\left(\frac{x+y}{2}, \frac{x+y}{2}\right) + u\left(\frac{x-y}{2}, \frac{x-y}{2}\right) - u(0,0) - \frac{1}{2}\iint F(\sigma,\tau)\, d\sigma\, d\tau$$

for solutions of (22), where the double integral is taken over the rectangle with sides $\sigma \pm \tau = 0$ and a vertex at (x,y). (See figure.) Show this problem has a unique solution in the upper half-plane, $y>0$.

(6.3) Find a representation for a solution of (22) whose value is specified on the lines $\sigma = 0$ and $\sigma = \tau$ (Goursat problem).



[6.2]



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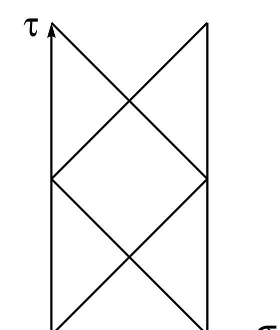


[6.4]

(6.4) Show that the mixed initial-boundary value problem of finding a solution of (22) in the rectangle $\{(x,y) : y > 0,\ 0 < x < 1\}$ for which the solution is specified on the sides $x = 0, 1$, and the solution and its first derivative are specified on the bottom, $y = 0$, is well-posed. (Use a Cauchy problem followed by a sequence of Goursat and Characteristic problems. See Figure.)

(6.5) Solve the Goursat problem (6.3) with u specified on a curve $\sigma = g(\tau)$, $g(0) = 0$, instead of the line $\sigma = 0$. Assume $|g'(\tau)| < 1$. Use this to discuss the mixed initial boundary value problem of finding a solution of (22) in the region $g_1(y) < x < g_2(y)$, $y > 0$, with the conditions

$$u(x,y) = \varphi(x)\ ,\quad u_y(x,0) = \psi(x)\ ,\quad g_1(0) < x < g_2(0)\ ;$$
$$u(g_1(y),y) = f_1(y)\ ,\quad u(g_2(y),y) = f_2(y)\ ,\quad y > 0\ .$$

We assume $g_1(y) < g_2(y)$, $|g_1'(y)| < 1$ and $|g_2'(y)| < 1$ for all $y > 0$.

7. The Wave Equation in 2-dimensions.

An attempt to obtain results analogous to Theorem 3 for the two-dimensional wave equation will necessarily fail. In fact, we shall show in this section that for the two-dimensional problem, the domain of influence of Cauchy data is very different from that in three-dimensions, and representations for solutions must exhibit this.

Before looking at the situation in two-dimensions, suppose we try to duplicate the discussion above which led to (25) and hence (26). Letting Δ_n denote the Laplace operator in the variables $(x_1, x_2, \ldots, x_n)$,

$$\Delta_n = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}\ ,$$

we obtain by the Chain rule the identity

$$\begin{aligned}\Delta_n &= u_{rr} + ((n-1)/r)u_r \\ &= \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial u}{\partial r}\right)\ .\end{aligned}$$

If $v = r^m u$, then

$$v_{rr} = r^m u_{rr} + 2mr^{m-1}u_r + m(m-1)r^{m-2}u\ ,$$

so we are led to choose $2m = n - 1$. Thus, if $v = r^{(n-1)/2}u$, then $\Delta_n u = u_{tt}$ if and only if

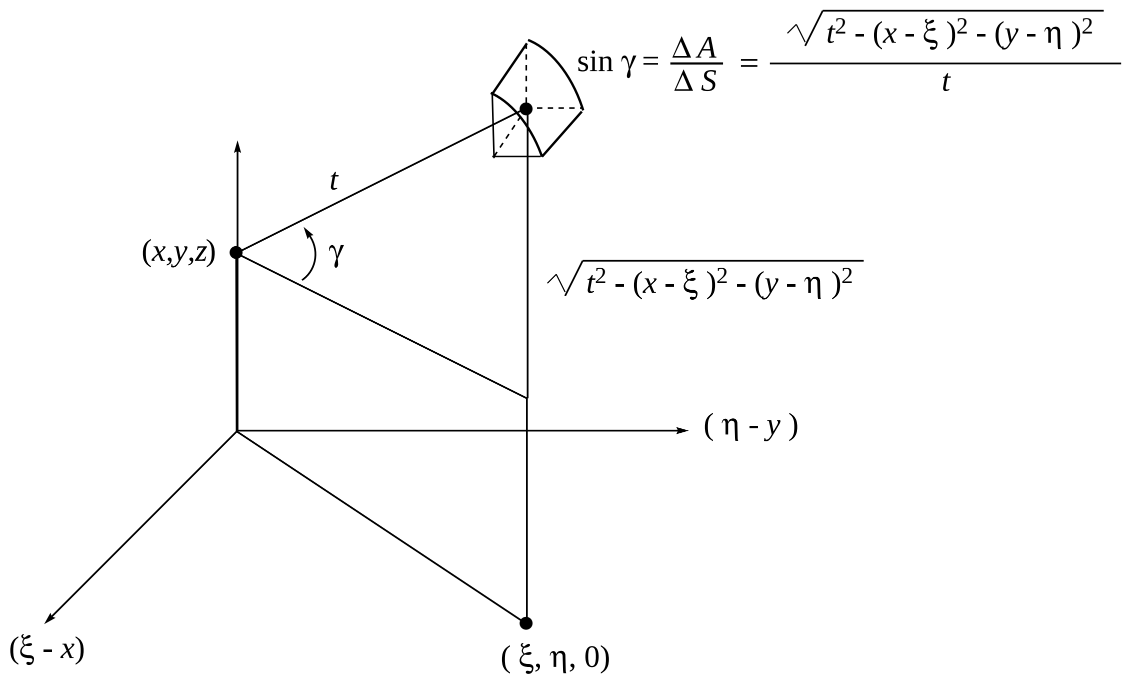
$$v_{rr} = v_{tt} + \frac{(n-1)(n-3)}{4r^2} v \ .$$

If n is odd, the discussion of Section 6 can be continued. The equation above shows that this is particularly simple only if $n = 1$ or 3. For n even, the attempt must fail, as the following discussion will show.

Consider now the function defined by (27) and suppose that the function φ is independent of z: $\varphi(x,y,z) = \varphi(x,y)$. Then $u(x,y,z)$ is independent of z. Hence, we can write $u(x,y,z,t) = u(x,y,t)$, and this function satisfies the two-dimensional Cauchy problem

$$\text{(31)} \qquad u_{xx} + u_{yy} - u_{tt} = 0 \ , \quad u(x,y,0) = 0 \ , \quad u_t(x,y,0) = \varphi(x,y) \ .$$

That is, we can obtain a solution of the Cauchy problem for the two-dimensional wave equation by regarding it as a special case of the three-dimensional problem. In order to obtain a representation like (27) but without the extra dimension, we denote the disc of radius t and center (x,y) by $D_t(x,y)$ and note that the integral in (27) taken over the surface $S_t(x,y,z)$ can be written as the sum of two integrals over $D_t(x,y)$, the projection of $S_t(x,y,z)$ onto the xy-plane. The ratio of surface area to the corresponding area of its projection onto $D_t(x,y)$ at a point $(\xi,\eta) \in D_t(x,y)$ is given by $t/(t^2 - (x-\xi)^2 - (y-\eta)^2)^{1/2}$, as the figure below indicates, and both the upper and lower hemispheres project onto the disk. Hence we obtain the representation



(32) $$u(x,y,t) = \frac{1}{2\pi} \int_{D_t(x,y)} \int \frac{\varphi(\xi,\eta)\, d\xi\, d\eta}{\sqrt{t^2 - (x-\xi)^2 - (y-\eta)^2}}$$

for the solution of (31) given by (27). (Note that a direct analogue of (27) in two-dimensions would be an integral around the circumference of the disk $D_t(x,y)$ instead of that given in (32).)

We compare (27) and (32). In (27), the value of the solution at a point depends only on the data on the boundary of a certain 3-sphere of radius t. Hence, any data which is non-zero only on a bounded set (φ is of bounded extent or has compact support) leads to a solution which may be non-zero at any given point for only some finite interval of time. By contrast, in (32) the solution at a point depends on the data everywhere in the disk (2-sphere), so if φ is non-zero on a bounded set, the solution may be non-zero at a given point for all t beyond some t_1. That the dependence of the solution of the wave equation on the initial data is different in two and three dimensions is known as Huygen's principle. In fact, the situation for $n = 2$ is indicative of the results in space of even dimension while that for $n = 3$ is typical for space of odd dimension at least 3. The D'Alembert formula (23) shows that the case $n = 1$ is exceptional and exhibits the phenomena of both odd and even dimension.

Exercises.

(7.1) For each of the Cauchy problems (24) and (31), find the domain of influence (Section 4) of a point on the initial plane or line, respectively.

(7.2) What are the analogues of (27) and (32) in one-dimension? Show that (23) contains both.

(7.3) Verify all the details in the preceding discussion which led to (31).

8. Energy Integrals.

In this final section we shall obtain uniqueness results for certain problems for the wave equation. The technique we use is that of constructing certain integrals containing the solution (or its derivatives) and then showing that these integrals vanish. This property of the integral will then imply the desired uniqueness result. We first consider Cauchy and mixed problems for the wave equation in one dimension, where our results are already rather complete. All computations are easy here, so this is a good setting in which to illustrate the technique. We next give a uniqueness result for the Cauchy problem in two dimensions. (The same technique applies to the corresponding three-dimensional problem but we leave the details as an exercise.) Finally, we demonstrate some uniqueness results for mixed problems for the n-dimensional wave equation.

First we remark that in all *linear* problems, to show that there is at most one solution to the problem with a given set of data is equivalent to showing that the zero function is the only solution to the problem with null data. This follows easily by considering the difference of two solutions to the problem. Thus, we shall assume in each of the following problems that we have a solution with null data and then attempt to show that this solution necessarily vanishes on some (maximal) set.

Our first example is the Cauchy problem

$$
\begin{aligned}
&u_{xx} = u_{tt} \ , \quad a(t) < x < b(t) \ , \quad t > 0 \ , \\
&u(x,0) = u_t(x,0) = 0 \ , \quad a(0) \le x \le b(0) \ .
\end{aligned}
$$

We seek minimal requirements on $\{a(t), b(t)\}$ to assure $u \equiv 0$ in the region $a(t) < x < b(t)$, $t > 0$. So, define the function

$$E(t) = (1/2) \int_{a(t)}^{b(t)} (u_x^2(s,t) + u_t^2(s,t))\, ds \ , \qquad t > 0 \ .$$

(Assume, hereafter, that all functions are sufficiently smooth for the indicated computations to be justified.) From the Leibnitz rule we obtain

$$E'(t) = (1/2)\left\{\left[u_x^2(b(t),t) + u_t^2(b(t),t)\right]b'(t) - \left[u_x^2(a(t),t) + u_t^2(a(t),t)\right]a'(t)\right\} + \int_{a(t)}^{b(t)} (u_x \cdot u_{xt} + u_t \cdot u_{tt})\,ds \ .$$

Since u satisfies the wave equation, the integrand above is just $(\partial/\partial x)(u_x \cdot u_t)$, so we evaluate the integral and obtain

$$E'(t) = (1/2)\{\left[u_x^2(b(t),t) + u_t^2(b(t),t)\right]b'(t) - \left[u_x^2(a(t),t) + u_t^2(a(t),t)\right]a'(t)\} + u_x(b(t),t)u_t(b(t),t) - u_x(a(t),t)u_t(a(t),t) \ .$$

We want to find conditions on the problem (e.g., on $a(t)$ and $b(t)$) which imply $E'(t) \le 0$. (Then we would have $0 \le E(t) \le E(0)$, thus $E(t) \equiv 0$, and this would imply $u \equiv 0$.) Our first such set of conditions is that

$$b'(t) \le -1 \ , \quad a'(t) \ge 1 \ .$$

Then we have

$$E'(t) \le -(1/2)\left\{[u_x(b(t),t) - u_t(b(t),t)]^2 + [u_x(a(t),t) - u_t(a(t),t)]^2\right\}$$

and thus $E'(t) \le 0$, and hence

$$u(x,t) = 0 \ \text{ for } \ a(t) \le x \le b(t) \ , \qquad t \ge 0 \ .$$

This result is the strongest when the region between $a(\cdot)$ and $b(\cdot)$ is maximal, and this occurs when $a(t) = a(0) + t$, $b(t) = b(0) - t$. The vanishing of Cauchy data on the interval $(a(0), b(0))$ implies the vanishing of the solution in the region bounded by the indicated *characteristic* lines through the endpoints. (Of course, this result was anticipated.)

To obtain another situation in which uniqueness occurs, suppose $a(\cdot)$ and $b(\cdot)$ are constants, hence the quantity $\{\dots\}$ in the computation of $E'(t)$ vanishes. Then we have $E'(t) \le 0$ if we require, e.g., that u satisfy one of the boundary conditions $u(s,t) = 0$ or $u_x(s,t) = 0$ for $t > 0$ at both of the endpoints, $s = a,$, $s = b$. Then we have $E'(t) \equiv 0$, so $E(t) = E(0) = 0$ and the solution necessarily vanishes on the cylinder $(a,b) \times (0,\infty)$. This same argument leads to uniqueness of solutions to a variety of mixed initial-boundary value problems for the wave equation in one-dimension. For example, at the right endpoint it suffices to require

$$\rho\, u_{tt}(b,t) + u_x(b,t) + +k\, u_t(b,t) + h\, u(b,t) = 0$$

where the coefficients ρ, k, h are all non-negative.

We consider now the Cauchy problem in 2-space. For each $t \geq 0$ we define the truncated cone

$$C(t) = \{(x,y,\tau) : x^2 + y^2 \leq (1-\tau)^2 \ ,\ 0 \leq \tau \leq t\} \ .$$

Suppose u is a solution of the wave equation

$$u_{xx} + u_{yy} = u_{tt}$$

in $C(t)$ and define for $t \geq 0$

$$E(t) = (1/2) \int_{x^2+y^2 \leq (1-t)^2} \int [u_x^2(\xi,\eta,t) + u_y^2(\xi,\eta,t) + u_t^2(\xi,\eta,t)]\, d\xi\, d\eta \ .$$

Since u satisfies the wave equation, the integrand in the first integral below vanishes identically and we can use the divergence theorem to obtain

$$\begin{aligned} 0 &\equiv \iiint_{C(t)} [(u_x u_t)_x + (u_y u_t)_y - (1/2)(u_x^2 + u_y^2 + u_t^2)_t]\, dV \\ &= \iint_{\partial C(t)} [(u_x u_t)\nu_1 + (u_y u_t)\nu_2 - (1/2)(u_x^2 + u_y^2 + u_t^2)\nu_3]\, dS \ . \end{aligned}$$

Here (ν_1, ν_2, ν_3) denotes the unit outward normal at each point of the boundary $\partial C(t)$ of $C(t)$. But this normal is just $(0,0,1)$ on the top and $(0,0,-1)$ on the bottom of $C(t)$, so we obtain

$$E(t) - E(0) = \iint [\ldots]\, dS \ ,$$

where the indicated double integral is taken over the sides of $C(t)$. But on the sides of $C(t)$ we have $\nu_3 = (\nu_1^2 + \nu_2^2)^{1/2}$, so this can be written

$$\begin{aligned} E(t) - E(0) &= \iint (1/2\nu_3)[2(u_x\nu_3)(u_t\nu_1) + 2(u_y\nu_3)(u_t\nu_2) - (u_x^2 + u_y^2 + u_y^2)\nu_3^2]\, dS \\ &= -\iint (1/2\nu_3)[(\nu_3 u_x - \nu_1 u_t)^2 + (\nu_3 u_y - \nu_2 u_t)^2]\, dS \ . \end{aligned}$$

The estimate $E(t) \leq E(0)$ follows from this, hence, a solution with zero Cauchy data on the base $C(0)$ must vanish on the cone $C(1)$.

Finally, we consider mixed initial-boundary value problems in n-space. Let G be a normal domain (see Appendix) with a unit outward normal $\nu = (\nu_1, \nu_2, \dots, \nu_n)$ at each point of the boundary ∂G. Let $x = (x_1, x_2, \dots, x_n)$ denote a point in $\mathbb{R}^n$ and suppose that the function $u(x,t)$ satisfies the n-dimensional wave equation $\Delta_n u = u_{tt}$ in $G \times (0, \infty)$, where

$$\Delta_n \equiv \sum_{j=1}^{n} \frac{\partial^2}{\partial^2 x_j^2}$$

is the Laplace operator in $\mathbb{R}^n$. Define the energy integral

$$E(t) = (1/2) \int_G \left\{ \sum_{j=1}^{n} u_{x_j}^2 + u_t^2 \right\} dx$$

for $t \geq 0$. Then we have by the divergence theorem

$$\begin{aligned} E'(t) &= \int_G \left[\sum_{j=1}^{n} u_{x_j} u_{x_j t} + u_t \Delta_n u \right] dx \\ &= \int_G \left[\sum_{j=1}^{n} (u_{x_j} u_t)_{x_j} \right] dx \\ &= \int_{\partial G} \sum_{j=1}^{n} u_{x_j} u_t \nu_j \, ds \ . \end{aligned}$$

We can write this in the form

$$E'(t) = \int_{\partial G} u_t \cdot \frac{\partial u}{\partial \nu} \, ds$$

where $\frac{\partial u}{\partial \nu} = \sum_{j=1}^{n} (\partial u / \partial x_j) \nu_j$ denotes the directional derivative of u along the normal. Hence, if we specify either of the conditions $u = 0$ or $\frac{\partial u}{\partial \nu} = 0$ at each point of $\partial G \times (0, \infty)$, then $E'(t) = 0$ and null Cauchy data on $G \times \{0\}$ will then give $u \equiv 0$ on $G \times (0, \infty)$. Other types of boundary conditions lead to uniqueness results and many of these can easily be obtained from above by inspection.

Exercises.

(8.1) Carefully state and prove the uniqueness result corresponding to each situation in the preceding discussion.

(8.2) Show there is at most one solution of the Cauchy problem (24) in the cone

$$C \equiv \{(x, y, z, t) : x^2 + y^2 + z^2 \leq (1-t)^2 \ , \ 0 \leq t \leq 1\} \ .$$

(8.3) Show there is at most one $u(x,t)$ defined for $(x,t) \in G \times (0, \infty)$, G normal in $\mathbb{R}^n$, which satisfies

$$\begin{aligned} &\Delta_n u(x,t) = u_{tt}(x,t) + f(x,t) \ , && (x,t) \in G \times (0,\infty) \ , \\ &u(x,0) = \varphi(x) \ , \quad u_t(x,0) = \psi(x) \ , && x \in G \ , \\ &a\frac{\partial u(s,t)}{\partial \nu} + bu(s,t) = g(s) \ , && (s,t) \in \partial G \times (0,\infty) \ , \end{aligned}$$

where a and b are non-negative real numbers and not both are zero.

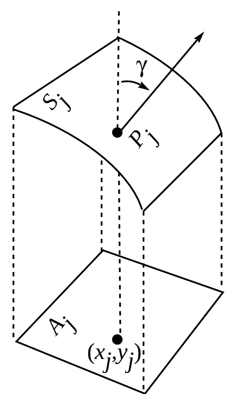
Appendix I. The Divergence Theorem.

We assume the reader is familiar with the Riemann integral of bounded functions on compact sets in $\mathbb{R}^n$, where (at least) $n = 1, 2, 3$. These are frequently called single, double, and triple integrals, respectively.

The notions we wish to discuss here depend on the concept of surface integrals. We shall restrict attention to surfaces which are (locally) determined by an equation

$$z = \varphi(x, y) \ , \qquad (x, y) \in A$$

where A is a bounded open connected set in the plane $\mathbb{R}^2$ and φ is $C^1(\overline{A})$. At each point on S we have a normal vector $(-\varphi_x, -\varphi_y, 1)$. If S is partitioned into elements S_j whose projections A_j on A are rectangles, then the ratio of the area of each S_j



to the area of A_j is just the magnitude of the normal vector above, $(\varphi_x^2 + \varphi_y^2 + 1)^{1/2}$, evaluated at some point in A_j. This quantity is also given by the secant of γ, where γ is the angle between the normal vector and the z-axis. Thus, the integral of the continuous function $f : S \to \mathbb{R}$ over the surface S is given by

(1) $$\iint_S f \, dS \equiv \iint_A f \cdot \sec(\gamma) \, dx \, dy \ .$$

This integral is obtained as the usual limit of Riemann sums of the form

$$\sum_{j=1}^{k} f(P_j) \Delta S_j$$

where $\Delta S_j = \sec(\gamma(P_j)) \cdot \Delta A_j$ is the area of S_j, ΔA_j is the area of A_j, and $P_j = (x_j, y_j, \varphi(x_j, y_j))$ is the point in S_j corresponding to (x_j, y_j) in A.

Surface integrals over the boundary of a region in $\mathbb{R}^3$ arise when we compute the (triple) integral of a derivative over such a region. This is essentially the Divergence

Theorem in $\mathbb{R}^3$. Corresponding results hold in $\mathbb{R}^n$ and the reader is encouraged to write out the details (at least for $\mathbb{R}^2$) as he follows the discussion below.

We consider first an important but special situation. Let A denote a compact set in $\mathbb{R}^2$ whose boundary ∂A is a simple closed curve which is smooth (determined by continuously differentiable functions). Let φ_1 and φ_2 be continuously differentiable in a neighborhood of A and assume $\varphi_1(x,y) < \varphi_2(x,y)$ at all (x,y) in the interior of A. Let

$$S_j = \{(x,y,z) : (x,y) \in A\ ,\ z = \varphi_j(x,y)\} \ \text{ for } \ j - 1, 2\ ,$$

and let

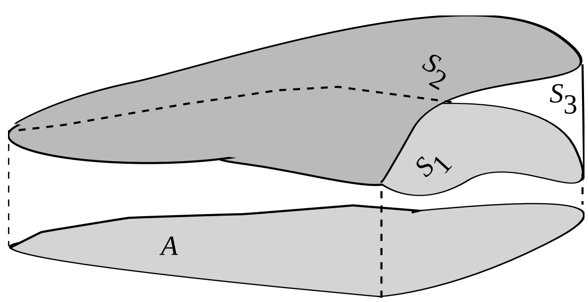
$$S_3 = \{(x,y,z) : (x,y) \in \partial A\ ,\ \varphi_1(x,y) < z < \varphi_2(x,y)\}\ .$$

(S_3 may be empty.) Then the region in $\mathbb{R}^3$ given by

$$G \equiv \{(x,y,z) : (x,y) \in A\ ,\ \varphi_1(x,y) \le z \le \varphi_2(x,y)\}$$

is called *z-standard* and its boundary is

(2) $$\partial G = S_1 \cup S_2 \cup S_3\ .$$



Suppose we are given a function f continuously differentiable in a neighborhood of G. We can integrate f_z over G by using an iterated integral and then the Fundamental Theorem of Calculus to obtain

$$\begin{aligned}\iiint\limits_G f_z\, dV &= \iint\limits_A \left\{\int_{\varphi_1(x,y)}^{\varphi_2(x,y)} f_z\, dZ\right\} dx\, dy \\ &= \iint\limits_A f(x,y,\varphi_2(x,y))\, dA - \iint\limits_A f(x,y,\varphi_1(x,y))\, dA\ .\end{aligned}$$

¿From (1) we have

$$\iiint\limits_G f_z\,dV = \iint\limits_{S_2} f\cdot\cos(\gamma)\,dS - \iint\limits_{S_1} f\cdot\cos\gamma\,dS\ .$$

Letting $\nu = (\nu_1, \nu_2, \nu_3)$ denote the unit outward normal at each point on ∂G, we have $\nu_3 = \cos(\gamma)$ on S_2 and $\nu_3 = -\cos(\gamma)$ on S_1, so the above becomes

(3) $$\iiint\limits_G f_z\,dV = \iint\limits_{\partial G} f\nu_3\,dS\ .$$

This is immediate if we use (2) to write the surface over ∂G as the sum of integrals over S_1, S_2 and S_3, notice that $\nu_3 = 0$ on S_3.

We can define x-standard and y-standard regions in $\mathbb{R}^3$ in an obvious way. Finally, we call a region *standard* if it is simultaneously x-standard, y-standard and z-standard. If P, Q and R are continuously differentiable functions in a neighborhood of the standard region G in $\mathbb{R}^3$, it follows that we have

(4) $$\iiint\limits_G (P_x + Q_y + R_z)\,dV = \iint\limits_{\partial G} (P\nu_1 + Q\nu_2 + R\nu_3)\,dS$$

The triple of functions (P, Q, R) is called a vector field on G. The quantity in the first integral in (4) is called the *divergence* of (P, Q, R). The second integrand can be written as a scalar product of the vectors (P, Q, R) and ν at each point, and (4) can be expressed in the vector form

$$\iiint\limits_G \nabla\cdot F\,dV = \iint\limits_{\partial G} F\cdot\nu\,dS\ ,$$

where $F = (P, Q, R)$ is the vector field with divergence

$$\nabla\cdot F = P_x + Q_y + R_z$$

The hypotheses on the region G are unnecessarily restrictive. If G can be cut up by a finite number of planes into a collection of standard regions,

$$G = G_1 \cup G_2 \cup \cdots \cup G_m$$

then we can apply (4) to each G_k. When the resulting equations are added, the left sides add to the left side of (4). The surface integrals add to the right side of (4) plus sums of pairs of integrals over the plane areas which are of the form $\partial G_j \cap \partial G_k$, where G_j and G_k are adjacent. Their respective outward normals are the negatives of each other at each point on the interface, so each pair of integrals over a common interface will add to zero. Again we obtain (4).

Define an open connected set G in $\mathbb{R}^n$ to be *normal* if its closure can be obtained as the union of a finite number of regions G_k, $k = 1, 2, \ldots, m$, each of which is x_j-standard in $\mathbb{R}^n$ for every $j = 1, 2, \ldots, n$. (The definition of x_j-standard is an obvious extension of the above.) Such a collection $\{G_k\}$ will be called a *regular partition* of G.

Divergence Theorem. *Let G be a normal domain in $\mathbb{R}^n$. Let P, Q, R be continuous on the closure of G, and let each have bounded and continuous derivatives of first order in the interior of each element of some regular partition of G. Then (4) holds.*

Other refinements and extensions are possible (and profitable), and one can consult Kellog, *Potential Theory* for a classical treatment. (Modern treatments are too numerous to name.)