Snow Plow Problem

Problem Statement

It starts snowing in the morning and continues steadily throughout the day. A snowplow that removes snow at a constant rate starts plowing at noon. It plows 2 miles in the first hour, and 1 mile in the second. What time did it start snowing?

Solution

Let t be the time measured in hours after noon.
Let $x(t)$ be the distance the snowplow has travelled.
Let $h(t)$ be the height of the snow at time t.
Let α be the constant rate of snow removal. (in any convenient unit)
Let k be the constant rate at which snow falls. (in any convenient unit)
Let b be the (unknown) number of hours before noon that it started snowing.

The change in height is given by the rate the snow falls,

$$\frac{d}{dt}h(t) = k \quad \Rightarrow \quad \begin{aligned} h(t) &= kt + c \qquad h(-b) = 0,\ c = kb. \\ h(t) &= k(t+b) \end{aligned}$$

The rate α is proportional to the cross-section of the snow being plowed and the speed of the truck. Let's assume the width of the road is a constant ω. Then $\alpha = \omega\, h(t) \frac{d}{dt}x(t)$. Rearranging, we find that

$$\frac{d}{dt}x(t) = \frac{C}{t+b} \qquad C = \frac{\alpha}{k\omega}$$

Which is a separable differential equation. Integrating both sides yields

$$x(t) = C\ln|t+b| + D$$

Plugging in the condition $x(0) = 0$ gives the constant of integration $D = -C\ln(b)$. Plugging in the other two conditions $x(1) = 1(\text{mi})$. and $x(2) = 2$, then lets us solve for $b = .618\text{hr}$. So the snow began about 37 minutes before noon. That is, at about **11:23 am.**