THE DIFFUSION EQUATION

1. Introduction.

We shall consider some problems for the *diffusion equation*

$$u_t = u_{xx} \tag{1}$$

and its non-homogeneous variations. Since the results on parabolic equations do not depend on the dimension, we shall restrict our attention to this case of one spatial variable.

Let's begin by seeking candidates for a well-posed problem for the diffusion equation. We use the energy integral method to get some elementary uniqueness results. Let the curves $a(t)$, $b(t)$ be given for $0 \leq t \leq T$, and let $u(x,t)$ be a solution of (1) in the region $a(t) < x < b(t)$, $0 < t < T$.

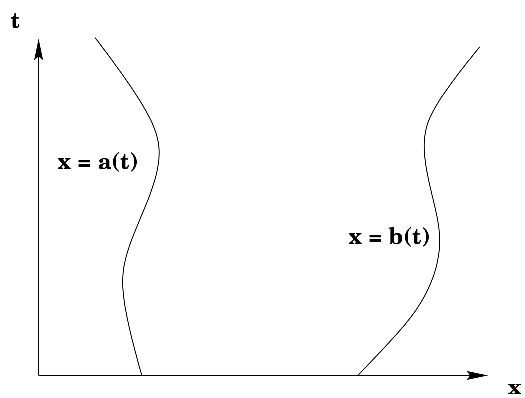

FIGURE 1

Integrate (1) over this interval to obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{a(t)}^{b(t)} u(x,t)\,dx = \\
&\int_{a(t)}^{b(t)} u_{xx}(x,t)\,dx + u(x,t)|_{x=b(t)}b'(t) - u(x,t)|_{x=a(t)}a'(t) = \\
&\qquad +u_x(x,t)|_{a(t)}^{b(t)} + u(x,t)|_{x=b(t)}b'(t) - u(x,t)|_{x=a(t)}a'(t)\,.
\end{aligned}$$

This shows that the total integral of the solution is changed by the addition of *flux* $u_x(x,t)$ at the endpoints or by the increase of the length of the interval. This yields an estimate of the *mean value* of the solution.

In order to get a different and possibly more substantial estimate of a solution, we

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first multiply (1) by u and then integrate over the interval $(a(t),\, b(t))$ and calculate

$$
\begin{aligned}
&\frac{d}{dt}\int_{a(t)}^{b(t)} u^2(x,t)\,dx = \\
&\int_{a(t)}^{b(t)} 2u(x,t)u_t(x,t)\,dx + u^2(x,t)|_{x=b(t)}b'(t) - u^2(x,t)|_{x=a(t)}a'(t) = \\
&\int_{a(t)}^{b(t)} 2u(x,t)u_{xx}(x,t)\,dx + u^2(x,t)|_{x=b(t)}b'(t) - u^2(x,t)|_{x=a(t)}a'(t) = \\
&-2\int_{a(t)}^{b(t)} (u_x(x,t))^2\,dx + 2u(x,t)u_x(x,t)|_{a(t)}^{b(t)} \\
&\qquad + u^2(x,t)|_{x=b(t)}b'(t) - u^2(x,t)|_{x=a(t)}a'(t)\,.
\end{aligned} \tag{2}
$$

Suppose that we are given the boundary conditions, $u(b(t),t) = u(a(t),t) = 0$. Then we have from (2)

$$\frac{d}{dt}\int_{a(t)}^{b(t)} u^2(x,t)\,dx \le 0\,,$$

so we obtain the estimate

$$\int_{a(t)}^{b(t)} u^2(x,t)\,dx \le \int_{a(0)}^{b(0)} u^2(x,0)\,dx\,, \quad a(t)<x<b(t),\, 0<t<T\,. \tag{3}$$

This easily yields the following *uniqueness* result.

Proposition 1. *There is at most one solution $u(x,t)$ in the region $a(t) < x < b(t),\, 0 < t < T$ of the Dirichlet initial-boundary-value problem*

$$u_t = u_{xx} + F(x,t), \quad a(t)<x<b(t), \tag{4.a}$$

$$u(a(t),t) = g_1(t)\,,\ u(b(t),t) = g_2(t), \quad 0<t<T, \tag{4.b}$$

$$u(x,0) = u_0(x)\,, \quad a(0)<x<b(0). \tag{4.c}$$

The preceding argument can be improved substantially to obtain results on the continuous dependence of the solution of (4) on the data. To this end, we first prove the following.

Lemma (Poincaré). *Let φ be a smooth function for $a < x < b$. Then*

$$\int_a^b \varphi^2(x)\,dx \le 4(b-a)^2\int_a^b \Big(\frac{d\varphi(x)}{dx}\Big)^2\,dx + 2(b-a)\varphi^2(b)$$

Proof. Start by evaluating the integral

$$\int_a^b \frac{d}{dx}((x-a)\varphi^2(x))\,dx = \int_a^b (\varphi^2(x))\,dx + \int_a^b (x-a)\frac{d}{dx}(\varphi^2(x))\,dx = (b-a)\varphi^2(b)$$

and then use the inequality $2\alpha\beta \le \frac{1}{2}\alpha^2 + 2\beta^2$ to obtain

$$\int_a^b \varphi^2(x)\,dx \le 2\int_a^b |(x-a)\varphi(x)\frac{d\varphi(x)}{dx}|\,dx + (b-a)\varphi^2(b)$$
$$\le \int_a^b (\frac{1}{2}\varphi^2(x) + 2(b-a)^2|\frac{d\varphi(x)}{dx}|^2)\,dx + (b-a)\varphi^2(b) \quad \square$$

Now let $u(x,t)$ be a solution of (4) with $g_1 = g_2 = 0$. (We shall investigate the dependence on the boundary data below.) Following the estimates (2) and using the preceding Lemma, we obtain

$$\frac{d}{dt}\int_{a(t)}^{b(t)} u^2(x,t)\,dx + \frac{1}{2(b(t)-a(t))^2}\int_{a(t)}^{b(t)} u^2(x,t)\,dx$$
$$\le \varepsilon \int_{a(t)}^{b(t)} u^2(x,t)\,dx + \frac{1}{\varepsilon}\int_{a(t)}^{b(t)} F^2(x,t)\,dx\,.$$

With the choice of $m \equiv \min\{\frac{1}{2(b(t)-a(t))^2} : 0 \le t \le T\}$ and $\varepsilon \equiv \frac{m}{2}$, the above gives

$$\frac{d}{dt}\big(\exp\{\frac{mt}{2}\}\int_{a(t)}^{b(t)} u^2(x,t)\,dx\big)$$
$$\le \frac{2}{m}\exp\{\frac{mt}{2}\}\int_{a(t)}^{b(t)} F^2(x,t)\,dx\,.$$

Continuing as before, we get

$$\int_{a(t)}^{b(t)} u^2(x,t)\,dx \le \exp\{\frac{-mt}{2}\}\int_{a(0)}^{b(0)} u_0^2(x)\,dx$$
(5)
$$+\exp\{\frac{-mt}{2}\}\int_0^t \frac{2}{m}\exp\{\frac{ms}{2}\}\int_{a(s)}^{b(s)} F^2(x,s)\,dx\,ds\,.$$

It follows that the solution depends continuously on F and u_0 in the sense of L^2 norm. Furthermore, if F decays appropriately as t increases, then (5) shows that the solution likewise decays exponentially in the L^2 norm as t increases.

Similar results follow likewise for other boundary conditions. For example, if the boundary points were constant, $a(t) = a,\ b(t) = b$, then we could replace the Dirichlet boundary conditions in (4.b) by the corresponding Neumann or Robin boundary conditions

(4.b')
$$u_x(a(t),t) - h_1\,u(a(t),t) = g_1(t)\,,\ \ u_x(b(t),t) + h_2\,u(b(t),t) = g_2(t), \quad 0 < t < T,$$

with $h_1 \ge 0,\ h_2 \ge 0$.

We illustrate some easy extensions of the preceding estimates. These are based on the multiplication of the equation by an appropriate function of u before integrating over the interval. Thus suppose that u is a solution of the initial-boundary-value problem (4). Let $\sigma(\cdot)$ be a real-valued function which is Lipschitz continuous, monotone, and $\sigma(0) = 0$. Let $\Sigma(\cdot)$ denote its antiderivative with $\Sigma(0) = 0$. Multiply the equation (4.a) by $\sigma(u(x,t))$ and integrate over the interval to obtain

$$
\begin{aligned}
(6)\quad \frac{d}{dt}\int_{a(t)}^{b(t)} \Sigma(u(x,t))\,dx &+ \int_{a(t)}^{b(t)} \sigma'(u(x,t))\,(u_x(x,t))^2\,dx \\
&= \int_{a(t)}^{b(t)} F(x,t)\,\sigma(u(x,t))\,dx + [\sigma(u(x,t))\,u_x(x,t)]\big|_{x=a(t)}^{x=b(t)} \\
&\qquad + [\Sigma(u(b(t),t))]b'(t) - [\Sigma(u(a(t),t))]a'(t)
\end{aligned}
$$

Consider the case of homogeneous boundary conditions, $g_1(t) = g_2(t) = 0$. Take $\sigma(r) \equiv |r|^{p-1} sgn(r)$ where $1 < p < +\infty$ and the *sign function* is given by $sgn(r) = 1$ for $x > 0$, $sgn(0) = 0$, and $sgn(r) = -1$ for $x < 0$. Then in (6) the boundary terms are zero and $\sigma'(u) \geq 0$, so it follows that

$$
\begin{aligned}
\frac{d}{dt}\frac{1}{p}\|u(t)\|^p_{L^p(a(t),b(t))} &\leq \int_{a(t)}^{b(t)} |F(x,t)|\,|u(x,t)|^{p-1}\,dx \\
&\leq \|F(t)\|_{L^p(a(t),b(t))}\|u(t)\|^{p-1}_{L^p(a(t),b(t))}\,.
\end{aligned}
$$

This leads to an explicit estimate on $\|u(t)\|_{L^p(a(t),b(t))}$ by means of the following inequality.

Lemma (Gronwall). *Assume $k \in L^1(0,T)$, $k \geq 0$, $0 < \alpha < 1$, and $w \in L^\infty(0,T)$ satisfies*

$$
w(t) \leq w_0 + \int_a^t k(s)w(s)^\alpha\,ds, \quad a \leq t \leq b.
$$

Then we have

$$
w(t)^{1-\alpha} \leq w_0^{1-\alpha} + \int_a^t k(s)\,ds, \quad a \leq t \leq b.
$$

Proof. Set $G(t) = \varepsilon + w_0 + \int_a^t k(s)w(s)^\alpha\,ds$ with $\varepsilon > 0$ and note that

$$
G'(t) \leq k(t)G(t)^\alpha,
$$

so we obtain

$$
\frac{d}{dt}G(t)^{1-\alpha} \leq (1-\alpha)k(t)\,.
$$

Integrate this over $[a,t]$ and let $\varepsilon \to 0$ to get the desired estimate. □

The preceding estimates extend to the case $p = 1$ without difficulty. One can either let $\sigma(\cdot)$ be a smooth approximation of $sgn(\cdot)$ and then take limits, or just let $p \to 1$ in the above to obtain the corresponding results.

The case $p = +\infty$ can be treated similarly, but we add to it an *order estimate* as follows. Assume that $u(x,t)$ is a solution of (4) with $g_1 \leq k$, $g_2 \leq k$ with $k \geq 0$.

Define $\sigma(r) \equiv (r-k)^+$, where $x^+ \equiv \max\{0, x\}$ denotes the *positive part* of x. Then we have $\Sigma(r) = \frac{1}{2}((r-k)^+)^2$. Then from (6) we get

$$\frac{d}{dt}\frac{1}{2}\|(u(t)-k)^+\|^2_{L^2(a,b)} \leq \int_a^b (F(x,t))^+ \, (u(x,t)-k)^+ \, dx$$
$$\leq \|F(t)^+\|_{L^2(a,b)} \|(u(t)-k)^+\|_{L^2(a,b)} \, .$$

The Gronwall inequality gives

$$\|(u(t)-k)^+\|_{L^2(a,b)} \leq \|(u_0-k)^+\|_{L^2(a,b)} + \int_0^t \|F(t)^+\|_{L^2(a,b)} \, dt \, .$$

From here we obtain the following *maximum principle.*

Proposition 2. *Let $u(x,t)$ be a solution in the region $a(t) < x < b(t)$, $0 < t < T$ of the Dirichlet initial-boundary-value problem (4). If $F(x,t) \leq 0$ in the region, then we have*

$$\max\{u(x,t)\}_{a(t)<x<b(t),0<t<T} \leq \max\{u_0(x), g_1(t), g_2(t), 0\}_{a(t)<x<b(t),0<t<T} \, .$$

Corollary. *If $F(x,t)$, $u_0(x)$, $g_1(t)$, $g_2(t)$ are all non-positive, then we have $u(x,t) \leq 0$ in the region.*

This gives another *uniqueness* result, and it also yields a *comparison principle.* If the data for a pair of initial-value problems is ordered, then the corresponding solutions have that same order. This is particularly useful in obtaining information about solutions from known solutions.

We seek an extension of the preceding maximum principle to cover the case of unbounded regions. Denote the region, its parabolic boundary, and its top by

$$\begin{aligned} G_T =& \{(x,t) : \ a(t) < x < b(t), \ 0 < t < T\} \, , \\ B_T =& \{(x,t) : \ t(x-a(t))(x-b(t)) = 0, \ 0 < t < T\} \, , \\ C_T =& \{(x,T) : \ a(T) < x < b(T)\} \, , \end{aligned}$$

respectively. We assume that $b(\cdot)$ is a *lower semicontinuous* extended real valued function, $b : [0,T] \to (-\infty, +\infty]$. That is, for every $t_0 \in [0,T]$ and $\alpha < b(t_0)$, there is a neighborhood of t_0 such that $\alpha < b(t)$ for all t in that neighborhood. Equivalently, $b^{-1}\{(\alpha, +\infty]\}$ is open in $[0,T]$ for each $\alpha \in \mathbb{R}$. Likewise, assume that $a(\cdot) : [0,T] \to [-\infty, \infty)$ is *upper semicontinuous.* It follows that G_T is open in $\mathbb{R}^2$. In particular, is is sufficient for $a(\cdot)$, $b(\cdot)$ to be continuous at each $t_0 \in [0,T]$ where their value is finite. However we have relaxed the assumptions of smoothness and boundedness of the domain here. We can still obtain the maximum principle here if we add an assumption of boundedness of the solution.

Theorem 1 (Maximum Principle). *Let $u(\cdot,\cdot)$ be an upper-bounded solution of (4) with $F \leq 0$ in G_T. Then $\sup_{G_T} u = \sup_{B_T} u$, i.e., if $u(s,t) \leq M$ for $(s,t) \in B_T$, then it follows that $u(x,t) \leq M$ for $(x,t) \in G_T$.*

Proof. Let $K = \sup\{u(x,t) : \ (x,t) \in G_T\}$ and define $v(x,t) = u(x,t) - \varepsilon(2t + x^2)$ on G_T. Then $v_t - v_{xx} \leq 0$ and

$$v(x,t) \leq K - \varepsilon x^2 \leq M \text{ if } x^2 \geq \frac{K-M}{\varepsilon} \, .$$

Define $G_T^A \equiv \{(x,t) \in G_T : \ |x| \le A \text{ and } t < A\}$. If $A \ge \sqrt{\frac{K-M}{\varepsilon}}$, then $v(s,t) \le M$ for $(s,t) \in B_T^A$, so by the Proposition 2 we get $v(x,t) \le M$ for $(x,t) \in G_T^A$. This holds for all $A \ge \sqrt{\frac{K-M}{\varepsilon}}$, so we have $v(x,t) \le M$ for $(x,t) \in G_T$. But this means that for all $(x,t) \in G_T$ we have $u(x,t) \le M + \varepsilon(2t + x^2)$ for each $\varepsilon > 0$, hence, $u(x,t) \le M$. $\square$

Corollary. *There is at most one bounded solution of*

$$u_t = u_{xx} + F(x,t), \quad (x,t) \in G_T\,, \qquad u(s,t) = g(s,t) \quad (s,t) \in B_T\,.$$

Exercises.

1. Prove the Corollary.

2. Use the technique of Proposition 2 to prove a maximum principle for the two-point boundary-value problem

$$u(x) - u''(x) = F(x), \quad a < x < b\,, \qquad u(a) = b_1,\ u(b) = b_2\,.$$

Hint: Start by subtracting k from both sides of the equation.
Prove a maximum principle for the problem

$$-u''(x) = F(x), \quad a < x < b\,, \qquad u(a) = b_1,\ u(b) = b_2\,.$$

Hint: Use the Poincaré Lemma.

3. Use the preceding technique to prove a maximum principle for the boundary-value problem in $\mathbb{R}^n$,

$$-\Delta_n u(x) = F(x),\ x \in G\,, \quad u(s) = g(s),\ s \in \partial G\,.$$

4. Prove a maximum principle for the initial-boundary-value problem

$$\begin{aligned} \frac{\partial u(x,t)}{\partial t} - \Delta_n u(x,t) &= F(x,t), \quad x \in G\,, \\ u(s,t) &= g(s,t), \quad s \in \partial G\,,\ 0 < t < T, \\ u(x,0) &= u_0(x), \quad x \in G\,. \end{aligned}$$

5. Show that the function $u(x,t) = \frac{x}{t^{\frac{3}{2}}} e^{-\frac{x^2}{4t}}$ satisfies (1) in the upper half-plane, $G = \{(x,t) : \ t > 0\}$. show that for every $x \in \mathbb{R}$, $\lim_{t\to 0} u(x,t) = 0$. Explain why the Corollary implies that $u(\cdot,\cdot)$ cannot be extended continuously to the closure, $\overline{G}$. Show this also directly.

2. The Initial-Value Problem.

As a first step to finding a solution of (1), consider the function $e^{\alpha x+\beta t}$. This is a solution of the diffusion equation (1.1) only if $\alpha^2 = \beta$, so we obtain a family of solutions, $e^{\alpha x+\alpha^2 t}$, $\alpha \in \mathbb{C}$. In order for these to be bounded on $t \geq 0$, we need each α to be pure imaginary, say, $\alpha = \mu i$, and the corresponding solutions, $e^{\pm\mu i x-\mu^2 t}$, $\mu \in \mathbb{R}$. The real parts of these are $e^{-\mu^2 t}\cos(\mu x)$, $\mu \in \mathbb{R}$. Finally, the continuous linear combination

$$u(x,t) = \int_0^\infty e^{-\mu^2 t}\cos(\mu x)\,d\mu$$

of all these solutions with uniform weight is a solution of (1) for $t > 0$, and it plays a special role.

Let's evaluate this integral. Introduce λ by the change of variable $t\mu^2 = \lambda^2$, $d\mu = \frac{1}{\sqrt{t}}d\lambda$ for which

$$u(x,t) = \frac{1}{\sqrt{t}}\int_0^\infty e^{-\lambda^2}\cos(\frac{x}{\sqrt{t}}\lambda)\,d\lambda\,.$$

Define $K(s) \equiv \int_0^\infty e^{-\lambda^2}\cos(s\lambda)\,d\lambda$ and then compute

$$K'(s) = -\int_0^\infty \lambda e^{-\lambda^2}\sin(s\lambda)\,d\lambda = -\int_0^\infty \frac{1}{2}e^{-\lambda^2}s\cos(s\lambda)\,d\lambda = -\frac{s}{2}K(s)\,.$$

Solve this to get $\ln K(s) = -\frac{s^2}{4} + c$, so $K(s) = Me^{-\frac{s^2}{4}}$ for some M. This is found to be

$$M = K(0) = \int_0^\infty e^{-\lambda^2}\,d\lambda = \frac{\sqrt{\pi}}{2}$$

so we have $K(s) = \frac{\sqrt{\pi}}{2}e^{-\frac{s^2}{4}}$ and therefore

$$u(x,t) = \sqrt{\frac{\pi}{4t}}e^{-\frac{x^2}{4t}}, \quad x \in \mathbb{R},\ t > 0\,.$$

After normalizing this function (see (e) below), we get the following.

Definition. The *fundamental solution* of (1.1) is defined to be the function

$$K(x,t) \equiv \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}, \quad x \in \mathbb{R},\ t > 0\,. \tag{1}$$

Lemma 1.

(a.) $K_t(x,t) - K_{xx}(x,t) = 0$.
(b.) $K(x,t) > 0$.
(c.) *If* $t > 0$, $K(x,t) \to 0$ *exponentially as* $|x| \to \infty$, *and the same holds for each derivative of* $K(x,t)$.
(d.) *For each* $\delta > 0$, $\lim_{t\searrow 0} K(x,t) = 0$ *uniformly on* $\{x : |x| \geq \delta\}$.
(e.) $\int_{-\infty}^{\infty} K(x,t)\,dx = 1$.
(f.) $\lim_{t\searrow 0}\int_{|x|\geq\delta} K(x,t)\,dx = 0$ *for each* $\delta > 0$.

Proof. For (a) it suffices to differentiate and check, and (b) is obvious. For (c), note that each derivative introduces a factor which is a rational function of x, t, so the result follows by l'Hopital's rule. For $|x| \geq \delta$, we have $K(x,t) \leq \frac{1}{\sqrt{4\pi t}} e^{-\frac{\delta^2}{4t}}$, so the uniform convergence of (d) follows. For (e) we compute

$$\frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{4t}}\, dx = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\mu^2} \sqrt{4t}\, d\mu = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-\mu^2}\, d\mu = 1\,.$$

A similar calculation gives

$$\int_{|x|\geq\delta} K(x,t)\, dx = \frac{1}{\sqrt{\pi}} \int_{|\mu|\geq \frac{\delta}{\sqrt{4t}}} e^{-\mu^2}\, d\mu \to 0$$

as $t \to 0$, so (f) follows. □

The fundamental solution is used to construct solutions of initial-boundary-value problems for the diffusion equation. We begin with the *initial-value problem.*

Theorem 2. *Let $f(\cdot) : \mathbb{R} \to \mathbb{R}$ be a bounded and continuous function. Define $u(\cdot,\cdot) : \mathbb{R} \times [0,\infty) \to \mathbb{R}$ by*

$$u(x,t) \equiv \begin{cases} \int_{-\infty}^{\infty} K(x-\xi,t) f(\xi)\, d\xi, & t > 0\,, \\ f(x), & t = 0\,. \end{cases} \tag{2}$$

Then $u(\cdot,\cdot)$ is bounded and continuous on $\{(x,t) : t \geq 0\}$, it is infinitely differentiable in $\{(x,t) : t > 0\}$, and it satisfies the initial-value problem

$$u_t(x,t) - u_{xx}(x,t) = 0, \quad x \in \mathbb{R},\ t > 0, \tag{3.a}$$

$$u(x,0) = f(x). \tag{3.b}$$

Proof. Set $M = sup\{|f(x)| : x \in \mathbb{R}\}$ and note that $u(x,t) \leq \int_{-\infty}^{\infty} K(x-\xi,t) M\, d\xi = M$ by Lemma 1.b and Lemma 1.e. From Lemma 1.c it follows that $u(\cdot,\cdot)$ is well defined and infinitely differentiable, and by Lemma 1.a it satisfies (3.a).

It remains only to demonstrate the continuity for $t \searrow 0$. To this end, let $x_0 \in \mathbb{R}$, $\varepsilon > 0$ and A be chosen with $A > |x_0|$. Since $f(\cdot)$ is uniformly continuous on $[-A, A]$, there is a $\delta > 0$ such that $(x_0 - 2\delta, x_0 + 2\delta) \subset [-A, A]$ and that for $x, y \in [-A, A]$, $|x - y| < 2\delta$ we have $|f(x) - f(y)| < \frac{\varepsilon}{2}$. We write

$$\begin{aligned} u(x,t) &= \int_{\eta\in\mathbb{R}} K(\eta,t)\, f(x+\eta)\, d\eta \\ &= \int_{|\eta|\geq\delta} K(\eta,t)\, f(x+\eta)\, d\eta + \int_{|\eta|\leq\delta} K(\eta,t)\, f(x+\eta)\, d\eta \end{aligned}$$

and then use Lemma 1.e to obtain

$$\begin{aligned} &u(x,t) - f(x_0) \\ &= \int_{|\eta|\geq\delta} K(\eta,t)\, (f(x+\eta) - f(x_0))\, d\eta + \int_{|\eta|\leq\delta} K(\eta,t)\, (f(x+\eta) - f(x_0))\, d\eta\,. \end{aligned}$$

If $|x - x_0| < \delta$, $|\eta| \le \delta$, then we have $x + \eta \in [-A, A]$, $|(x+\eta) - x_0| < 2\delta$, so $|f(x+\eta) - f(x_0)| < \frac{\varepsilon}{2}$. Hence, when $|x - x_0| < \delta$ we have

$$\left|\int_{|\eta|\le\delta} K(\eta,t)\,(f(x+\eta) - f(x_0))\,d\eta\right| < \frac{\varepsilon}{2}\,.$$

Also, from Lemma 1.f it follows that there is a $t^* > 0$ for which

$$\left|\int_{|\eta|\ge\delta} K(\eta,t)\,(f(x+\eta) - f(x_0))\,d\eta\right| \le 2M \int_{|\eta|\ge\delta} K(\eta,t)\,d\eta < \frac{\varepsilon}{2}$$

whenever $0 < t < t^*$. In summary, whenever $|x - x_0| < \delta$ and $0 < t < t^*$, we have $|u(x,t) - f(x_0)| < \varepsilon$. □

Corollary 1. *The convergence* $\lim_{t \searrow 0} u(x,t) = f(x)$ *is uniform on compact subsets of* $\mathbb{R}$.

We consider next a case in which the data is *not continuous*, specifically, the case of the Heaviside function, $f(x) = H(x)$ where $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x \le 0$. With this initial data the formula (2) defines the function

$$\begin{aligned}
u(x,t) =& \int_0^\infty K(x-\xi,t)\,d\xi \\
=& \frac{1}{\sqrt{4\pi t}} \int_0^\infty e^{-\frac{(x-\xi)^2}{4t}}\,d\xi = \frac{1}{\sqrt{4\pi t}} \int_{-x}^\infty e^{\frac{-\eta^2}{4t}}\,d\eta \\
=& \frac{1}{\sqrt{\pi}} \int_{\frac{-x}{\sqrt{4t}}}^\infty e^{-\xi^2}\,d\xi = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{4t}}} e^{-\xi^2}\,d\xi\,.
\end{aligned}$$

It is straightforward to check that this is, in fact, the solution of the initial-value problem (3) as before, even with the discontinuity at the origin. Only the *uniformity* of the convergence as $t \searrow 0$ is (necessarily) lost here.

Now we can use this special case to develop a formula for the solution arising from an initial function $f(\cdot)$ which is bounded and continuous except at the origin, where we assume it has both left and right sided limits,

$$f(0^+) = \lim_{x \searrow 0} f(x), \quad f(0^-) = \lim_{x \nearrow 0} f(x)\,.$$

Denote the jump in $f(\cdot)$ at the origin by $\sigma \equiv f(0^+) - f(0^-)$. Then we can write $f(\cdot)$ as the sum of its continuous part and its jump by

$$f(x) = (f(x) - \sigma H(x)) + \sigma H(x)\,.$$

Now we can apply Theorem 2 to the first part and the above example to the second and obtain by linearity the solution of (3) with the more general initial-value $f(\cdot)$. This solution $u(\cdot,\cdot)$ takes the limiting value at the origin determined by the mean value of the left and right limits, i.e.,

$$\lim_{(x,t)\to(0,0)} u(x,t) = f(0^-) + \frac{1}{2}(f(0^+) - f(0^-)) = \frac{f(0^+) + f(0^-)}{2}\,.$$

The corresponding result holds for an initial function with discontinuity at any point on the axis.

Corollary 2. *Let the function $f(\cdot)$ be bounded and piecewise continuous on $\mathbb{R}$. Then the function $u(\cdot,\cdot)$ given by (2) in $\{(x,t) : t > 0\}$ is bounded, infinitely differentiable, satisfies the diffusion equation (3.a) in $\{(x,t) : t > 0\}$, and it satisfies the initial condition*

$$\lim_{(x,t)\to(x_0,0)} u(x,t) = \frac{f(x_0^+) + f(x_0^-)}{2}\,.$$

Of course this is just $f(x_0)$ as before at each point x_0 where $f(\cdot)$ is continuous.

Remark. We showed above that the fundamental solution is given by

$$K(x,t) = \frac{1}{\pi}\int_0^\infty e^{-\mu^2 t}\cos(\mu x)\,d\mu.$$

Using this form in (2) with $\cos\mu(x-\xi) = \cos(\mu x)\cos(\mu\xi) + \sin(\mu x)\sin(\mu\xi)$ gives the representation

$$u(x,t) = \int_{-\infty}^\infty \int_0^\infty \frac{1}{\pi} e^{-\mu^2 t}(\cos(\mu x)\cos(\mu\xi) + \sin(\mu x)\sin(\mu\xi)) f(\xi)\,d\mu\,d\xi\,.$$

By interchanging the order of integration, we obtain

$$u(x,t) = \int_0^\infty e^{-\mu^2 t}(a(\mu)\cos(\mu x) + b(\mu)\sin(\mu x))\,d\mu\,, \tag{4}$$

where

$$a(\mu) \equiv \frac{1}{\pi}\int_{-\infty}^\infty f(\xi)\cos(\mu\xi)\,d\xi\,, \qquad b(\mu) \equiv \frac{1}{\pi}\int_{-\infty}^\infty f(\xi)\sin(\mu\xi)\,d\xi\,.$$

When $t = 0$, (2) is the *Fourier integral theorem.* Although it is obtained here for a very different class of functions, this calculation suggests that the Fourier theorem is extremely useful in computing solutions of the initial-value problem.

Theorem 2 shows that for each function $f(\cdot)$ in a certain class, X, there is exactly one solution $u(\cdot,\cdot)$ of the initial-value problem. In particular, X is the class of continuous and bounded functions on $\mathbb{R}$. Moreover, for each $t > 0$, the function $u(\cdot,t)$ belongs to that class, and it is given by (2). Define the *operator* which gives the solution at time $t > 0$ as a function of the initial data by $S(t)$. That is we define $S(t) : X \to X$ by $S(t)f(\cdot) \equiv u(\cdot,t)$ for each $t > 0$, and $S(0) \equiv I$, the identity. These operators are well-defined by Theorem 2. From the representation (4) we see that $S(t)$ can be described as follows: expand the function $f(\cdot)$ in its Fourier representation and then multiply the μ^{th} term by $e^{-\mu^2 t}$ to obtain the expansion of $S(t)f(\cdot)$. This suggests the similarity of the operators $S(t)$ to (the action of) exponential functions. This similarity is actually quite profound. In fact, if we define the differential operator A by

$$A(u) = f \text{ in } X \iff u,\ f \in X \text{ and } -u''(x) = f(x),\ x \in \mathbb{R},$$

then the initial-value problem (3) is of the form

$$u'(t) + A(u(t)) = 0, \quad u(0) = f \text{ in } X.$$

The solution of this problem is (formally) given by the *exponential operators* $u(t) = e^{-At}f,\ t > 0$, so we have $e^{-At} = S(t)$. From (3.a) we have $S'(t) = -A\,S(t)$.

Let $S(\cdot)$ denote the family of solution operators given as above by Theorem 2. Since the equation is linear, if $f_1,\ f_2 \in X$ are given, then the function $S(t)f_1 + S(t)f_2$ is a solution of the initial value problem with initial data $f_1 + f_2$. By uniqueness, it follows that $S(t)f_1 + S(t)f_2 = S(t)(f_1 + f_2)$. A similar argument holds for multiples of $f \in X$, so we conclude that each $S(t)$ is a *linear operator* on X. Now let $\tau > 0$ be fixed. If $u(t) = S(t)f$, then the function $t \mapsto u(t+\tau)$ is the solution of the equation (3.a) with initial value $u(\tau) = S(\tau)f$. That is, $u(t+\tau) = S(t)u(\tau)$, so we have $S(t+\tau)f = S(t)\,S(\tau)f$ for each $f \in X$, and this shows that

$$S(t+\tau) = S(t)\,S(\tau)\,, \quad t,\ \tau \geq 0\,.$$

This is the *semigroup property* of the operators $S(t)$. We have shown that $S(\cdot) : \mathbb{R} \to L(X)$ is a *semigroup homomorphism* from the non-negative real numbers with addition to the linear operators on X with composition.

Consider now the *non-homogeneous* initial-value problem

$$u_t(x,t) - u_{xx}(x,t) = F(x,t), \quad x \in \mathbb{R},\ t > 0, \tag{5.a}$$

$$u(x,0) = f(x). \tag{5.b}$$

We can write this (formally) as before,

$$u'(t) + A(u(t)) = F(t), \quad u(0) = f \text{ in } X. \tag{6}$$

If $u(\cdot)$ is any solution of (6) and $S(\cdot)$ is the semigroup of operators, then for each $t > 0$ we consider the map $s \mapsto S(t-s)u(s),\ 0 < s < t$. The derivative is given by

$$\begin{aligned} \frac{d}{ds} S(t-s)u(s) &= S(t-s)u'(s) + S'(t-s)u(s) \\ &= S(t-s)(-Au(s) + F(s)) - A\,S(t-s)u(s) = S(t-s)F(s). \end{aligned}$$

Integrating this yields

$$u(t) = S(t)f + \int_0^t S(t-s)F(s)\,ds, \quad t > 0,$$

and it suggests the following result.

Theorem 3 (Duhamel). *Let $f(\cdot)$ be bounded and continuous on $\mathbb{R}$, and let $F(\cdot,\cdot)$ be bounded and continuous on $\mathbb{R} \times [0,T]$ with $T > 0$. Define $u(\cdot,\cdot) : \mathbb{R} \times (0,T] \to \mathbb{R}$ by*

$$\begin{aligned} u(x,t) \equiv &\int_{-\infty}^{\infty} K(x-\xi,t) f(\xi)\,d\xi \\ &+ \int_0^t \int_{-\infty}^{\infty} K(x-\xi,t-s) F(\xi,s)\,d\xi\,ds \quad x \in \mathbb{R},\ 0 < t \leq T\,. \end{aligned} \tag{7}$$

Then $u(\cdot,\cdot)$ is bounded and continuous on $\{(x,t) : 0 \le t \le T\}$, it is infinitely differentiable in $\{(x,t) : 0 < t \le T\}$, and it satisfies the initial-value problem

$$u_t(x,t) - u_{xx}(x,t) = F(x,t), \quad x \in \mathbb{R},\ 0 < t \le T, \tag{8.a}$$

$$u(x,0) = f(x). \tag{8.b}$$

Proof outline. It suffices to check that the second term in (7) is a solution of (8.a). (It clearly vanishes at $t = 0^+$.) We compute

$$\begin{aligned}&\frac{\partial}{\partial t}\int_0^t\int_{-\infty}^{\infty} K(x-\xi,t-s)F(\xi,s)\,d\xi\,ds \\ &= \int_0^t\int_{-\infty}^{\infty}\frac{\partial}{\partial t}K(x-\xi,t-s)F(\xi,s)\,d\xi\,ds + \lim_{s\to t}\int_{-\infty}^{\infty} K(x-\xi,t-s)F(\xi,s)\,d\xi\end{aligned}$$

and pay special atention to the singularity of $K(x-\xi,t)$ at $t = 0^+$. From Lemma 1.a we get

$$\int_0^t\int_{-\infty}^{\infty}\frac{\partial^2}{\partial x^2}K(x-\xi,t-s)F(\xi,s)\,d\xi\,ds + \lim_{\tau\to 0}\int_{-\infty}^{\infty} K(x-\xi,\tau)F(\xi,t)\,d\xi\,.$$

Now take the derivatives outside the integral in the first term and use Theorem 2 for the second to obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\int_0^t\int_{-\infty}^{\infty} K(x-\xi,t-s)F(\xi,s)\,d\xi\,ds \\ &\qquad = \frac{\partial^2}{\partial x^2}\int_0^t\int_{-\infty}^{\infty} K(x-\xi,t-s)F(\xi,s)\,d\xi\,ds + F(x,t)\,d\xi\,.\end{aligned}$$

This is the desired result, namely, (8.a). □

3. Green's Function.

We have obtained existence-uniqueness results for the initial-value problem (2.8) on the upper half-plane, $\{(x,t) : t \ge 0\}$. Also we have established uniqueness for the initial-boundary-value problem (1.4) for a rather general domain. Here we shall develop a representation of the solution on such general domains as an integral operator on the data. Recall that for the Laplace equation, which is *self-adjoint*, we used a particular family of (singular) solutions of the (adjoint) equation to construct a kernel for an integral representation based on the divergence theorem. The definition of the adjoint equation is essentially a divergence requirement, as we shall see below.

Denote the diffusion operator by $L[u] \equiv u_t - u_{xx}$. We want to define an operator $M[\cdot]$ by the requirement that the expression $vL[u] - uM[v]$ be a divergence. Thus, we compute

$$\begin{aligned}vL[u] = v(u_t - u_{xx}) &= (vu)_t - v_t u - (vu_x - uv_x)_x + uv_{xx} \\ &= uM[v] - \nabla\cdot(vu_x - uv_x, uv)\end{aligned}$$

where $M[v] \equiv -v_t - v_{xx}$. Recall that the fundamental solution was defined by

$$K(x,t) = \frac{1}{(4\pi t)^{\frac{1}{2}}} \exp^{\frac{-x^2}{4t}}, \quad t > 0.$$

We note that for each fixed pair (ξ, τ), the function $K(x-\xi, t-\tau)$ satisfies $K_t - K_{xx} = 0$ where $t > \tau$, and for each fixed pair (x,t), this function satisfies the adjoint equation, $K_\tau + K_{\xi\xi} = 0$ where $\tau < t$. These observations will be useful in our development of an integral representation for the solution of the problem.

The *initial-boundary-value problem* is given as follows. Let $a(t) < b(t)$ be given for $t_0 < t < T$, and define $G \equiv \{(x,t) : a(t) < x < b(t)\,,\ t_0 < t < T\}$. We denote the lateral boundary by $B_T \equiv \{(x,t) : x = a(t) \text{ or } x = b(t)\,,\ t_o < t < T\}$ and the initial characteristic line by $C_{t_0} \equiv \{(x, t_0) : a(t_0) < x < b(t_0)\}$. Let the functions $F,\ f$, and $g_1,\ g_2$ be given, respectively, on $G,\ C_{t_0}$, and on the inverval $[t_0, T]$. We seek a function u on G for which u_{xx} and u_t are continuous, and u satisfies

(1.a) $$L[u] = F(x,t), \quad (x,t) \in G,$$

(1.b) $$u(a(t),t) = g_1(t),\ u(b(t),t) = g_2(t), \quad t \in [t_0, T],$$

(1.c) $$u(x,t_0) = f(x), \quad a(t_0) < x < b(t_0).$$

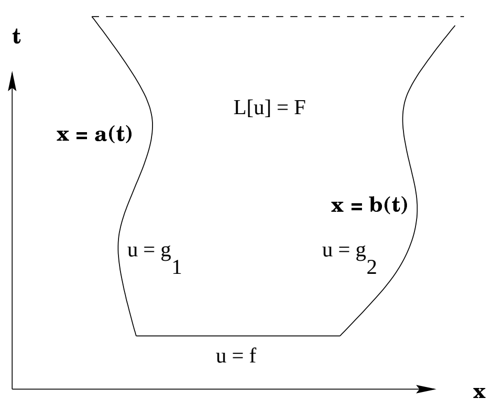

FIGURE 2

Let $(x,t) \in G$ and consider the region $G_t = \{(\xi,\tau) \in G : \tau < t\}$ obtained by passing a characteristic (τ = constant) through (x,t). We would like to use the identity

(2) $$\int\int_{G_t} (vL[u] - uM[v]) = -\int_{\partial G_t} (vu_x - uv_x, uv) \cdot \vec{n}\, ds$$

with u a solution of (1) and with $v(\xi,\tau) = K(x-\xi, t-\tau)$. However this choice of v is singular at $(\xi,\tau) = (x,t)$. For the elliptic problem we isolated the singularity by a sphere and let the radius of the sphere decrease to zero. Because the singularity here is on the *boundary* of the region, we can move it outside by a small distance and then let that distance decrease to zero.

Let $(x,t) \in G$, and define G_t as above and $B_t = B \cap \overline{G_t}$, $C_t = \overline{G_t} \cap \{\tau = t\}$. Let $\varepsilon > 0$ and define the function $K_\varepsilon(\xi,\tau) = K(x-\xi, t+\varepsilon-\tau)$. Note that K_ε has its singularity at the point $(x, t+\varepsilon) \notin \overline{G_t}$.

Let u be a solution in $C^2(G) \cap C^1(\overline{G})$ of the initial-boundary-value problem (1). Set $v = K_\varepsilon$, and note that $M[v] = 0$, so from (2) we obtain

$$\int\int_{G_t} K_\varepsilon(\xi,\tau) f(\xi,\tau)\, d\xi d\tau = -\int_{B_t} ((K_\varepsilon u_\xi - u(K_\varepsilon)_\xi)n_1 - (uK_\varepsilon)n_2)\, ds \\ - \int_{C_{t_0}} uK_\varepsilon\, ds + \int_{C_t} uK_\varepsilon\, ds\,,$$

where (n_1, n_2) is the unit outward normal along the boundary ∂G_t. We shall consider the limit of this identity as $\varepsilon \searrow 0$. Since the distance from B_t to the points $(x, t+\varepsilon) : \varepsilon > 0$ is positive, the integrands in the first two terms converge uniformly to the obvious limits. In order to examine the last term, we set $\varphi(\xi) = u(\xi,t)$, $\alpha = a(t)$, $\beta = b(t)$, so it may be written as

$$\int_{C_t} uK_\varepsilon\, ds = \int_\alpha^\beta \varphi(\xi) K(x-\xi, \varepsilon)\, d\xi\,.$$

But this we recognize as the solution of the initial-value problem with initial data given as φ extended as 0 off of $[\alpha,\beta]$, so we know it has the limit $\varphi(x)$ as $\varepsilon \searrow 0$. Thus, we obtain the representation

$$(3)\quad u(x,t) = \int_{B_t} (K(x-\xi,t-\tau)u_\xi(\xi,\tau) - u(\xi,\tau)K_\xi(x-\xi,t-\tau))n_1\, ds_{(\xi,\tau)} \\ - \int_{B_t} u(\xi,\tau)K(x-\xi,t-\tau)n_2\, ds_{(\xi,\tau)} + \int_{C_{t_0}} uK(x-\xi,t-t_0))\, ds \\ + \int\int_{G_t} K(x-\xi,t-\tau)F(\xi,\tau)\, d\xi d\tau$$

for any solution of the diffusion equation (1.a).

The formula (3) does not serve to represent a solution of the initial-boundary-value problem (1) in terms of its data, since the first term contains the derivative of the solution on the lateral boundary. This quantity is not immediately available from (1). We shall eliminate that term from the representation (3) by using in place of the singular solution $K(\cdot,\cdot)$ an appropriate modification. This modified kernel for (3) will depend not only on the equation but also on the domain G on which the problem is posed.

Definition. The *Green's function* for the initial-boundary-value problem (1) on the region G is given by

$$G(x,t,\xi,\tau) = K(x-\xi,t-\tau) - w(x,t,\xi,\tau)$$

where $K(\cdot,\cdot)$ is the fundamental solution and, for each $(x,t) \in G$, the function $w(x,t,\cdot,\cdot) \in C^2(G_t) \cap C^1(\overline{G_t})$ satisfies

$$(4)\quad \begin{aligned} M[w(x,t,\xi,\tau)] &= -w_\tau - w_{\xi\xi} = 0, \quad (\xi,\tau) \in G_t\,, \\ w(x,t,\xi,\tau) &= K(x-\xi,t-\tau), \quad (\xi,\tau) \in B_t\,, \\ w(x,t,\xi,t) &= 0, \quad (\xi,t) \in C_t\,. \end{aligned}$$

Note that the operator $M[\cdot]$ is equivalent to $L[\cdot]$ under the time-reversing change of variable, $\tau = -t$, so it follows from our uniqueness results that there is at most one such Green's function. By repeating the discussion which led to (3), but with $K(\cdot,\cdot)$ replaced by $G(x,t,\xi,\tau)$, we obtain

$$(5)\quad u(x,t) = -\int_{B_t} u(\xi,\tau) G_\xi(x,t,\xi,\tau)\, n_1\, ds_{(\xi,\tau)} + \int_{C_{t_0}} G(x,t,\xi,t_0)\, d\xi + \int\int_{G_t} G(x,t,\xi,\tau) F(\xi,\tau)\, d\xi d\tau\,.$$

Note that in the first integral we have $n_1\, ds_{(\xi,\tau)} = \pm d\tau$ and $\xi = a(\tau)$ or $\xi = b(\tau)$. This holds in general for any solution $u \in C^2(G) \cap C^1(\overline{G})$ of (1), and if the boundary ∂G can be smoothly approximated from inside G, we need only to assume that $u \in C^2(G) \cap C(\overline{G})$. All of the information required for the right side of (5) is available as data in the initial-boundary-value problem (1). Finally, note that if we (formally) set $F = \delta_{(\xi,\tau)}$, the point source located at (ξ,τ), then $u(x,t) = G(x,t,\xi,\tau)$ is the corresponding solution arising from that point source. Thus, the Greens function is frequently called the *source function* or *influence function* for the problem.

The preceding characterization motivates the construction of the Greens function by *reflection* about simple boundaries, and we shall develop this approach in a couple of examples to follow.

Example 1: Half-plane. Let $G \equiv \{(x,t) : \ t > 0\}$, the upper half-plane. Since $C_t = \mathbb{R} \times \{t\}$, it follows that we have $w = 0$, so

$$G(x,t,\xi,\tau) = K(x-\xi, t-\tau)\,.$$

Note also that $B_T = \emptyset$. Thus, the representation (5) agrees with the Duhamel formula (2.7).

Example 2: Quarter-plane. Let $G \equiv \{(x,t) : \ x > 0,\ t > 0\}$, the upper quarter-plane. The initial-boundary-value problem is to find a function u on G for which

$$(6.a)\qquad u_t - u_{xx} = F(x,t), \quad x > 0,\ t > 0\,,$$

$$(6.b)\qquad u(0,t) = g(t), \quad t > 0\,,$$

$$(6.c)\qquad u(x,0) = f(x), \quad 0 < x\,.$$

We want to find the function w which satisfies (4). First note that with (x,t) fixed in G, any function of the form $w(x,t,\xi,\tau) = \alpha K(-x-\xi, t-\tau) = \alpha K(x+\xi, t-\tau)$ satisfies the partial differential equation in G_t and the initial condition on C_t, since $x > 0$ and the singularity is at the point $(-x,t) \notin \overline{G}$. By choosing $\alpha = 1$ we attain the boundary condition on $B_t = \{(0,\tau) : \ 0 < \tau < t\}$. Thus, the Green's function is given by

$$G(x,t,\xi,\tau) = K(x-\xi, t-\tau) - K(x+\xi, t-\tau)\,.$$

Note that we could have anticipated this formula by regarding it as the source function arising from two sources, one with a charge of $+1$ at (ξ,τ) and another with a charge of -1 at $(-\xi, \tau)$, so they balance out on the boundary. In order to use the representation (5), we need to compute

$$G_\xi(x,t,\xi,\tau) = \frac{1}{(4\pi(t-\tau))^{\frac{1}{2}}}\left(e^{\frac{-(x-\xi)^2}{4(t-\tau)}}\frac{2(x-\xi)}{4(t-\tau)} - e^{\frac{-(x+\xi)^2}{4(t-\tau)}}\frac{-2(x+\xi)}{4(t-\tau)}\right)$$

and evaluate it on the boundary to obtain

$$G_\xi(x,t,0,\tau) = \frac{x}{\sqrt{4\pi}(t-\tau))^{\frac{3}{2}}}e^{\frac{-(x)^2}{4(t-\tau)}}\,.$$

The representation (5) is given by

$$\begin{aligned} u(x,t) &= \int_0^t \frac{x}{\sqrt{4\pi}(t-\tau))^{\frac{3}{2}}}e^{\frac{-(x)^2}{4(t-\tau)}}\,g(\tau)\,d\tau \\ &+ \int_0^\infty (K(x-\xi,t)-K(x+\xi,t))\,f(\xi)\,d\xi \\ &+ \int_0^t\int_0^\infty (K(x-\xi,t-\tau)-K(x+\xi,t-\tau))\,F(\xi,\tau)\,d\xi d\tau\,. \end{aligned}$$

The analogue of Theorem 2.7 holds here if the boundary data $g(\cdot)$ is bounded and continuous.

Example 3: Cylinder. Let $G \equiv \{(x,t):\ 0 < x < b,\, 0 < t < T\} = (0,b)\times(0,T)$, the indicated cylinder in the plane. The initial-boundary-value problem is to find a function u on G for which

(7.a) $$u_t - u_{xx} = F(x,t), \quad 0 < x < b,\, 0 < t < T\,,$$

(7.b) $$u(0,t) = g_1(t),\ u(b,t) = g_2(t) \quad 0 < t < T\,,$$

(7.c) $$u(x,0) = f(x), \quad 0 < x < b\,.$$

Let's try to obtain the Green's function for this problem directly by placing charges at appropriate points in the plane. We start with a charge of $+1$ located at the point $(\xi,\tau)\in G$. To balance this about the boundary line $\xi = 0$, we place a charge of -1 at the point $(-\xi,\tau)$. Then to balance these two with respect to the boundary line $\xi = b$, we place charges -1 at $(-\xi+2b,\tau)$ and $+1$ at $(\xi+2b,\tau)$.

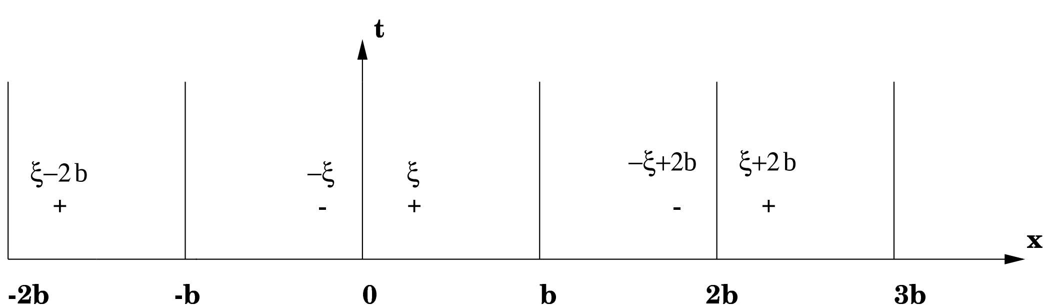

Figure 3

These are balanced about $\xi = 0$ by charges -1 and $+1$ at $(-\xi - 2b, \tau)$ and $(\xi - 2b, \tau)$, respectively, and so on. This leads to a sequence of charges to obtain the balance simultaneously about both boundary points (see Figure 3) and the corresponding Green's function representation

$$G(x,t,\xi,\tau) = \sum_{n=-\infty}^{n=+\infty} (K(x-\xi-2nb, t-\tau) - K(x+\xi-2nb, t-\tau))$$

$$= \frac{1}{(4\pi(t-\tau))^{\frac{1}{2}}} \sum_{n=-\infty}^{n=+\infty} \left(\exp^{\frac{-(x-\xi-2nb)^2}{4(t-\tau)}} - \exp^{\frac{-(x+\xi-2nb)^2}{4(t-\tau)}} \right).$$

We will develop below an alternative representation of this Green's function by separation of variables.

Example 3: Wedge. We would like to construct the Green's function for the initial-boundary-value problem on a wedge

$$u_t(x,t) - u_{xx}(x,t) = F(x,t) \quad x > a\,t,\ t > 0\,,$$
$$u(a\,t, t) = 0\,, \quad u(x,0) = 0\,.$$

We shall accomplish this by making an appropriate change of variable to reduce it to the quarter-plane problem

$$w_s(y,s) - w_{yy}(y,s) = H(y,s)\,, \qquad y > 0,\ s > 0\,.$$

Thus, let

$$y = x - a\,t, \quad s = t,$$
$$x = y + a\,s, \quad t = s,$$

be the indicated change of variable, and note that by the chain rule we obtain

$$w_t + a\,w_x - w_{xx} = H(x - a\,t, t)\,, \qquad x > a\,t,\ t > 0\,.$$

In order to eliminate the second term, we set $w(y,s) = e^{v(x,t)}u(x,t)$:

$$u_t + v_t u + a(u_x + v_x u) = u_{xx} + 2v_x u_x + (v_x^2 + v_{xx})u + e^{-v}H\,.$$

Thus, we would like to choose v so that $v_t + av_x = v_x^2 + v_{xx}$ and $a = 2v_x$. That is, $v = \frac{a}{2}x - \frac{a^2}{4}t$, so we have $u(x,t) = e^{-\frac{a}{2}(x - \frac{a}{2}t)}w(x - a\,t, t)$. This function then satisfies

$$u_t(x,t) - u_{xx}(x,t) = e^{-\frac{a}{2}(x-\frac{a}{2}t)}H(x - a\,t, t) = F(x,t) \quad x > a\,t,\ t > 0\,,$$

as we desired. The solution of the quarter-plane problem can be written

$$w(y,s) = \int_0^s \int_0^{+\infty} (K(y-\xi, s-\tau) - K(y+\xi, s-\tau))H(\xi,\tau)\,d\xi d\tau$$

so we obtain the solution of the problem on the wedge in the form

$$u(x,t) = e^{-\frac{a}{2}(x-\frac{a}{2}t)} \int_0^t \int_0^{+\infty} (K(x-a\,t-\xi, t-\tau) - K(x-a\,t+\xi, t-\tau)) e^{\frac{a}{2}(\xi+\frac{a}{2}\tau)} F(\xi + a\,\tau, \tau)\, d\xi d\tau\,.$$

After a change of variable this can be written as

$$u(x,t) = \int_0^t \int_{a\,\tau}^{+\infty} G(x,t,\xi,\tau) F(\xi,\tau)\, d\xi d\tau$$

where

$$G(x,t,\xi,\tau) = e^{-\frac{a}{2}((x-\xi)-\frac{a}{2}(t-\tau))}(K(x-a\,t-\xi+a\,\tau, t-\tau) - K(x-a\,t+\xi-a\,\tau, t-\tau))$$

is necessarily the corresponding Green's function.