CONTINUOUS DEFORMATION OF THE BOWEN-SERIES MAP ASSOCIATED TO A COCOMPACT TRIANGLE GROUP

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Abstract. In 1979, for each signature for Fuchsian groups of the first kind, Bowen and Series constructed an explicit fundamental domain for one group of the signature, and from this a function on $\mathbb{S}^1$ tightly associated with this group. In general, their fundamental domain enjoys what has since been called the 'extension property'. We determine the exact set of signatures for cocompact triangle groups for which this extension property can hold for any convex fundamental domain, and verify that for this restricted set, the Bowen-Series fundamental domain does have the property.

To each Bowen-Series function in this corrected setting, we naturally associate four continuous deformation families of circle functions. We show that each of these functions is aperiodic if and only if it is surjective; and, is finite Markov if and only if its natural parameter is a hyperbolic fixed point of the triangle group at hand.

Contents

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1. Introduction

In 1979, for each signature for Fuchsian groups of the first kind, Bowen-Series [6] constructed an explicit fundamental domain for one group of the signature, and from this, a so-called boundary map, a function on $\mathbb{S}^1$, tightly associated with this group. In the cocompact setting, they showed that their function is uniquely ergodic with respect to a probability measure equivalent to Lebesgue measure. These functions have been revisited many times, most notably in the surface group setting by, among others, Alder-Flatto [3], Morita [11], Pit [12], Los and co-authors [10, 4], and S. Katok and co-authors [9, 1, 2]. Many interval maps have the flavor of boundary maps, in particular regular continued fractions and the underling interval map are well-known to be related to the Fuchsian group $\mathrm{PSL}_2(\mathbb{Z})$. The well-studied α-continued fractions of Nakada [13] give a one-parameter deformation of the regular continued fraction map. In an analogous manner, there is a geometrically natural way to form one-parameter deformations of the Bowen-Series maps, as [9, 10] do in the surface group setting.

One naturally expects that many of the results for the surface groups will succeed in the setting of the cocompact triangle groups. Some aspects must be simpler, since any two cocompact triangle Fuchsian groups of the same signature are conjugate in $\mathrm{PSL}_2(\mathbb{R})$. (In particular, the use by Bowen-Series of quasi-conformal maps, to extend their results so as to apply to all Fuchsian groups of a given signature, is unnecessary in this setting.) We were surprised to find that a central feature of the Bowen-Series construction fails for certain cocompact triangle signatures, see Theorem 1. We restrict to the signatures where this 'extension property' does hold, and consider one-parameter deformations of the Bowen-Series function for each. Another surprise arose: Functions so created can fail to be ergodic with respect to a probability measure equivalent to Lebesgue measure. Indeed, one finds functions that fail to be surjective! In Theorem 2 we identify exactly when our functions are surjective.

Recall that aperiodicity of a function with respect to a partition is a type of transitivity property. As Adler-Flatto [3] prove (see their 'Folklore Theorem'), when a sufficiently smooth piecewise monotone eventually expanding function is both aperiodic and Markov with respect to a finite partition, it has is an ergodic probability measure equivalent to Lebesgue measure. We show, see again Theorem 2, that any surjective member of the deformation families which is Markov is also aperiodic. The (finite) Markov property holds exactly for those parameter values which are hyperbolic fixed points of the Fuchsian group at hand, see Theorem 3.

1.1. Main Results.

We give a correction to the initial sentence of [[6], Section 3]. Let "$\mathcal{F}$ is a fundamental domain for the signature" mean that there is some Fuchsian group of the given signature which has $\mathcal{F}$ as a fundamental domain. To simplify notation throughout, we write (m_1, m_2, m_3) to represent the standard signature $(0; m_1, m_2, m_3)$ when speaking of Fuchsian triangle groups.

Theorem 1. *Suppose that (m_1, m_2, m_3) is the signature of a cocompact hyperbolic Fuchsian triangle group. If more than one m_i is odd, then no convex fundamental domain for the signature has the extension property. Otherwise, the Bowen-Series fundamental domain for this signature does have the extension property.*

We define $\mathscr{E}$ be the set of those signatures (m_1, m_2, m_3) corresponding to a cocompact hyperbolic Fuchsian triangle group such that the Bowen-Series fundamental domain is a non-degenerate quadrilateral with the extension property, and insist on a particular ordering of the m_i; see Definition 15. The Bowen-Series function is defined in terms of the side pairing elements of the Bowen-Series fundamental domain, $T_i, 1 \leq i \leq 4$; see Subsection 2.1. For $(m_1, m_2, m_3) \in \mathscr{E}$, the function f is eventually expanding since each T_i is applied on a subset of its isometric circle. Set $(n_1, n_2, n_3, n_4) = (m_3, m_2/2, m_3, m_1/2)$. These latter values control aspects related to the partition $\mathcal{P}$ of $\mathbb{S}^1$ associated to f, see Subsection 2.3.

Recall that a function g is called *aperiodic* with respect to a partition of its domain of definition if there is a finite compositional power of the function that maps the closure of each partition element onto the whole domain. Note that we use the term *Markov* following the convention that it means what is sometimes called *finite Markov.* In our setting, Bowen-Series showed that their eventually expansive function f is both aperiodic and Markov; and therefore has an invariant probability measure equivalent with Lebesgue measure. There are four corresponding *overlap intervals* $\mathscr{O}_i$ where appropriately replacing $f(x) = T_i x$ by $x \mapsto T_{i-1}x$ for all x in some subinterval determined by a parameter α leads to an eventually expansive function f_α, see Subsection 4.1.

Theorem 2. *Fix $(m_1, m_2, m_3) \in \mathscr{E}$ and let f denote its Bowen-Series function. Then for $\alpha \in \mathscr{O}_i$ the function f_α of* (16) *is surjective if and only any of the following conditions holds:*

(i) $n_i > 2$,
(ii) $n_i = 2$ *and* $n_{i+2} > 2$,
(iii) α *belongs to the closure of the set of points* $x \in \mathscr{O}_i$ *such that* $f^{n_i}(x) = f^{n_i - 1}(T_{i-1}x)$

Furthermore if f_α is Markov with respect to $\mathcal{P}_\alpha$, defined in Subsection 4.1, then f_α is aperiodic with respect to $\mathcal{P}_\alpha$ if and only if f_α is surjective.

Theorem 3. *Let Γ be the group from which f is constructed. The map f_α is Markov if and only if $\alpha \in \mathscr{O}$ is a hyperbolic fixed point of Γ.*

Further results about these maps are given in the Ph.D. dissertation of the second named author, [7].

2. Background

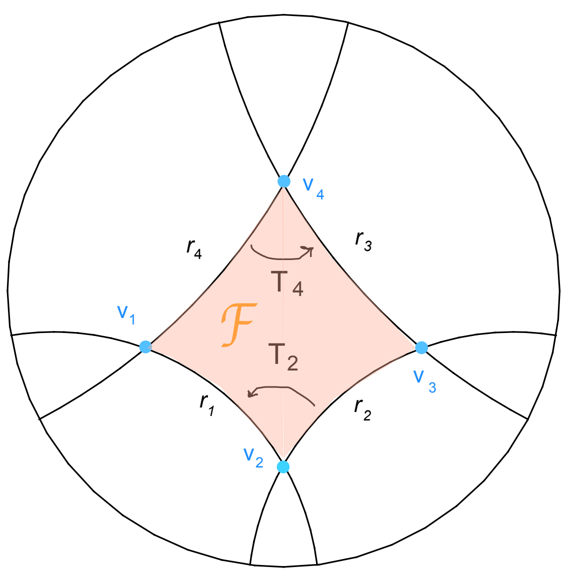

FIGURE 1. The Bowen-Series fundamental domain $\mathscr{F}$, here $(m_1, m_2, m_3) = (6, 6, 3)$.

2.1. **The Bowen-Series fundamental domain.** Bowen-Series [6] construct one fundamental domain, within the disk model $\mathbb{D}$ of the hyperbolic plane, per each signature of Fuchsian groups. As already stated, in the setting of triangle groups, where since the genus is zero, we use a simplified signature of (m_1, m_2, m_3) and will simply call this the signature. Recall that permutations of the indices result in signatures of isomorphic groups.

Given (m_1, m_2, m_3), their construction results in a fundamental domain $\mathscr{F}$ which is a quadrilateral; note that we follow the convention that a 'side' cannot be paired with itself, as in [5] but not as in [6] or [12]. We normalize and label, see Figure 1, so that $\mathscr{F}$ has vertices v_1, v_3 each of internal angle π/m_3 and v_2, v_4 of respective internal angles $2\pi/m_2, 2\pi/m_1$ with v_2, v_4 lying on the central vertical line and where the v_i are in counter clockwise order.

In all that follows, indices $i \pm 1$ are to be the appropriate representative of $i \pm 1 \pmod 4$. Redundantly labeling the sides as $r_i, \ell_i, 1 \le i \le 4$ where r_i is the side meeting v_i at its right (facing inward) and ℓ_i at its left, we denote by T_i the side pairing taking r_i to its paired side. Thus, letting

$$\sigma(i) := \begin{cases} i-1 & \text{if } i \text{ is even;} \\ i+1 & \text{if } i \text{ is odd,} \end{cases}$$

we have $T_i(r_i) = r_j$ whenever $\sigma(i) = j$. Similarly, we define

$$\rho(i) := \sigma(i) + 1 \tag{1}$$

and find that $\rho(i) = j$ implies that $T_i(v_i) = v_j$. Accordingly, for all i we have

$$\begin{aligned} T_{\sigma(i)} T_i &= I \\ T_{\rho(i)} T_{i-1} &= I \\ T_{i-1}(v_i) &= v_{\rho(i)} \text{ and } T_i(v_i) = v_{\rho(i)}\,. \end{aligned} \tag{2}$$

In particular, $T_2(v_2) = v_2$, $T_4(v_4) = v_4$ while (up to orientation) $T_1(r_1) = r_2$, $T_3(r_3) = r_4$ and $T_2 = T_1^{-1}$, $T_4 = T_3^{-1}$. We have that $\{v_1, v_3\}$ forms a cycle, as $T_1(v_1) = v_3$, $T_3(v_3) = v_1$. We find that T_2T_4 fixes v_1 and rotates through an angle of $2\pi/m_3$.

Key to their use of $\mathscr{F}$, the side r_i is contained in the isometric circle of T_i for each i.

2.2. **The net of a fundamental domain.** Suppose that $\mathcal{F}$ is a convex fundamental domain for some Fuchsian group Γ. Let S be the set of the sides of $\mathcal{F}$ and for each $s \in S$ let $g(s)$ be the geodesic containing s. Following Adler-Flatto [3], we say that $\mathcal{F}$ has the *extension property* if

$$g(s) \cap \bigcup_{T \in \Gamma} T(\mathring{\mathcal{F}}) = \emptyset \;\; \forall s \in S.$$

Following [6], the *net* N of $\mathcal{F}$ is the set of images under Γ of the the sides of $\mathcal{F}$. That is,

$$N = \bigcup_{T \in \Gamma, s \in S} T(\,s\,).$$

The union of the sides of $\mathcal{F}$ is its boundary hence N is the union of the images of the boundary of $\mathcal{F}$. Since the images of $\mathcal{F}$ tesselate, it follows that any geodesic which avoids the images of the interior of $\mathcal{F}$ must be contained in N, and vice versa. More formally, we have the following, showing that the second part of Condition $(*)$ in [6] is equivalent to the extension property.

Lemma 4. *Suppose that $\mathcal{F}$ is a convex fundamental domain for a Fuchsian group Γ. Then the extension property holds for $\mathcal{F}$ if and only if*

$$g(s) \subset N \;\; \forall s \in S.$$

The following is an observation of Adler-Flatto [3]; see the comment below their Theorem 1.1, p. 239.

Lemma 5. *Suppose that $\mathcal{F}$ is a convex fundamental domain for a Fuchsian group Γ. If the boundary of $\mathcal{F}$ projects to closed geodesics, then the extension property holds for $\mathcal{F}$.*

Proof. Suppose first that some side s projects to a closed geodesic. Then $g(s)$ projects to (wrap around) this same geodesic. Equivalently, $g(s)$ is contained in the Γ-orbit of s. That is, $g(s) \subset N$.

Similarly, if there is a sequence of sides $s_1, \dots, s_n$ which projects to a closed geodesic, then each of the $g(s_i)$ projects to this geodesic. Thus, each $g(s_i)$ is contained in N. □

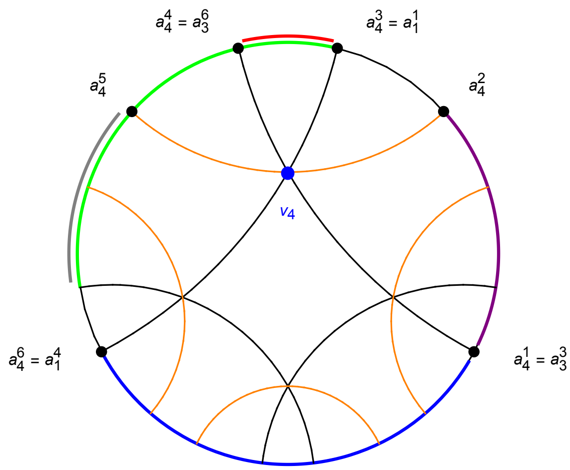

FIGURE 2. The Bowen-Series fundamental domain for $(m_1, m_2, m_3) = (6, 6, 3)$ has the extension property. Note that here, $n_i = 3$ for all i. In green, the Bowen-Series map, see (3), f is such that $f(x) = T_4(x)$ on $[a_4^{n_4}, a_1^{n_1})$. Expansivity is preserved upon replacing the action of T_4 by T_3 in the overlap interval $\mathscr{O}_4 = L_3(v_4)$, hinted by red; see (8). Left- and right intervals in accordance with Definition 8: In blue, $R_1(v_4)$; in purple, $R_2(v_4)$; continuing counter clockwise, one finds $R_3(v_4)$ and then the $L_j(v_4)$ in increasing order of j. Hinted in gray, $\mathcal{A}_4$, see (5).

Following [12], for each vertex $v \in \mathcal{F}$ we let N_v be the set of geodesics passing through v that contain the image of some side of $\mathcal{F}$. That is,

$$N_v = \{g \text{ geodesic} \mid v \in g;\ \exists T \in \Gamma, \exists s \in S, Ts \in g\}.$$

Suppose now that $\mathcal{F}$ has the extension property. Then Lemma 4 implies that $N_v \subset N$. If furthermore v is not an ideal point (that is, $v \notin \partial\mathbb{D}$), then N_v is a finite set: The tesselation by images of $\mathcal{F}$ meets v in finitely of these images, none of whose interiors the $g \in N_v$ can meet.

The following is a variant of a step in the proof of [[6], Lemma 2.3], see also [[12], Proposition 2.2].

Lemma 6. *Suppose that $\mathcal{F}$ is a convex fundamental domain for a Fuchsian group Γ of finite covolume. Assume that $\mathcal{F}$ is not a degenerate quadrilateral. If $\mathcal{F}$ satisfies the extension property, then for any $g, g' \in N$, we have that $g \cap g'$ is either: empty, a vertex of $\mathcal{F}$, or equal to $g(s)$ for some side s.*

Proof. The other cases being trivially verified, we take g, g' to be distinct and even such that there are distinct vertices v, v' of $\mathcal{F}$ with $g \in N_v, g' \in N_{v'}$. We argue by contradiction. Suppose v, v', g, g' are such that $g \cap g' = P$ with the point P not lying on a side of $\mathcal{F}$. Let $\Delta := vPv'$ be the closed triangle with these three points as vertices. Since neither of g, g' can meet the interior of $\mathcal{F}$, convexity shows that Δ contains a consecutive vertex v'' of one of v, v'. Relabel such that v, v'' are consecutive vertices. If $v'' \neq v'$, then let $v''' \neq v$ also be consecutive to v'' and s'' their shared side. Convexity also shows that $g(s'')$ meets g. That is, by replacing v' by v'' and g' by $g(s'')$ as necessary, we may and do assume that v, v' are consecutive vertices.

Now let $U \in \Gamma$ be the side pairing element such that $U^{-1}(s)$ is also a side. Then $U(\mathcal{F}) \cap \Delta \neq \emptyset$. Since the sides of Δ cannot intersect the interior of $U(\mathcal{F})$, we have that $U(\mathcal{F}) \subset \Delta$. By convexity, the sides t, t' of $U(\mathcal{F})$ meeting s at v, v' respectively must be such that $g(t') \cap g(t'')$ meet in a point of Δ. But, by [[6], Lemma 2.2] this is impossible (under their running assumption of the extension property). $\square$

Remark 7. The proof of [6] for their Lemma 2.2 does not explicitly mention a special case that could be of interest in the setting of cocompact triangle groups. Given some triangle vPv' as in the proof above, they implicitly find a sequence $(r_k)_{k \geq 1}$, with r_k the intersection of $g(t_k), g(t'_k)$ for distinct sides t_k, t'_k (separated by exactly one side, say s_k) of some $U_k(\mathcal{F})$ such that for $k \geq 2$, r_k is in the triangle of endpoints P_{k-1} and the endpoints of t_{k-1}, t'_{k-1} meeting s_{k-1}. See [[6], Figure 1]. A contradiction is raised, since the area of the original triangle is finite.

The case that they do not explicitly treat is where the sequence of the r_k is eventually constant. In that case, we can simply assume that the sequence is constant. We can thus suppose that $g(t_1), g(t'_1)$ and $g(t_2), g(t'_2)$ all meet at P_1. As in that proof, one also has that each $U_k(\mathcal{F})$ is the image of $U_{k-1}(\mathcal{F})$ by way of a side pairing to s_{k-1}, and that $s_k \neq s_{k-1}$. Hence, $U_2(\mathcal{F})$ is a quadrilateral and thus so is each $U_k(\mathcal{F})$. Therefore, the complement of $U_1(\mathcal{F})$ in its triangle is again a triangle. Similarly, each union $\cup_{i=1}^{k} U_i(\mathcal{F})$ gives a complement in this first triangle that is again a triangle. Thus, the initial triangle cannot be the union of finitely many of these quadrilaterals, but again by area considerations it cannot contain infinitely many of them; a contradiction is reached in this case as well.

2.3. **The Bowen-Series function.** Fix a cocompact Fuchsian triangle group signature (m_1, m_2, m_3). Assume that the Bowen-Series fundamental domain $\mathscr{F}$ has the extension property. Theorem 1, below, shows that this is a restrictive assumption.

2.3.1. *Definition of function f and its partition $\mathcal{P}$.* To simplify notation, let $N_i = N_{v_i}$; let also $n_i = \#N_i$. Since T_4 fixes v_4 and rotates through an angle of $2\pi/m_1$, it is always the case that $2n_4 \geq m_1$ and certainly our assumption of the extension property then shows that equality must hold: $2n_4 = m_1$; similarly, $2n_2 = m_2$ and $n_1 = n_3 = m_3$.

We label the endpoints of the $g \in N_i$, see Figure 2. For each i, let a_i^1 be the endpoint of $g(\ell_i)$ which is on the same side of v_i as is v_{i-1}. Then continue labeling the endpoints of the $g \in N_i$ in counter clockwise order as $a_i^2, \cdots, a_i^{2n_i}$. Following [6], let $W_i = \{a_i^1, \cdots, a_i^{2n_i}\}$.

Of course $a_i^{n_i+1}$ is the other endpoint of $g(\ell_i)$. Since $g(\ell_{i+1}) \cap g(\ell_i) = v_i$, by Lemma 6 no other $g \in N$ can have an endpoint between a_{i+1}^1 and $a_i^{n_i+1}$. Since $g(\ell_{i+1}) \in N_i$, one has the key identities $a_i^{n_i} = a_{i+1}^1$ and $a_i^{2n_i} = a_{i+1}^{n_{i+1}+1}$. We find that the intervals $[a_i^1, a_i^{n_i})$, $1 \leq i \leq 4$ partition $\mathbb{S}^1$.

In our notation, the function defined in [6] is

$$f : \mathbb{S}^1 \to \mathbb{S}^1 \qquad (3)$$
$$x \mapsto T_{i-1}(x) \quad \text{if } x \in [a_i^1, a_i^{n_i}\,).$$

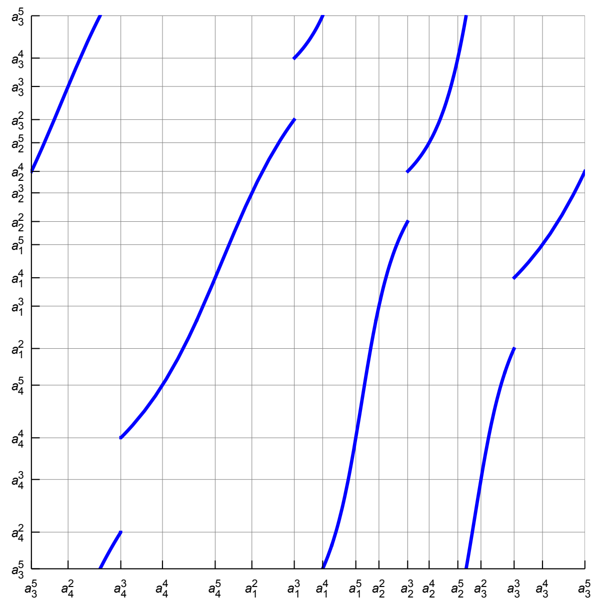

FIGURE 3. Plot representing the Bowen-Series function f for signature $(6, 6, 3)$. The partition $\mathcal{P}$, consisting of with the 16 intervals indicated along the axes, is a Markov partition for f. The true plot lives on a torus, here the torus is cut, using 'argument' functions. Note that, on the torus, there are four connected components of the graph of f.

The map f is (eventually) expansive: For each i, the side r_{i-1} lies on the isometric circle of T_{i-1} and thus T_{i-1} increases Euclidean distances on the arc subtended by the geodesic containing $g(r_{i-1}) = g(\ell_i)$. This arc is $(a_i^1, a_i^{n_i+1}\,)$. Since $[a_i^1, a_i^{n_i}\,)$ is a subset of the arc, the eventual expansivity of f easily follows. Let W be the union of the W_i; Bowen-Series show that f is a Markov map with respect to the *partition* $\mathcal{P}$ of $\mathbb{S}^1$ whose endpoints form W. In particular, $f(W) = W$.

2.3.2. *More partitions.* The following partitions of $\mathbb{S}^1$ are also from [6].

Definition 8. Let

$$L_j(v_i) = [a_i^{2n_i-j}, a_i^{2n_i-j+1}),\ R_j(v_i) = [a_i^{j-1}, a_i^j), \qquad \text{where } 1 \le j \le n_i,$$

and we take $j - 1 = 2n_i$ when $j = 1$.

It follows from (6), that each of the $L_r(v_i), 2 \le r \le n_i$ and $R_s(v_i), 3 \le s \le n_i$ is itself an element of $\mathcal{P}$. As well, each of $L_1(v_i), R_1(v_i), R_2(v_i)$ is a union of elements of $\mathcal{P}$. Furthermore, for each i, the 'left' and 'right' intervals partition $\mathbb{S}^1$,

$$\mathbb{S}^1 = \bigsqcup_{j=1}^{n_i} \big(L_j(v_i) \sqcup R_j(v_i) \big)\,. \qquad (4)$$

We use further notation from [6]: For each i, let

$$\mathcal{A}_i = L_1(v_i) \setminus L_{n_{i+1}}(v_{i+1}). \tag{5}$$

Thus, $f(x) = T_i x$ holds exactly on $A_i \cup \bigcup_{k=2}^{n_i} L_k(v_i)$.

By the conformality of each T_i, we have for all i and $1 \le k \le 2n_i$

$$T_{i-1}(a_i^k) = a_{\rho(i)}^{k-1} \quad \text{and} \quad T_i(a_i^k) = a_{\rho(i)}^{k+1}\ , \tag{6}$$

with values taken appropriately modulo $2n_i$.

Using (6) and the conformality of the transformations in Γ, for each i,

$$\begin{aligned} f\,|_{L_r(v_i)} &= T_i, \qquad f(L_r(v_i)) = L_{r-1}(v_{\rho(i)}) \quad \text{for } 2 \le r \le n_i\,;\\ f\,|_{R_s(v_i)} &= T_{i-1}, \quad f(R_s(v_i)) = R_{s-1}(v_{\rho(i)}) \quad \text{for } 2 \le s \le n_i\,. \end{aligned} \tag{7}$$

2.4. **Matching f-orbits of x and $T_{i-1}(x)$ for $x \in \mathscr{O}$.** In this subsection, we continue to establish notation, and make more precise a result of [6].

At the expense of more doubling of notation, for each $i, 1 \le i \le 4$, define the corresponding *overlap* interval

$$\mathscr{O}_i = [a_i^{n_i}, a_i^{n_i+1}). \tag{8}$$

That is, $\mathscr{O}_i = L_{n_i}(v_i)$, and hence $\mathcal{A}_i = L_1(v_i) \setminus \mathscr{O}_{i+1}$. Let $\mathscr{O} = \cup_{i=1}^4 \mathscr{O}_i$.

With our usual conventions, let

$$\theta(i) = \rho^{n_i - 1}(i),$$

where ρ is given in (1). With i fixed, let

$$i_0 = i \text{ and } i_k = \theta(i_{k-1}) + 1,\ \forall k \in \mathbb{N}. \tag{9}$$

Given $x \in \mathscr{O}_i$, let

$$\begin{cases} x_0 = x, & \text{and} \ \ x_{k+1} = f^{n_{i_k}-1}(x_k)\ \forall k \ge 0;\\ y_0 = T_{i-1}x, & \text{and} \ \ y_{k+1} = f^{n_{i_k}-1}(y_k)\ \forall k \ge 0. \end{cases} \tag{10}$$

Thus, for each $\ell \in \mathbb{N}$ letting

$$r_\ell = \sum_{k=0}^{\ell-1} (n_{i_k} - 1),$$

for $k \ge 2$, we have $x_k = f^{r_k}(x) = f^{r_k - r_{k-1}}(x_{k-1})$ and similarly for y_k. We think of $x_k \mapsto x_{k+1}$ as a *giant step* in the orbit of x and each $f^j(x_k) \mapsto f^{j+1}(x_k)$ with $j \le n_{i_k} - 1$ as a *baby step*, and similarly for maps on the f-orbit of of y_0.

Still with i fixed, for each $\ell \in \mathbb{N}$, let

$$M_\ell = M_\ell^{(i)} = \{x \in \mathscr{O}_i \,|\, f(x_\ell) = y_\ell, \text{ with minimal } \ell\}.$$

Thus, for $x \in M_\ell$ setting $p = r_\ell$ we have that

$$f^{p+1}(x) = f^p(\,T_{i-1}(x)\,).$$

In Lemma 10, below, we show that the M_ℓ partition $\mathscr{O}_i$.

Fix i. For $x \in \mathscr{O}_i$ by (7) and the definition of the Bowen-Series function f,

$$f^{n_i}(x) = \begin{cases} T_{\theta(i)} f^{n_i-1}(x) & \text{if } f^{n_i-1}(x) \in \mathcal{A}_{\theta(i)} = L_1(v_{\theta(i)}) \setminus \mathscr{O}_{\theta(i)+1};\\ \\ T_{\theta(i)+1} f^{n_i-1}(x) & \text{if } f^{n_i-1}(x) \in \mathscr{O}_{\theta(i)+1}. \end{cases} \tag{11}$$

The first statement of the following shows that for every $x \in \mathscr{O}_i$ one of the two conditions above does indeed hold.

This lemma and its proof are extracted from the proof of [[6], Lemma 2.4].

Lemma 9. *The map f^{n_i-1} restricted to $\mathscr{O}_i$ gives a homeomorphism onto $L_1(v_{\theta(i)})$. Furthermore, for all $x \in L_{n_i}(v_i)$,*

$$T_{\theta(i)} f^{n_i-1}(x) = f^{n_i-1}(\,T_{i-1}(x)\,). \tag{12}$$

Moreover, $f^{n_i}(x) = f^{n_i-1}(\,T_{i-1}(x)\,)$ holds on the inverse image of $\mathcal{A}_{\theta(i)}$ under the homeomorphism.

Sketch. The transitivity of the equalities in the top line of (7) gives

$$\begin{aligned} f^{n_i-1} : L_{n_i}(v_i) &\xrightarrow{\sim} L_1(v_{\rho^{n_i-1}(i)}) \\ x &\mapsto T_{\rho^{n_i-2}(i)} \cdots T_{\rho(i)} T_i(x). \end{aligned} \tag{13}$$

By definition, $\theta(i) = \rho^{n_i-1}(i)$ and thus the first statement holds. Due to (11), the first and second statements imply the third statement.

To show that (12) holds, first note that

$$\theta(i) = \begin{cases} i & \text{if } 2 \mid i; \\ i & \text{if } 2 \nmid i,\, 2 \nmid n_i\,; \\ \rho(i) & \text{if } 2 \nmid i,\, 2 \mid n_i\,. \end{cases} \tag{14}$$

Using (6) one finds $T_{i-1}(x) \in R_{n_i}(v_{\rho(i)})$ and, similarly to the above (7) gives

$$f^{n_i-1}(T_{i-1}(x)) = T_{\rho^{n_i-1}(i)-1} \cdots T_{\rho^2(i)-1} T_{\rho(i)-1} T_{i-1}(x) \in R_1(v_{\rho^{n_i}(i)}). \tag{15}$$

Using (2), one now verifies the result in each of the cases of (14). □

Lemma 10. *Fix $i \in \{1, \ldots, 4\}$. With notation as above,*

$$\mathscr{O}_i = \cup_{\ell=1}^{\infty} M_\ell\,.$$

Furthermore, for $x \in M_\ell$ and $k < \ell$, we have $x_k \in \mathscr{O}_{i_k}$. Moreover, the f-orbit segment from x to $f^{r_\ell - 1}(x)$ meets $\mathscr{O}$ in exactly the $x_k, k < \ell$.

Proof. Set $i_0 = i$. Lemma 9 shows that M_1 is the inverse image of $\mathcal{A}_{i_1-1}$ under the restriction of f^{r_1} to $\mathscr{O}_i$. This is a non-empty subinterval of $\mathscr{O}_i$ whose complement is an interval. Applying the lemma again shows that on this complement f^{r_2} is the composition of homeomorphisms and hence is a homeomorphism onto its image. Hence, $M_1 \cup M_2$ is a non-empty subinterval whose complement is an interval to which f^{r_3} restricts to be a homeomorphism. We can clearly repeat this argument indefinitely: Each M_ℓ is non-empty, and their union over all ℓ gives a subinterval of $\mathscr{O}_i$ whose complement is an interval.

We now have that $\mathscr{O}_i \setminus \cup_{\ell=1}^{\infty} M_\ell$ is either empty or a finite length interval, say J. Since f is eventually expansive (and f itself is never contractive), the length of $f^{r_\ell}(J)$ is unbounded as $\ell \to \infty$. However, $f^{r_\ell}(J) \subset L_{n_{i_\ell}}(v_{i_\ell})$ for every ℓ. This contradiction shows that $\mathscr{O}_i \setminus \cup_{\ell=1}^{\infty} M_\ell$ is in fact empty.

By definition, $x_0 \in \mathscr{O}_{i_0}$. For $1 \leq k < \ell$, Lemma 9 shows that $x_k \in \mathscr{O}_{\theta(i_{k-1})+1}$. Since $\theta(i_{k-1}) + 1 = i_k$, we have that x_k does belong to $\mathscr{O}_{i_k}$.

Finally, the proof of Lemma 9 shows that the f orbit of x up to x_ℓ visits $\mathscr{O}$ in exactly the x_k. □

2.5. **Strong orbit-equivalence.** Orbit equivalence between a function g and a group G acting on $\mathbb{S}^1$ holds when for any pair $(x,y) \in \mathbb{S}^1 \times \mathbb{S}^1, \exists T \in G$ such that $y = Tx$ if and only if $g^p(x) = g^q(y)$ for some $p, q \geq 0$. Bowen and Series show the orbit equivalence of their function f and the group action of Γ, at least up to a finite number of exceptions in $\mathbb{S}^1 \times \mathbb{S}^1$. Morita [11] shows that the exceptional set is empty.

We use notation introduced in [12]. With g, G as above, for each x, let $\gamma_1[x] = \gamma[x] \in G$ such that $g(x) = \gamma[x]x$, and for $p \in \mathbb{N}$ let $\gamma_p[x] = \gamma[\,\gamma_{p-1}[x]x\,]$. Thus, for any $p \in \mathbb{N}$, one has $g^p(x) = \gamma_p[x]\,x$. A property observed by Morita to hold for Bowen-Series functions, was named by Pit [12] *strong orbit-equivalence*: $\forall x \in \mathbb{S}^1, \forall T \in G,\ \exists p, q \geq 0$ such that $\gamma_p[x] = \gamma_q[\,Tx]\,T$ in G.

We now mildly generalize some results in [12].

Lemma 11. *Suppose that a function g is expansive, given on intervals by elements of a Fuchsian group G acting on $\mathbb{S}^1$. Then every point of finite g-orbit is a fixed point of G.*

Proof. Suppose that $x \in \mathbb{S}^1$ is periodic, with say $g^p(x) = x$. Then $\gamma_p[x] \in G$ fixes x. Since g is given piecewise by elements of G, there is a non-empty interval I containing x such that g^p agrees with the action of $\gamma_p[x]$ on I. Since g is expansive, g^p here is not the identity. We conclude that x is a fixed point for G. If y is any point of finite g-orbit, then y must be have a periodic point x in its forward orbit. Say $g^q(y) = x$. We then find that $(\gamma_q[y])^{-1} \cdot \gamma_p[x] \cdot \gamma_q[y]$ fixes y. □

Lemma 12. *Suppose that a function g is given on left closed, right open intervals by elements of a Fuchsian group G acting on $\mathbb{S}^1$, and that orbit equivalence holds. Then strong orbit equivalence holds for g, G.*

Proof. Let $x \in \mathbb{S}^1, T \in G$. By orbit equivalence, there are $p, q \geq 0$ such that $g^p(x) = g^q(Tx)$. Thus, $\gamma_p[x]x = \gamma_q[Tx]\,Tx$. Since g is piecewise defined, we have that $\gamma_p[x]x' = \gamma_q[Tx]y'$ for all x', y' in sufficiently small on left closed, right open intervals containing x and Tx, respectively. The action of T is continuous and orientation preserving, hence by possibly shrinking the interval containing x, we may assume that T sends it to the interval containing Tx. That is, $\gamma_p[x]$ and $\gamma_q[Tx]\,T$ agree on an interval. Only the identity element of G can fix any interval, and hence $\gamma_p[x] = \gamma_q[Tx]\,T$. □

Lemma 13. *Suppose that a function g is given on intervals by elements of a Fuchsian group G acting on $\mathbb{S}^1$, and strong orbit equivalence holds for g, G. Then every hyperbolic fixed point of G has finite g-orbit.*

Proof. Let x be a fixed point of the hyperbolic element $T \in \Gamma$. There are $p, q \geq 0$ such that $\gamma_p[x] = \gamma_q[\,Tx]\,T = \gamma_q[\,x]\,T$. Since T is not the identity, we have $p \neq q$. Now, $g^p(x) = \gamma_p[x]x = \gamma_q[\,x]\,Tx = \gamma_q[\,x]\,x = g^q(x)$. It follows that x has a finite g-orbit. □

3. The extension property: failure and success; proof of Theorem 1

Proof of Theorem 1. By [[5], Theorem 10.5.1], a convex fundamental domain $\mathcal{F}$ for a triangle group Γ is either a quadrilateral or a hexagon. In the latter case, there must be an accidental cycle of length three. As Pit [12] observed, such a cycle always causes a failure of the extension property; to see this, let u_1, u_2 and u_3 be the vertices of this accidental cycle and θ_1, θ_2 and θ_3 their corresponding internal angles. There exist $T_1, T_2 \in \Gamma$ such that $T_1(u_1) = u_3$ and $T_2(u_2) = u_3$; hence $u_3 \in \mathcal{F} \cap T_1(\mathcal{F}) \cap T_2(\mathcal{F})$. But, $\theta_1 + \theta_2 + \theta_3 = 2\pi$ and for each i, $\theta_i < \pi$. It follows that the geodesic that contains the side in the intersection of $T_1(\mathcal{F})$ and $T_2(\mathcal{F})$ meets the interior of $\mathcal{F}$. Clearly, the extension property fails.

If $\mathcal{F}$ is a quadrilateral and (at least) two of the m_i are odd, then there is a vertex $v \in \mathcal{F}$ whose internal angle measures $2\pi/m$ with m one of these odd values. See Figure 4 for a specific case of this. Furthermore, there is a side pairing $T \in \Gamma$ that rotates by $2\pi/m$ about v. In particular, $U = T^{\frac{m+1}{2}}$ is an element of Γ that rotates by $(m+1)\pi/m$. Letting s be the side paired to $T(s)$ by T, and s' the geodesic arc extending s beyond v, we find that U sends s' to the geodesic arc agreeing with a rotation of $2\pi + \pi/m$ of s. This arc meets the interior of $\mathcal{F}$; indeed, it lies on the angle bisector of the sides meeting at v. Again, the extension property fails.

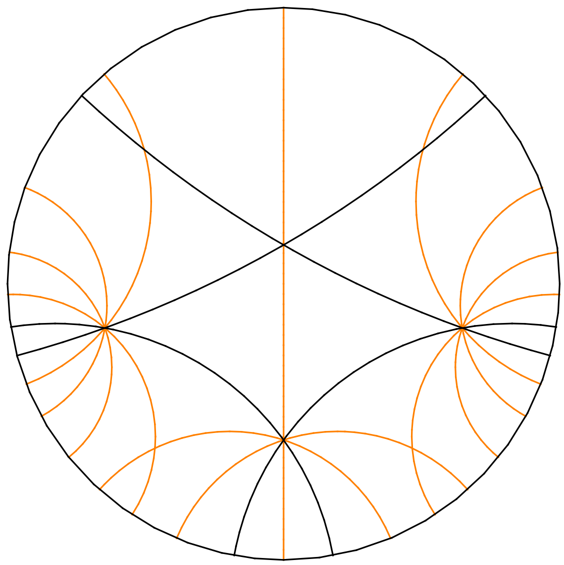

FIGURE 4. The Bowen-Series fundamental domain for $(m_1, m_2, m_3) = (3, 5, 6)$ fails to have the extension property.

We now to turn to the Bowen-Series fundamental domain $\mathscr{F}$, although we relabel as necessary so that m_1, m_2 are both even. We show that the sides of $\mathscr{F}$ do appropriately project to closed geodesics, and thus the result will follow from Lemma 5. First suppose that also m_3 is even. Then each of the four vertices is fixed by an elliptic element of even order, and taking appropriate powers, each is fixed by an elliptic element of order two. Hence, any side lies on the axis of the hyperbolic element formed by the product of elliptic elements of order two corresponding to its endpoints. In particular the side itself projects to a closed geodesic (on the orbifold, the closed geodesic is given by traversing the projected path twice, once in each direction). Thus, the extension property holds in this case.

If $m_3 = 2k+1$ is odd, then we claim that each union of opposite sides of $\mathscr{F}$ projects to a closed geodesic. For ease, we begin with r_4 as emanating from v_4. Following the unit tangent vectors along this side until v_1, we then apply $(T_2T_4)^k$ thus rotating the unit tangent vector by $k \cdot 2\pi/m_3 = 2k\pi/(2k+1) = \pi - \pi/(2k+1)$. An application of T_4 now sends this to the unit tangent vector based at v_2 and pointing in the negative direction of r_2. Following this side until reaching v_2, we apply the appropriate power of T_2 so as to turn the unit tangent vector through an angle of π. We now retrace the steps all the way back to v_4, where the arriving unit tangent vector has the opposite direction from that of the initial departing vector; we thus apply the appropriate power of T_4 so as to turn the unit tangent vector through an angle of π. In summary, we find that the union of r_4 and $-r_2 = \ell_3$ projects to the closed geodesic that is the projection of the axis of the hyperbolic element that this the product of $[T_4(T_2T_4)^k]^{-1}\,T_2^{m_2/2}\,[T_4(T_2T_4)^k]$ with $T_4^{m_1/2}$. By symmetry, a similar result holds for the remaining two sides, and we find that the extension property holds also in this case. □

Remark 14. The proof above shows that the fundamental domain constructed in [6] in the setting of genus $g = 0$ and $n \geq 3$ singularities fails to have the extension property whenever at most one of the indices is even.

Naturally enough, in what follows we will restrict to the setting where the extension property holds. In order to also ensure that eventual expansivity holds, we further insist that $\mathcal{F}$ is not a degenerate quadrilateral. (That is, there are no vertices of angle π.)

Definition 15. Let $\mathscr{E} \subset \{(m_1, m_2, m_3) \in \mathbb{N}^3\}$ such that

(1) $\sum_{i=1}^{3} m_i^{-1} < 1$;
(2) $2 \mid m_1, 2 \mid m_2$;
(3) $2 \in \{m_1, m_2, m_3\} \Rightarrow m_3 = 2$.

4. Continuous deformations of the Bowen-Series map

4.1. **Definition of function and partition.** Fix a signature in $\mathscr{E}$ and its Bowen-Series function f, defined in (3). For each $\alpha \in \mathscr{O}$, we define

$$f_\alpha(x) := \begin{cases} T_{i-1}(x), & x \in [a_i^{n_i}, \alpha); \\ f(x), & \text{otherwise}, \end{cases} \tag{16}$$

where $\alpha \in \mathscr{O}_i$. Note that each f_α is expansive, since f itself is expansive and $[a_i^{n_i}, \alpha)$ lies within the isometric circle of T_{i-1}. We call $\mathscr{D} = \mathscr{D}_\alpha = [a_i^{n_i}, \alpha)$ the *differing interval*; indeed, the functions f, f_α differ exactly on $\mathscr{D}$.

Let

$$W_\alpha = W \cup \{f_\alpha^k(\alpha)\}_{k\geq 0} \cup \{f_\alpha^k(T_{i-1}\alpha)\}_{k\geq 0}.$$

Since W is f-invariant and f, f_α agree on W except at $a_i^{n_i}$, the following implies the f_α-invariance of W_α. We have $f_\alpha(a_i^{n_i}) = T_{i-1}(a_i^{n_i})$, by (6) this equals $a_{\rho(i)}^{n_i-1}$ and thus belongs to W. Let $\mathcal{P}_\alpha$ be the partition of $\mathbb{S}^1$ given by the left-closed, right-open intervals whose endpoints are the elements of W_α.

Remark 16. Note that one could take the four-dimensional family defined by choosing a parameter in each $\mathscr{O}_i$ and simultaneously making changes as above to define a new function. We intend to return to this setting in upcoming work.

4.2. **More on f-dynamics.**
Our study of the various f_α requires various details about the dynamics of f itself.

Lemma 17. *Each $\mathscr{O}_i$ is a leftmost subinterval of each of $R_2(v_{i+1})$, a rightmost subinterval of $L_1(v_{i-1})$ and is contained in $R_1(v_{i+2})$. No other $R_j(v_k)$ or $L_j(v_k)$ meets $\mathscr{O}_i$.*

Proof. This is verified directly from the definition of the intervals. □

Lemma 18. *For each i, the image under f of $\mathcal{A}_i$ is the interval $R_1(v_{\rho(i)}) \setminus R_{n_{\rho(i)-1}}(v_{\rho(i)-1})$. The image of $L_1(v_i)$ under f is given by $f(\mathcal{A}_i) \cup L_{n_{i+1}-1}(v_{\rho(i+1)})$.*

Proof. The restriction of $f(x)$ to $\mathcal{A}_i = L_1(v_i) \setminus L_{n_{i+1}}(v_{i+1})$ is given by the action of T_i, see (5). From (6), T_i sends $L_1(v_i)$ to $R_1(v_{\rho(i)})$ and also $L_{n_{i+1}}(v_{i+1})$ to $R_{n_{i+1}}(v_{\rho(i+1)})$. Since for all i both $n_i = n_{\rho(i)}$ and $\rho(i+1) = \rho(i) - 1$ hold, the first result holds. The second follows from (7). □

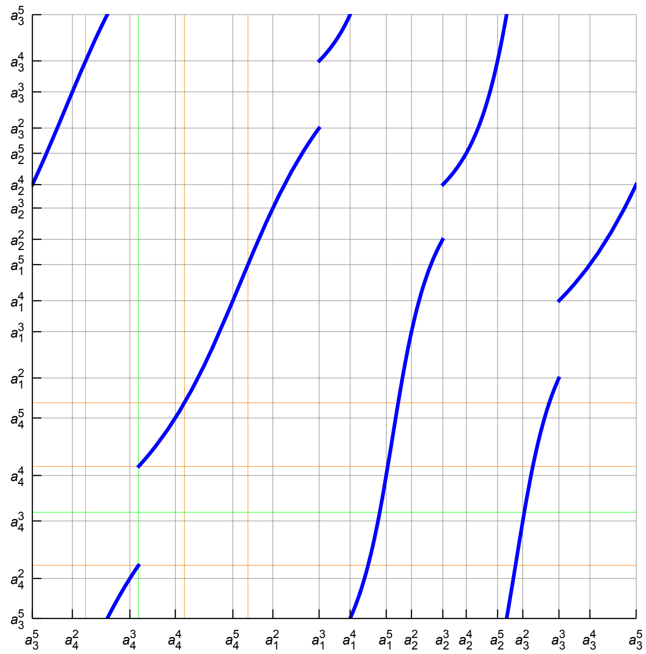

FIGURE 5. Plot representing the function f_α for signature $(6,6,3)$ with α the fixed point of $T_4^2T_2^2T_3T_1T_4^2T_1T_4^3$. Here, $\alpha \in M_1 \cap \mathscr{O}_4$, and is marked by the light green lines.

The following lemma is a direct implication of Lemma 9 and its proof.

Lemma 19. *Fix i. Then for any $x \in L_{n_i}(v_i)$, we have*

$$\begin{cases} f^j(x) \quad\quad\; \in L_{n_i-j}(v_{\rho^j(i)}), & 0 \le j \le n_i - 1, \\ \\ f^j(\,T_{i-1}x) \in R_{n_i-j}(v_{\rho^{1+j}(i)}), & 0 \le j \le n_i - 1. \end{cases}$$

Lemma 20. *Fix i. Then for any $x \in L_{n_i}(v_i)$, setting $y = T_{i-1}(x)$ we have*

$$f^{n_i-1}(\,y\,) \in \begin{cases} R_1(v_{\rho^{n_i}(i)}) \setminus R_{n_{\rho(\theta(i)+1)}}(v_{\rho(\theta(i)+1)}) & \text{iff } f^{n_i-1}(x) \in \mathcal{A}_{\theta(i)}; \\ \\ R_{n_{\rho(\theta(i)+1)}}(v_{\rho(\theta(i)+1)}) & \text{iff } f^{n_i-1}(x) \in L_{n_{\theta(i)+1}}(v_{\theta(i)+1}). \end{cases}$$

Proof. From Lemma 9, we have $f^{n_i-1}(y) = T_{\theta(i)}f^{n_i-1}(x)$. The result thus follows from Lemma 18. $\square$

The parity of m_3 determines the period length of the sequence of indices $(i_k)_{k\ge 0}$. Recall that for $k > 0$ we have $i_k = \theta(i_{k-1}) + 1$ where $\theta(i) = \rho^{n_i-1}(i)$ for each i.

Lemma 21. *Fix any $i_0 \in \{1,2,3,4\}$. The sequence $(i_k)_{k\ge 0}$ is purely periodic. Furthermore, the period length is 2 if m_3 is even and 4 otherwise. Taken in order, the period alternates between even and odd values.*

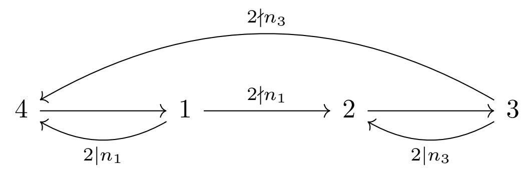

FIGURE 6. An automaton which produces the sequence of the i_k, and shows that this sequence is always periodic. Since $n_1 = n_3 = m_3$, the sequence is always purely periodic, of period length either 2 or 4.

Proof. From (14), we have

$$i_{k+1} = \begin{cases} i_k + 1 & \text{if } 2 \mid i_k; \\ i_k + 1 & \text{if } 2 \nmid i_k,\ 2 \nmid n_{i_k}\,; \\ i_k - 1 & \text{if } 2 \nmid i_k,\ 2 \mid n_{i_k}\,. \end{cases} \tag{17}$$

Therefore, the sequence $(i_k)_{k \geq 0}$ is always periodic; see Figure 6. Since $m_3 = n_1 = n_3$, we find that the period length is 2 when $2 \mid m_3$ and 4 otherwise. The final statement follows from the fact that i_{k+1} always has the opposite parity of i_k. □

5. APERIODICITY; THE PROOF OF THEOREM 2

Recall, say from [3], that a function g is called *aperiodic* with respect to a partition of its domain of definition if there is a finite compositional power of the function that maps the closure of each partition element onto the whole domain. Each Bowen-Series function f is shown in [6] to satisfy $\bigcup_{r=0}^{\infty} f^r(I) = \mathbb{S}^1$ for each I of $\mathcal{P}$. Aperiodicity itself follows from the finiteness of $\mathcal{P}$ and the fact that f is eventually expansive and nowhere contractive.

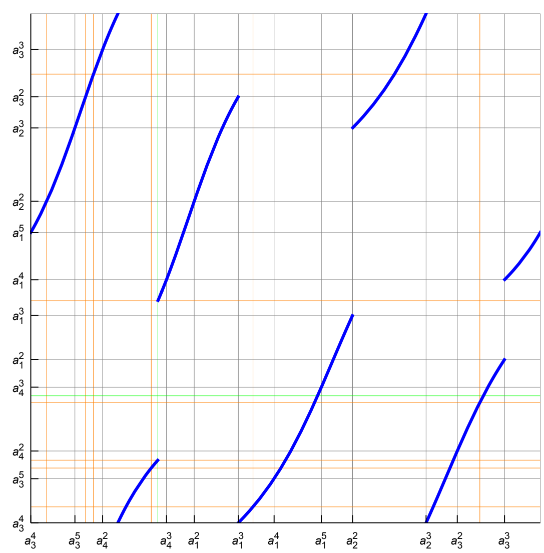

FIGURE 7. Plot of the function f_α for the signature $(4, 4, 3)$, with α the fixed point of $T_3T_2T_4T_1T_3T_1T_4T_1T_3T_2^2T_3T_1T_4$. This function is not aperiodic with respect to its Markov partition $\mathcal{P}_\alpha$; it is not even a surjective function!

5.1. **Surjectivity of f_α.** We first identify a condition that holds exactly when f_α is surjective.

Lemma 22. *Suppose $\alpha \in \mathscr{O}_i$. The function f_α is surjective if and only if there is a union J of elements of $\mathcal{P} \setminus \{\mathscr{O}_i\}$ such that $f(\mathscr{D}) \subseteq f(J)$.*

Proof. Since $f_\alpha(\mathscr{D}) \cap f(\mathscr{D}) = \emptyset$ and also f is given by T_i on all of $\mathscr{O}_i \supset \mathscr{D}$, the map f_α is surjective if and only if $f(\mathscr{D})$ is contained in the image of $\mathbb{S}^1 \setminus \mathscr{O}_i$ under f. Since $\mathbb{S}^1 \setminus \mathscr{O}_i$ admits the partition $\mathcal{P} \setminus \{\mathscr{O}_i\}$, the condition can be relaxed to $f(\mathscr{D})$ being contained in the f-image of some union of elements of $\mathcal{P} \setminus \{\mathscr{O}_i\}$. □

We now give two further results about f-dynamics before showing, using the reduction of the previous lemma, that the conditions of Theorem 2 are those identifying the α for which f_α is surjective.

Lemma 23. *Fix i. The interval $f(\mathcal{A}_{\rho(i-1)})$ contains $\mathcal{A}_i$ and each of $L_k(v_i), 2 \le k \le n_i - 1$. When $n_i > 2$, a leftmost subinterval of $f(\mathcal{A}_{\rho(i-1)})$ is formed by $L_{n_i-1}(v_i)$; otherwise, $n_i = 2$ and $\mathcal{A}_i$ forms a leftmost subinterval of $f(\mathcal{A}_{\rho(i-1)})$.*

Furthermore, $\mathcal{A}_{\rho(i-1)}$ meets none of the $L_j(v_{\rho(i)}), 2 \le j \le n_i$.

Proof. From Definition 8, one finds

$$\mathcal{A}_i \cup \bigcup_{k=2}^{n_i-1} L_k(v_i) \subseteq R_1(v_{i-1}) \setminus R_{n_{i-2}}(v_{i-2}).$$

(The case of equality occurs exactly when $n_{i-2} = 2$.) Since ρ^2 is the identity, by (7) and Lemma 18 we have the containment in $f(\mathcal{A}_{\rho(i-1)})$. Definition 8 then also implies the statement on the leftmost subintervals.

Finally, again by the definition of the intervals, $\mathcal{A}_{\rho(i-1)} = L_1(v_{\rho(i-1)}) \setminus L_{n_{\rho(i-1)+1}}(v_{\rho(i-1)+1})$ fully contains no $L_j(v_k)$. But, each $L_j(v_{\rho(i)}), 2 \le j \le n_i$ is an element of $\mathcal{P}$ and $\mathcal{A}_{\rho(i-1)}$ is a union of elements of $\mathcal{P}$. Therefore, by the Markov property of f, it must be that $\mathcal{A}_{\rho(i-1)}$ does not intersect any of the $L_{k+1}(v_{\rho(i)}), 2 \le k \le n_i - 1$. □

Lemma 24. *Fix i. The overlap interval $\mathscr{O}_i$ is contained in $f(\mathcal{A}_{\rho(i+2)})$ if and only if $n_{i+1} > 2$.*

Proof. By Lemma 18, $f(\mathcal{A}_{\rho(i+2)}) = R_1(v_{i+2}) \setminus R_{n_{i+1}}(v_{i+1})$. By Lemma 17, this set fails to fully contain $\mathscr{O}_i$ exactly when $n_{i+1} = 2$. □

We now show that the conditions of Theorem 2 are those identifying the α for which f_α is surjective. We use an overline over the notation for a set to indicate the closure of the set.

Lemma 25. *Suppose $\alpha \in \mathscr{O}_i$. There exists a union J of elements of $\mathcal{P} \setminus \{\mathscr{O}_i\}$ with $f(\mathscr{D}) \subset f(J)$ if and only if any of the following conditions is satisfied:*

- *(i)* $n_i > 2$,
- *(ii)* $n_i = 2$ *and* $n_{i+2} > 2$,
- *(iii)* $\alpha \in \overline{M_1}$.

When any of these conditions holds, J may be taken to be $\mathcal{A}_{i+1}$.

Proof. ($\Leftarrow$) From (7), $f(\mathscr{D}) \subseteq f(\mathscr{O}_i) = L_{n_i-1}(v_{\rho(i)})$. From Lemma 17 $\mathcal{A}_{i+1}$ is disjoint from $\mathscr{O}_i$.

When $n_i > 2$, Lemma 23 shows that $f(\mathcal{A}_{\rho(\rho(i)-1)})$ contains $f(\mathscr{O}_i)$. One easily verifies the identity $\rho(\rho(i) - 1) = i + 1$. Thus, $f(\mathcal{A}_{i+1}) \supset f(\mathscr{D})$.

If $n_i = 2$, then Lemma 23 shows that $f(\mathcal{A}_{i+1})$ contains the subset $\mathcal{A}_{\rho(i)}$ of $f(\mathscr{O}_i)$. Since here $\theta(i) = \rho(i)$, by Lemma 9 $f(\mathscr{D}) \subsetneq \mathcal{A}_{\rho(i)}$ occurs exactly when there is matching at the first giant step for α; namely, when $\alpha \in M_1$. Since f is orientation preserving, $f(\mathscr{D}) = \mathcal{A}_{\rho(i)}$ holds exactly when $\alpha \in \overline{M_1}$.

Suppose now that $n_i = 2$ and $n_{i+2} > 2$; note that this implies $2 \mid i$. From the previous paragraph, we are reduced to the case that $f(\mathscr{D})$ is not fully contained in $\mathcal{A}_{\rho(i)}$. By definition of the $\mathcal{A}_j$, we have $f(\mathscr{O}_i) = \mathcal{A}_{\rho(i)} \cup \mathscr{O}_{\rho(i)+1}$. Since $n_{\rho(i)+2} = n_{i+2} > 2$, Lemma 24 shows that $\mathscr{O}_{\rho(i)+1}$ is contained in $f(\mathcal{A}_{\rho(\rho(i)+1+2)}) = f(\mathcal{A}_{i+1})$. We have already seen that $\mathcal{A}_{\rho(i)} \subseteq f(\mathcal{A}_{i+1})$. Therefore,

$$f(\mathcal{A}_{i+1}) \supset \mathcal{A}_{\rho(i)} \cup \mathscr{O}_{\rho(i)+1} \supset f(\mathscr{D}).$$

($\Rightarrow$) When none of the conditions hold, we have $n_i = n_{i+2} = 2$ and $\alpha \notin \overline{M_1}$. Since $n_i = 2$, from the above we now have both that $f(\mathscr{D}) \supset \mathcal{A}_{\rho(i)}$ and $f(\mathscr{D}) \cap \mathscr{O}_{\rho(i)+1} \neq \emptyset$. Suppose that J is a union of elements of $\mathcal{P} \setminus \{\mathscr{O}_i\}$ such that $f(J) \supset f(\mathscr{D})$. Since $\mathscr{O}_{\rho(i)+1} \in \mathcal{P}$, by the Markov property of f, we have $f(K) \supset \mathscr{O}_{\rho(i)+1}$ for some partition element K of the union J.

Recall from (4) that for each k the partition given by the $R_j(v_k)$ and $L_j(v_k)$ with $1 \le j \le n_k$ is coarser than $\mathcal{P}$. Since $\mathcal{P}$ is a Markov partition, K is contained in some such interval, say K'. By Lemma 17, the right or left intervals containing $\mathscr{O}_{\rho(i)+1}$ are exactly $L_1(v_{\rho(i)}), R_1(v_{\rho(i)+3})$, and $R_2(v_{\rho(i)+2})$. There are three 'obvious possibilities' for K': $L_2(v_i) = \mathscr{O}_i; R_2(v_{i+1})$; and, $R_3(v_{i+2})$ were it to exist, however since $n_{i+2} = 2$ it does not. Since $L_2(v_i)$ is itself in $\mathcal{P}$, we find that K would be equal to this interval, but this contradicts K avoiding $\mathscr{O}_i$. Similarly, $R_2(v_{i+1}) \supset \mathscr{O}_i$ and furthermore, f on $R_2(v_{i+1}) \setminus \mathscr{O}_i$ avoids $\mathscr{O}_{\rho(i)+1}$; this remaining obvious possibility is ruled out.

We now rule out any K' being any $L_1(v_k)$ or $R_1(v_k)$. It is easily verified that any $R_1(v_k)$ is the union of various left intervals, hence we need only consider the possibility that $K' = L_1(v_k)$ for some k. By Lemma 18, if $f(L_1(v_k)) \supset \mathscr{O}_{\rho(i)+1}$ then $f(\mathcal{A}_k) \supset \mathscr{O}_{\rho(i)+1}$. But by this same lemma combined with Lemma 24, $f(\mathcal{A}_k) \supset \mathscr{O}_{\rho(i)+1}$ if only if both is $k = \rho(\rho(i) + 2)$ and $n_k > 2$. However, $\rho(\rho(i) + 2) = i + 2$ and thus here $n_k = 2$. That is, the containment does not occur. We have shown that there is no possible $K' \supseteq K$.

Therefore, there is no possible union of elements of $\mathcal{P}$ which both avoids $\mathscr{O}_i$ and has image containing $f(\mathscr{D})$. $\square$

5.2. **From surjectivity to aperiodicity; completion of the Proof of Theorem 2.**

Proof of Theorem 2. A non-surjective function cannot be aperiodic, thus when the conditions of Theorem 2 are not fulfilled, Lemma 25 shows that f_α is certainly not aperiodic with respect to $\mathcal{P}_\alpha$.

Now assume that the conditions are fulfilled, and thus $f_\alpha(\mathcal{A}_{i+1}) = f(\mathcal{A}_{i+1}) \supseteq f(\mathscr{D})$.

Main Step: f_α-orbit of $\mathscr{D}$ meets $\mathcal{A}_{i+1}$. By definition of f_α, we have $f_\alpha(\mathscr{D}) \subset R_{n_i}(v_{\rho(i)})$, as a leftmost subinterval. As usual, let $i = i_0$. By Lemma 19, we have f^{n_i-2} sends $R_{n_i}(v_{\rho(i)})$ to $R_2(v_{\theta(i)}) = R_2(v_{i_1-1})$. By Lemma 17, $\mathscr{O}_{i_1-2}$ is a leftmost subinterval of $R_2(v_{i_1-1})$. Since $i_1 \neq i+2$ by (17), $\mathscr{O}_{i_1-2} \neq \mathscr{O}_i$. From Lemma 19 and (11), the f-orbit of a leftmost subinterval of $\mathscr{O}_{i_1-2}$ reaches, and contains, $\mathcal{A}_{\theta(i_1-2)}$ before encountering any other overlap intervals.

We now describe a sequence of $\mathcal{A}_{j_k}, k \ge 0$, beginning at any $\mathcal{A}_{j_0}$, where for each k, there is a leftmost subinterval of $\mathcal{A}_{j_k}$ having f-orbit reaching, and containing, $\mathcal{A}_{j_{k+1}}$ before encountering $\mathscr{O}$. Indeed, by Lemma 23, for general j, either $f(\mathcal{A}_j) \supset \mathcal{A}_{\rho(j)+1}$ or else $f(\mathcal{A}_j)$ has a leftmost subinterval given by $L_{n_{\rho(j)+1}-1}(v_{\rho(j)+1})$. In this latter case, by Lemma 9, applying $f^{n_{\rho(j)+1}-2}$ sends a leftmost subinterval surjectively to $\mathcal{A}_\ell$, where $\ell = \rho^{n_{\rho(j)+1}-2}(\rho(j) + 1)$; note that $\mathscr{O}$ is not encountered during the intermediate steps of applying f. In fact, this single formula for ℓ applies in both of these cases: we pass from a leftmost subinterval of $\mathcal{A}_j$ to full containment of $\mathcal{A}_\ell$. That is, inductively defining $j_{k+1} = \rho^{n_{\rho(j_k)+1}-2}(\rho(j_k) + 1)$ results in our desired sequence.

One easily verifies that reversing the arrows in the automaton for $(i_k)_{k\ge0}$, given in Figure 6, results in an automaton giving the sequence $(j_k)_{k\ge0}$. If $2 \nmid m_3$, then the sequence takes on all values in $\{1,2,3,4\}$. In this case, with $j_0 = \theta(i_1 - 2)$, there is some k such that $j_k = i+1$. Otherwise, direct calculation shows that $j_0 = \theta(i_1 - 2)$ is itself equal to $i+1$. Therefore, in both cases, the f_α-orbit of this subinterval reaches $\mathcal{A}_{i+1}$.

Step 2: f_α-orbit of $\mathscr{D}$ contains $f(\mathscr{D})$. We have shown that the f-orbit of a leftmost subinterval of $f_\alpha(\mathscr{D})$ reaches a leftmost subinterval of $\mathcal{A}_{i+1}$. We can now follow the periodic sequence of the $\mathcal{A}_{j_k}$ and from the expansive aspect of f deduce that this orbit eventually contains all of $\mathcal{A}_{i+1}$. Since $\mathscr{O}$ is avoided while doing this, we deduce that the f_α-orbit of $\mathscr{D}$ contains $f(\mathscr{D})$.

Step 3: Aperiodicity holds. (This is only in this step that we make use of the hypothesis that f_α is Markov with respect to $\mathcal{P}_\alpha$.) We aim to show that the f_α-orbit of any $I' \in \mathcal{P}_\alpha$ includes all of $\mathbb{S}^1$. Let $I' \in \mathcal{P}_\alpha$. By hypothesis, the partition endpoint set W_α is finite. Thus, the left endpoint of I' has a finite f_α-orbit, and hence contains a periodic point. Let $J' \in \mathcal{P}_\alpha$ have this periodic point as its left endpoint. Then by the expansive aspect of f_α, applying f_α to compositional powers that are multiples of the minimal period length of the endpoint, the image of J' must eventually contain some $I \in \mathcal{P}$. Since f is known to be aperiodic with respect to $\mathcal{P}$, all of $\mathbb{S}^1$ is contained in a finite union of powers of f applied to I.

If all but possibly the final power of f so applied are such that their image of I avoids $\mathscr{D}$, then f_α would agree with f throughout, and we find that all of $\mathbb{S}^1$ is contained in a finite union of powers of f_α applied to I. Otherwise, each time that a piece of the f-orbit of I intersects $\mathscr{D} \subset \mathscr{O}_i$, by the Markov property of f, that piece of the f-orbit meets all of $\mathscr{D}$. By Step 2, we can thus follow the f_α-orbit of $\mathscr{D}$ until we agree with $f(\mathscr{D})$ and thereafter continue following the f-orbit. Therefore, the f_α-orbit of I, and hence that of I', contains all of $\mathbb{S}^1$. □

6. Markov values; the proof of Theorem 3

6.1. **Markov values are hyperbolic fixed points.** Arguing as in [6], the function f_α is Markov with respect to $\mathcal{P}_\alpha$ if and only if W_α is a finite set. The finiteness of W_α implies that α is a hyperbolic fixed point.

Lemma 26. *If f_α is Markov with respect to $\mathcal{P}_\alpha$ then α is a hyperbolic fixed point.*

Proof. For f_α to be Markov, W_α must be finite. In particular, the f_α-orbit of α must be finite. By Lemma 11 we have that α is a fixed point of Γ. Since Γ has no parabolic elements, this must be a hyperbolic fixed point. □

6.2. **Orbit equivalence and hyperbolic fixed points.** Under orbit equivalence, the converse of the previous result is easily shown to hold.

Lemma 27. *Suppose that f_α is orbit equivalent to the action of Γ on $\mathbb{S}^1$. If α is a hyperbolic fixed point of Γ, then f_α is Markov with respect to $\mathcal{P}_\alpha$.*

Proof. The set W_α is finite if and only if each of α and $T_{i-1}\,\alpha$ have finite f_α-orbits. Since α is a fixed point, then so is its image $T_{i-1}\,\alpha$. Lemmas 12 and 13 now show that both α and $T_{i-1}\,\alpha$ have finite f_α-orbit. □

We argue more directly, since our maps in general do *not* enjoy the property of being orbit equivalent to the action of Γ, see the second author's PhD dissertation, [7].

6.3. **Hyperbolic fixed points give Markov maps, general setting.** To prove the remaining direction of Theorem 3, we will in fact prove the following.

Theorem 28. *Fix $\alpha \in \mathscr{O}$. Suppose that $x \in \mathbb{S}^1$ has infinite f_α-orbit. Then there are infinitely many values of j such that the f-orbit of x contains either (1) $f_\alpha^j(x)$, (2) $f \circ f_\alpha^j(x)$, or (3) $f^2 \circ f_\alpha^j(x)$.*

Theorem 28 implies completion of proof of Theorem 3. By the Pigeonhole Principle, we can then assume one of the three cases occurs for infinitely many values of j. Since f is a finite-to-one function, if the f-orbit of x is finite, then there are only finitely many preimages under f or f^2 of this finite set. It then follows that whenever the f-orbit of x is finite, there are some $j \neq k$ such that $f_\alpha^j(x) = f_\alpha^k(x)$. The finiteness of the f_α-orbit of x then follows. Due to the strong orbit equivalence of f and Γ, every hyperbolic fixed point of Γ has finite f-orbit. Given that α is a hyperbolic fixed point of Γ then so is $T_{i-1}(\alpha)$, it follows that both of these values also have finite f_α orbits. This in turn implies that f_α is Markov, and thus the proof of Theorem 3 will indeed be complete. □

Of course if the f_α-orbit of x never meets the differing interval $\mathscr{D}$, then this orbit *is* the f-orbit of x. Theorem 28 thus trivially holds for such x. Each time that the f-orbit of x enters $\mathscr{D} \subset \mathscr{O}_i$, it does so in some M_ℓ. Hence our proof will focus on f-orbits of points in M_ℓ.

6.3.1. *Proof of Theorem 28 when m_3 is even.* In this subsection we show that Theorem 28 holds in the case of 'easy case' of even m_3.

Lemma 29. *Suppose that m_3 is even. Fix $i = i_0$ and let $x = x_0 \in M_\ell$. Then for $0 \leq k < \ell$ we have $y_k \notin \mathscr{O}_{i_0}$.*

Proof. The case of $k = 0$ is clear, since $\mathscr{O}_{i_0}$ has empty intersection with its image under T_{i_0-1}.

For $0 < k < \ell$, arguing as in the proof of Lemma 10, allows Lemma 20 to show that $y_k \in R_{n_{\rho(i_k)}}(v_{\rho(i_k)})$. In view of Lemma 17, since every $n_j > 1$ always holds, only if both $n_{\rho(i_k)} = 2$ and $\rho(i_k) = i_0 + 1$ could y_k possibly lie in $\mathscr{O}_{i_0}$. Since $\rho = (13)$ as an element of the permutation group S_4, we have $\rho(i_k) = i_0 + 1$ if and only if $(i_0, i_k) \in \{(1,2),(2,1),(3,4),(4,3)\}$. Since m_3 is even, $(i_k)_{k\geq 0}$ has period length two with period a permutation of one of $(1,4)$ and $(2,3)$. It follows that $y_k \notin \mathscr{O}_{i_0}$. □

Lemma 30. *Suppose that m_3 is even. Fix $i = i_0$ and let $x = x_0 \in M_\ell$ for some $\ell \in \mathbb{N}$. Then for all $0 \leq k < \ell$ and $0 < j < n_{i_k} - 1$, we have $f^j(y_k) \notin \mathscr{O}_{i_0}$.*

Proof. First notice that if $n_{i_k} = 2$, then the condition $0 < j < n_{i_k} - 1$ is never fulfilled. We assume that k is such that the voidness of the condition on j is avoided.

In view of Lemmas 19 and 17, the only non-trivial case is $j = n_{i_k} - 2$, where we then know that $f^{n_{i_k}-2}(y_k) \in R_2(v_{\theta(i_k)})$, an interval which meets $\mathscr{O}$ exactly in $\mathscr{O}_{\theta(i_k)-1}$. Thus, if we can show that $i_0 \neq \theta(i_k) - 1$ we are done. From (9), $\theta(i_k) - 1 = i_{k+1} - 2$.

By Lemma 21, the sequence $(i_k)_{k\geq 0}$ is purely periodic of length two, with one odd value and one even. Now, whatever the value of $i_0 \in \{1,2,3,4\}$ we cannot have that $i_{k+1} - 2 = i_0$. The result thus holds. □

We now show that Theorem 28 holds in this case of even m_3.

Lemma 31. *Suppose that m_3 is even and that $x \in \mathbb{S}^1$ has infinite f_α-orbit. Then there are infinitely many values of j such that the f-orbit of x contains $f_\alpha^j(x)$.*

Proof. As mentioned above, if the f_α-orbit of x never enters $\mathscr{D}$, then this orbit and the f-orbit agree. Thus in this case, the result clearly holds. Otherwise, by temporarily renaming some element of this f_α-obit as x as necessary, we may suppose that there is some ℓ such that $x \in M_\ell \cap \mathscr{D}$. Set $y = f_\alpha(x) = T_{i_0-1}(x)$. By the previous two lemmas, we have that the f-orbit of y avoids $\mathscr{O}_{i_0}$ and hence certainly $\mathscr{D}$, until at least $f^{r_\ell}(y)$. In particular, $f^{r_\ell}(y) = f_\alpha^{r_\ell}(y)$. Now, by definition of M_ℓ we have that $f^{r_\ell+1}(x) = f^{r_\ell}(y)$. Since $f^{r_\ell+1}(x) = f_\alpha^{r_\ell}(y) = f_\alpha^{r_\ell+1}(x)$, we have a matching associated to this entry into $\mathscr{D}$.

If there are only finitely many entries into $\mathscr{D}$ of the f_α-orbit of x, then there is a last matching as above. We take the values of j giving the infinite tail that begins with this matching of the orbits. Finally, if the orbit returns infinitely often to $\mathscr{D}$ then we take those j giving the corresponding matching values. □

6.3.2. *Odd* m_3. In this subsection we show that Theorem 28 holds in the remaining case of odd m_3.

Lemma 32. *Suppose that* m_3 *is odd. Fix* $i = i_0$ *and let* $x = x_0 \in M_\ell$. *If*

(i) $2 \nmid i$ *or*

(ii) $2 \mid i$ *and at least one of* m_1, m_2 *is not equal to 4,*

then for all $0 \le k < \ell - 1$ *and* $0 < j < n_{i_k} - 1$, *we have* $f^j(y_k) \notin \mathscr{O}_{i_0}$.

Moreover, if $k = \ell - 1$ *then for* $0 < j < n_{i_k} - 1$ *the containment of* $f^j(y_{\ell-1})$ *in* $\mathscr{O}_{i_0}$ *is only possible if* $\ell \equiv 2 \pmod 4$ *and* $j = n_{i_k} - 2$.

Proof. The result is clear for $k = 0$. Let $0 < k \le \ell - 1$. As in the proof of Lemma 30, if $i_0 \ne i_{k+1} - 2$ then the result holds. Since m_3 is odd, the period length of $(i_k)_{k\ge0}$ is four. Furthermore, $i_0 = i_{k+1} - 2$ (with our usual modulus 4 convention) exactly when $k + 1 \equiv 2 \pmod 4$. Therefore if $k \not\equiv 1 \pmod 4$, we have $f^j(y_k) \notin \mathscr{O}_{i_0}$, where $0 < j < n_{i_k} - 1$.

Suppose $k \equiv 1 \pmod 4$, thus $i_{k+1} = i + 2$. Moreover, suppose that $f^j(y_k) \in \mathscr{O}_{i_0} \subset L_{n_i}(v_i)$. Again as in the proof of Lemma 30, we must then have $j = n_{i_k} - 2$; it follows from Lemma 19 that $y_{k+1} \in L_{n_i-1}(v_{\rho(i)})$. On the other hand, Lemma 20 gives two cases as to where y_{k+1} is to be found. Upon replacing x there by x_k, these cases correspond to the end of a giant step. But, by Lemma 9 exactly the first case implies that $f(x_{k+1}) = y_{k+1}$. By the minimality of ℓ, we must then have $k = \ell - 1$. In particular, the final statement of our result holds.

Now with $k \equiv 1 \pmod 4$ and $k < \ell - 1$, we are reduced to considering the possibility of $y_{k+1} \in R_{n_{\rho(i_{k+1})}}(v_{\rho(i_{k+1})}) = R_{n_{\rho(i+2)}}(v_{\rho(i+2)})$. We now argue that it is impossible for y_{k+1} to be simultaneously in both $L_{n_i-1}(v_{\rho(i)})$ and $R_{n_{\rho(i+2)}}(v_{\rho(i+2)})$, compare with Table 1. If $2 \mid i$, then by hypothesis at least one of m_1 and m_2 is not 4, and thus we have that at least one of n_2 and n_4 is greater than 2. If $2 \nmid i$ then recall that m_3 being odd implies $n_1, n_3 > 2$. With this, applying Definition 2.5 then shows that in each case $L_{n_i-1}(v_{\rho(i)}) \cap R_{n_{\rho(i+2)}}(v_{\rho(i+2)}) = \emptyset$. That is, $f^j(y_k) \notin \mathscr{O}_{i_0}$ when $0 \le k < \ell - 1$. □

	$L_{n_i-1}(v_{\rho(i)})$	$R_{n_{\rho(i+2)}}(v_{\rho(i+2)})$
$2 \mid i$	$L_{n_i-1}(v_i)$	$R_{n_{i+2}}(v_{i+2})$
$2 \nmid i$	$L_{n_i-1}(v_{\rho(i)})$	$R_{n_i}(v_i)$

TABLE 1. Candidates for the location of y_{k+1} if $k \equiv 1 \pmod 4$ in the proof of Lemma 32

Lemma 33. *Suppose that m_3 is odd and $m_1 = m_2 = 4$. Suppose that $i = i_0$ is even and that $x = x_0 \in M_\ell$. Then for all $0 \le k < \ell - 2$ and $0 < j < n_{i_k} - 1$, we have $f^j(y_k) \notin \mathscr{O}_{i_0}$.*

Moreover, if $\ell - 2 \le k \le \ell - 1$ then for $0 < j < n_{i_k} - 1$ the containment of $f^j(y_k)$ in $\mathscr{O}_{i_0}$ is only possible if $k \equiv 1 \,(mod\ 4)$ and $j = n_{i_k} - 2$.

Proof. Firstly, we note that since $m_1 = m_2 = 4$, we have $n_2 = n_4 = 2$. By the proof above, it suffices to check the case where both $k \equiv 1 \,(\mathrm{mod}\ 4)$ and $i_{k+1} = i + 2$, where we again find that only if $y_{k+1} \in L_{n_i - 1}(v_{\rho(i)}) \cap R_{n_{\rho(i+2)}}(v_{\rho(i+2)})$ can the result fail to hold. By our hypotheses, we have $L_{n_i-1}(v_{\rho(i)}) = L_1(v_i)$ and $R_{n_{\rho(i+2)}}(v_{\rho(i+2)}) = R_2(v_{i+2})$. By Definition 2.5, $L_1(v_i) \cap R_2(v_{i+2}) = L_{n_{i+1}}(v_{i+1})$.

Now, if $y_{k+1} \in L_1(v_i)$, then from Lemma 20 we have the following two cases:

$$\begin{cases} x_{k+1} \in L_1(v_{i+1}) \setminus L_2(v_{i+2}) & \text{and } y_{k+1} \in L_1(v_i) \setminus L_{n_{i+1}}(v_{i+1}), \\ x_{k+1} \in L_2(v_{i+2}) & \text{and } y_{k+1} \in L_{n_{i+1}}(v_{i+1}). \end{cases}$$

In the first of these cases, matching occurs in that $f(x_{k+1}) = y_{k+1}$ and hence $k = \ell - 1$. In the second case, we find that

$$y_{k+2} = f^{n_{i_{k+1}} - 1}(y_{k+1}) = f^{n_{i+2}-1}(y_{k+1}) = f(y_{k+1}) \in L_{n_{i-1}-1}(v_{i-1}) \subset R_1(v_{i+2}) \setminus R_{n_{i+1}}(v_{i+1}).$$

Therefore, y_{k+2} is in the first case in Lemma 20, and Lemma 9 then gives that $f(x_{k+2}) = y_{k+2}$. That is $k = \ell - 2$. □

Lemma 34. *Suppose that m_3 is odd. Fix $i = i_0$ and let $x = x_0 \in M_\ell$. Then for all $0 \le k < \ell - 1$, $y_k \notin \mathscr{O}_{i_0}$. Moreover, $y_{\ell-1} \in \mathscr{O}_{i_0}$ implies all of the following: $2 \nmid i$; $n_{i+1} = 2$; and, $\ell \equiv 2 \pmod 4$.*

Proof. Since $\mathscr{O}_{i_0}$ has empty intersection with its image under $T_{i_0 - 1}$, we certainly have that $y_0 \notin \mathscr{O}_{i_0}$.

For $0 < k < \ell$, as in the proof of Lemma 29, only if both $n_{\rho(i_k)} = 2$ and $\rho(i_k) = i_0 + 1$ could y_k possibly lie in $\mathscr{O}_{i_0}$. From this, we must have $n_{i_0+1} = 2$. Since m_3 is odd, both n_1 and n_3 are greater than two. Therefore, $y_k \notin \mathscr{O}_{i_0}$ when $2 \mid i_0$.

Also as in the proof of Lemma 29, $\rho(i_k) = i_0 + 1$ if and only if $(i_0, i_k) \in \{(1,2), (2,1), (3,4), (4,3)\}$. Since m_3 is odd, $(i_k)_{k \ge 0}$ has period length four with period a permutation of $(1,2,3,4)$. Thus, $y_k \in \mathscr{O}_{i_0}$ implies $k \equiv 1 \pmod 4$.

Now, $y_k \in \mathscr{O}_i, 2 \nmid i, n_{i+1} = 2$ and $k \equiv 1 \pmod 4$ imply both $y_k \in R_2(v_{i+1})$ and $n_{i_k} = 2$. Thus,

$$y_{k+1} = f^{n_{i_k} - 1}(y_k) = f(y_k) \in R_1(v_{i+1}).$$

Also, if $y_k \in \mathscr{O}_i$ then $y_k \in L_{n_i}(v_i)$ and $y_{k+1} \in L_{n_i - 1}(v_{\rho(i)}) \subset R_1(v_{i+1}) \setminus R_{n_i}(v_i)$. That is, y_{k+1} is in the first case of Lemma 20 and hence matching occurs. Therefore, $k + 1 = \ell$. □

By arguing as at the end of the proof of Lemma 36, the following shows that Theorem 28 holds when m_3 is odd.

Lemma 35. *Suppose that m_3 is odd. Fix $i = i_0$ and let $x = x_0 \in M_\ell \cap \mathscr{D} \subset \mathscr{O}_{i_0}$. Then*

$$f^{r_\ell + 1}(x) = \begin{cases} f_\alpha^{r_\ell + 1}(x) & \textit{or} \\ \\ f \circ f_\alpha^{r_\ell}(x) & \textit{or} \\ \\ f^2 \circ f_\alpha^{r_\ell - 1}(x). \end{cases}$$

Furthermore, in the last two cases, the corresponding $f_\alpha^{r_\ell + a}(x) \in \mathscr{D}$.

Proof. Since $x \in M_\ell$ we have

$$f^{r_\ell+1}(x) = f^{r_\ell}(y). \tag{18}$$

We now consider various cases.

Case of odd i. By Lemma 32, $f^j(y_k) \notin \mathscr{O}_{i_0}$ for all $0 \le k < \ell - 1$ and $0 < j < n_{i_k} - 1$. If $n_{i+1} \neq 2$ then Lemma 34 implies that $y_k \notin \mathscr{O}_{i_0}$ for all $0 \le k < \ell$. Arguing as in the proof of Lemma 32, the only remaining possibility for an 'early' entrance to $\mathscr{D}$ is at the final baby step before matching. That is, $f^s(y) \notin \mathscr{D}$ for all $0 \le s < r_{\ell-1} + n_{r_{\ell-1}} - 2 = r_\ell - 1$. Hence, $f^{r_\ell - 1}(y) = f_\alpha^{r_\ell - 1}(y) = f_\alpha^{r_\ell}(x)$. Therefore, (18) gives

$$f^{r_\ell+1}(x) = f \circ f_\alpha^{r_\ell}(x). \tag{19}$$

Now, if $f^{r_\ell-1}(y) \notin \mathscr{D}$, then f and f_α agree on this value, and one finds $f^{r_\ell+1}(x) = f_\alpha^{r_\ell+1}(x)$.

In the subcase of $n_{i+1} = 2$, the only possible early entry into $\mathscr{D}$ is of $y_{\ell-1}$ if also $\ell \equiv 2 \mod 4$. Arguing as above, we find

$$f^{r_{\ell-1}}(y) = f_\alpha^{r_{\ell-1}}(y) = f_\alpha^{r_{\ell-1}+1}(x).$$

Since $r_{k+1} = r_k + n_{i_k} - 1$, here we find $f^{r_\ell}(y) = f^{r_{\ell-1}+1}(y) = f \circ f_\alpha^{r_{\ell-1}+1}(x) = f \circ f_\alpha^{r_\ell}(x)$. Of course, if there is no early entry in this subcase, then we argue as above. Note, throughout this case, (19) always holds.

Case of Even i. Other than the subcase of $m_1 = m_2 = 4$ one again finds the same two possibilities for $f^{r_\ell+1}(x)$. Suppose now that $m_1 = m_2 = 4$ and thus $n_2 = n_4 = 2$. Lemmas 33 and 34 combine to show that entry of the f-orbit of y into $\mathscr{D}$ before matching can occur at most once; it is possible only for $f^{n_{i_k}-2}(y_k)$ with $k \in \{\ell-2, \ell-1\}$ and $k \equiv 1 \pmod 4$.

The case of $k = \ell - 1$ is argued as above, except that here only the form as given in (19) is possible. Consider now the case of $k = \ell-2$, with of course $\ell > 2$. Here, $f^{r_{\ell-1}-1}(y) = f_\alpha^{r_{\ell-1}-1}(y)$ holds. Note that also $n_i = n_{i_{\ell-1}} = 2$, and hence $r_\ell = r_{\ell-1} + n_{i_{\ell-1}} - 1$ gives

$$f^{r_\ell}(y) = f^{n_{i_{\ell-1}}} \circ f^{r_{\ell-1}-1}(y) = f^2 \circ f^{r_\ell - 2}(y) = f^2 \circ f_\alpha^{r_\ell-2}(y). \tag{20}$$

Again, (18) applies to give the result. □

We now show that Theorem 28 holds in this case of odd m_3.

Lemma 36. *Suppose that m_3 is odd and that $x \in \mathbb{S}^1$ has infinite f_α-orbit. Then there are infinitely many values of j such that the f-orbit of x contains at least one of $f_\alpha^j(x), f \circ f_\alpha^j(x)$, $f^2 \circ f_\alpha^j(x)$.*

Proof. As mentioned above, if the f_α-orbit of x never enters $\mathscr{D}$, then this orbit and the f-orbit agree. Thus in this case, the result clearly holds. Otherwise, by temporarily renaming some element of this f_α-orbit as x as necessary, we may suppose that there is some ℓ such that $x \in M_\ell \cap \mathscr{D}$. By Lemma 35, we have that $f^{r_\ell+1}(x) \in \{f_\alpha^{r_\ell+1}(x), f \circ f_\alpha^{r_\ell}(x), f^2 \circ f_\alpha^{r_\ell-1}(x)\}$. Hence, if there are infinitely many entries of the f_α-orbit into $\mathscr{D}$ then we are done.

If there are only finitely many entries into $\mathscr{D}$ of the f_α-orbit of x, then we replace x by its final visit. Since the last two cases of Lemma 35 result in a further visit to $\mathscr{D}$, here it can only be the case that $f^{r_\ell+1}(x) = f_\alpha^{r_\ell+1}(x)$. Since there is no further visit to $\mathscr{D}$, the infinite tails of these orbits agree. □

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