LOW HEIGHT GEODESICS ON $\Gamma^3\backslash\mathcal{H}$: HEIGHT FORMULAS AND EXAMPLES

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For Marvin Knopp, our teacher and loyal friend for $17 + 42$ *years.*

Abstract. The Markoff spectrum of binary indefinite quadratic forms can be studied in terms of heights of geodesics on low-index covers of the modular surface. The lowest geodesics on $\Gamma^3\backslash\mathcal{H}$ are the simple closed geodesics; these are indexed up to isometry by Markoff triples of positive integers (x, y, z) with $x^2+y^2+z^2 = 3xyz$, and have heights $\sqrt{9 - 4/z^2}$. Geodesics considered by Crisp and Moran have heights $\sqrt{9 + 4/z^2}$; they conjectured that these heights, which lie in the "mysterious region" between 3 and the Hall ray, are isolated in the Markoff Spectrum.

In previous work, we classified the low height-achieving non-simple geodesics of $\Gamma^3\backslash\mathcal{H}$ into seven types according to the topology of highest arcs. Here, we obtain explicit formulas for the heights of geodesics of the first three types; the conjecture holds for approximation by closed geodesics of any of these types. Explicit examples show that each of the remaining types is realized.

Contents

1. Introduction and Statement of Results

There is a natural relationship between the Markoff spectrum formed by the minima on the integer lattice of indefinite binary quadratic integral forms and the heights (that is, penetration into cusp) of closed geodesics of low-index covers of the modular surface. (See the introduction to [SS3] for further discussion of the history and context of the geometric approach to the Markoff spectrum.) Indeed, to a hyperbolic matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ one associates the quadratic form $cx^2 + (d - a)xy - by^2$, arising from the fixed-point equation for the matrix acting fractional

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linearly. Should one prefer, one can pass from a given closed geodesic to the (conjugacy class of its) corresponding hyperbolic matrix, and then to an appropriately reduced word in chosen generators for the group. For these subgroups of the modular group, this then allows one to relate a closed geodesic to a periodic continued fraction (up to cyclic permutation). For this well-known relationship between closed geodesics and periodic continued fraction expansions, see [CF] as well as the introduction to [CM]. An advantage of the geometric approach is that there is a natural partition of the set of closed geodesics of a fixed surface by topological type. The Markoff values (heights) of geodesics of the same type fall into families that often have attractive formulas. To find these families beginning with the conditions on the periodic continued fraction expansions can be challenging, see for example [R] for the case of the various simple geodesics.

We focus on $\Gamma^3\backslash\mathcal{H}$, an index three cover of simple geometry: it is a cusped sphere, with three elliptic singularities.

The height of a geodesic reports its penetration into a cusp; normalizing so that this cusp lifts to ∞, the height of a geodesic is the supremum of the Euclidean diameters of lifts to $\mathcal{H}$ of the geodesic. The set of all heights of geodesics of $\Gamma^3\backslash\mathcal{H}$ gives the Markoff spectrum. A geodesic with a 'highest arc' is called a height-achieving geodesic, in [SS3], we showed the (presumably well-known) result that the height of any geodesic is achieved as the height of some height-achieving geodesic. We thus study only these latter. In [SS3], we show that there are eight types of high arcs of geodesics of $\Gamma^3\backslash\mathcal{H}$ of height between h=3 and h=6; of these, only seven can be highest arcs. In this paper, we study the first three types of geodesics in some detail, and give brief examples of the remaining types.

The height of a simple closed geodesic on $\Gamma^3\backslash\mathcal{H}$ is $\sqrt{9-4/z^2}$, where (x,y,z) is some Markoff triple of positive integers, thus satisfying $x^2+y^2+z^2=3xyz$, with $z\geq y\geq x$. In the setting of the modular hyperbolic torus, Crisp and Moran [CM] showed that geodesics of the form that they called Proper Single Self-Intersecting, and denoted *PSSI*, have heights $\sqrt{9+4/z^2}$. In [SS2], we indicated a geometric connection between simple closed geodesics and the avatars of the PSSI on $\Gamma^3\backslash\mathcal{H}$. In particular, this leads to the Markoff-indexed fundamental domains reviewed in § 4. These domains facilitate insight into the geometry of large families of geodesics. As an example of this, we have the following result. See also Figure 6 on page 17.

Theorem 1. *If γ is a Type 3 geodesic, then γ is a closed geodesic connecting an elliptic fixed point to itself. Furthermore, there exists a natural number n and a Markoff triple (x,y,z) such that the height of γ is given by*

$$h(\gamma)=\sqrt{9+\frac{4}{(a_nz)^2}}\,,$$

where the sequence $a_n:=a_n(z)$ satisfies

$$a_{-1}=0\,;\;\;a_0=1\,;\;\;a_n=3za_{n-1}-a_{n-2}\,,\;\textit{for}\;n\geq 1\,.$$

The portion of the Markoff spectrum between 3 and $\sqrt{13}$ remains of most interest, see [CF]. This is thus the range that we discuss. Particularly of interest is the topology of this set: its isolated points, limit points and the like. We hence discuss certain limits in our final section.

1.1. **Remark on Generality: Teichmüller Space Context.** The classification given in [SS3] applies mutatis mutandis to any hyperbolic punctured sphere with three elliptic points, see [Sh] or [Sch] for the necessary background for this generalization. We give explicit geodesics on $\Gamma^3\backslash\mathcal{H}$; one can adjust our techniques to give corresponding geodesics on the general element of its Teichmüller space. However, out of concern for the length of these papers, we restrict ourselves to the case of number theoretic interest.

1.2. **Outline.** After further clarifying background and motivation in § 2 and § 3, we proceed to the main parts to the paper: §§ 4 – 7, study the first three types of geodesics and give various examples of geodesics of the remaining types; results on limit values are given in § 8.

2. Markoff Spectrum and Lifts of Geodesics

Note that all but the final paragraph of this section appears in [SS3].

Recall that the geodesics of the Poincaré upper half-plane, $\mathcal{H}$, are the vertical (half-)lines and the semi-circles centered on the real line. Following tradition, we refer to these as *h-lines*, and their intersections with the (extended) real line as *feet*, with the left foot of a non-vertical h-line being less than its right foot. The highest point of a non-vertical h-line is called its *apex*.

Let $\mathcal{S}$ be a Riemann surface with hyperbolic structure and a distinguished cusp. One can uniformize $\mathcal{S}$ such that the cusp of $\mathcal{S}$ is given by ∞. A *high point* of a geodesic arc on $\mathcal{S}$ is a point on the arc that lifts to the apex of some h-line; in other words, the unit tangent vector to the geodesic based at this point can naturally be viewed as being horizontal. The *height* of the point is the Euclidean radius of this h-line. A *highest point* of a geodesic is a point along the geodesic that lifts to be as least as high as any other point on the geodesic. The *height* of a geodesic is the supremum of the Euclidean diameters of the h-lines that cover it. (Warning: a point $p \in \mathcal{H}$ has height equal to $\Im(p)$, the h-line of apex p has height $2\,\Im(p)$.)

Suppressing further reference to our particular cusp, we define the *Markoff spectrum* of $\mathcal{S}$ to be the set of all heights of geodesics of $\mathcal{S}$. The related notion of the *geometric Markoff value*, see [H], of a geodesic γ is defined as the area of the largest horocycle centered at the cusp which is disjoint from γ. Up to a normalizing constant, the height of γ is the inverse of its geometric Markoff value.

We say that a geodesic γ *achieves its height* if there is a lift of γ that has diameter equal to the height of γ. Recall that Lemma 1 of [SS3] states that any height of a geodesic on a hyperbolic surface or orbifold is in fact realized by a height-achieving geodesic.

For the standard formal definitions of the classical Markoff and Lagrange spectra, see say [CF]. In our language, the Markoff value of a geodesic of $\Gamma^3\backslash\mathcal{H}$ is the supremum of the heights of its lifts to the hyperbolic plane; the Lagrange value is the limit superior of these heights. The spectra are then the sets of these corresponding values when all geodesics are considered. Both of the spectra are closed subsets of the real numbers, with the Markoff spectrum strictly containing the Lagrange spectrum. Indeed, the Lagrange spectrum is the closure of the suprema of the heights of closed geodesics (for which, of course, the limsup and the supremum agree); the Markoff spectrum is the closure of the geodesics which in each direction laminate on some closed geodesic — that is, the corresponding bi-infinite word in the generators T_i of Γ^3 (see §3.1) is eventually periodic in each direction. All of this follows from various results reported in [CF], see in particular Theorems 2 and 3 there, and from the connection between geodesics, words in the group, and continued fractions.

3. Geometry of the Surface

The modular group is $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$. It acts on the Poincaré upper half-plane $\mathcal{H}$ by way of möbius transformations; the modular surface is $\Gamma\backslash\mathcal{H}$. Let Γ' be the commutator subgroup of Γ. Then $\Gamma'\backslash\mathcal{H}$ is a punctured torus, called the *modular (punctured hyperbolic) torus.*

3.1. **Group and Surface Basics.** Due to its simpler geometry, we focus upon a different cover of the modular surface — the quotient of $\Gamma'\backslash\mathcal{H}$ by its elliptic involution. The elliptic involution has three fixed points, the Weierstrass points. The quotient of this hyperbolic surface by this order two automorphism is a hyperbolic orbifold of genus zero with three singularities

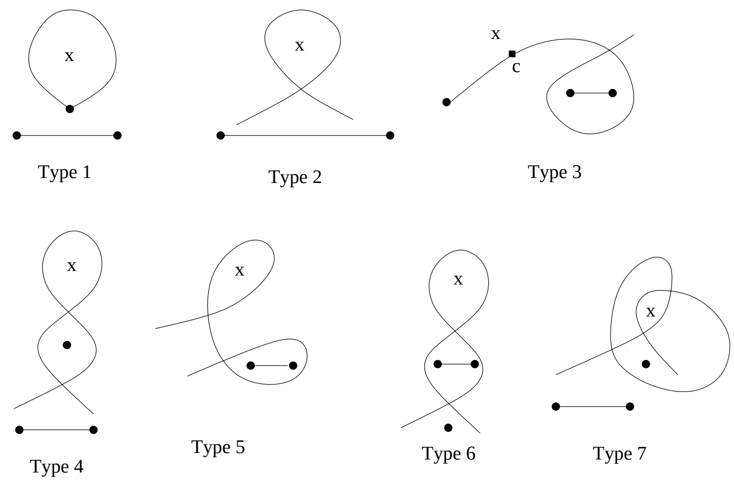

FIGURE 1. The seven types of low height-achieving non-simple geodesics.

and a single puncture. This quotient is $\Gamma^3\backslash\mathcal{H}$ — where, as usual, Γ^3 denotes the subgroup of Γ generated by its cubes — the group Γ^3 thus has signature $(0; 2, 2, 2; \infty)$ and contains Γ' as an index two subgroup (for a discussion of this, see, for example, [Sh]). The singularities are the projections of elliptic fixed points of order two for Γ^3, we refer to them simply as elliptic points. It is traditional to refer to the puncture as the *cusp*. We call all other points of $\Gamma^3\backslash\mathcal{H}$ *regular* points. Each simple closed geodesic on $\Gamma^3\backslash\mathcal{H}$ connects a pair of distinct elliptic points of order two, see [Sh] (especially, Figure 1.3b there); this follows from the fact that each simple closed geodesic of a hyperbolic punctured torus meets two distinct Weierstrass points.

The group of isometries of $\Gamma^3\backslash\mathcal{H}$ is of order six. The order three subgroup of orientation preserving isometries is generated by the element induced by the fundamental translation of Γ, i.e. by the action on $\mathcal{H}$ of $S : w \mapsto w + 1$. Adjoining the standard non-orientation preserving map $w \mapsto -\bar{w}$ then generates the full isometry group.

The element of Γ, $T : w \mapsto -1/w$ is in Γ^3. Indeed, Γ^3 is generated by T_j with $j \in \{0, 1, 2\}$ where $T_j := S^j T S^{-j}$. The fixed points of the T_j project to the three elliptic points of $\Gamma^3\backslash\mathcal{H}$. Note also that $S^3 = T_2T_1T_0$ is in Γ^3 . It is the presence of this translation in the group that ensures that any geodesic on $\Gamma^3\backslash\mathcal{H}$ of height greater than three to have a self-intersection.

3.2. **Seven Types of Geodesics.** The main result of [SS3] is that each height-achieving geodesic of $\Gamma^3\backslash\mathcal{H}$ of height strictly between 3 and 6 has a highest arc of (at least) one of seven types. See Figure 1 for a resume of these types. The explicit definitions are given in Definition 5 of [SS3].

4. Type 1 Geodesics and Markoff Triples

Crisp and Moran [Cr], [CM], [CM2] showed that there are two types of closed single self-intersecting geodesics of $\Gamma'\backslash\mathcal{H}$. One of these has a monogon about the cusp, and is always of large height; the second forms a bigon about the cusp, a geodesic of this type is of height $\sqrt{9 + 4/z^2}$ with integers (x, y, z) a *Markoff triple*: solving $x^2 + y^2 + z^2 = 3xyz$. Crisp and Moran called this second type of geodesics the *Proper Single Self-Intersecting* geodesics, and referred to them as *PSSI*.

Each PSSI of $\Gamma'\backslash\mathcal{H}$ projects to a Type 1 geodesic — this was stated in [SS2], a brief proof follows from the fact that each PSSI meets exactly one Weierstrass point and, in terms of [Sh], has exactly one parabolic intersection; its projection to $\Gamma^3\backslash\mathcal{H}$ forms a path about the cusp, joining an elliptic point to itself. (This phenomenon was already pictured in Figure 1.3 of [Sh].) We sketch a more detailed proof: Crisp and Moran [CM] showed that each PSSI is the projection from $\mathcal{H}$ of the axis of A^2B^2 or of one of its images under the automophisms of Γ', where in the present notation, $A = T_1T_2T_1T_0$ and $B = T_0T_1$. Let $M = S^3T_0S^{-3}T_0$, then $M = B^{-1}A^{-2}B^{-1}$. But, this latter is conjugate to the inverse of A^2B^2. Thus, its axis determines a PSSI. (In fact, this is the highest PSSI, of height $\sqrt{13}/2$.) Since M is the product of two elliptic elements of order two in Γ^3, it is a primitive element of Γ^3. From the form of M, the axis of M projects to $\Gamma^3\backslash\mathcal{H}$ as a geodesic which joins, by a loop about the cusp, the projection of i to itself. One uses a result of Nielsen – the action of $\mathrm{Aut}(\Gamma')$ can be represented by homeomorphisms, see §1.1.1 of [SS2] for discussion – to complete the proof.

Lemma 1. [SS2] *Each Type 1 geodesic of $\Gamma^3\backslash\mathcal{H}$ admits a highest lift on $\mathcal{H}$ that connects the fixed point of an elliptic element of Γ^3 to its own translate by $S^3 : w \mapsto w + 3$.*

The natural ramified double covering of $\Gamma^3\backslash\mathcal{H}$ by $\Gamma'\backslash\mathcal{H}$ induces a 1-to-1 correspondence between the Crisp-Moran PSSI and the Type 1 geodesics. Since we work only with $\Gamma^3\backslash\mathcal{H}$, there should be no confusion in our recognizing the prior work of Crisp and Moran with the following definition.

Definition 1. We call a Type 1 geodesic of $\Gamma^3\backslash\mathcal{H}$ a (Crisp-Moran) *PSSI.*

4.1. **Markoff-indexed Fundamental Domains.** In [SS2], we showed that the PSSI and simple closed geodesic pairing can be expressed purely geometrically: a PSSI is paired with the unique simple closed geodesic which it does not intersect. Furthermore, the high points of a PSSI-simple closed geodesic pair are aligned. That is, the PSSI and its associated simple can be simultaneously lifted to (Euclidean) concentric h-lines. In fact, to each such pair there is a naturally associated fundamental domain of Γ^3. The following two results can be compared with Lemma 2 and Theorem 1 of [SS2].

Theorem 2. (Markoff Equation and Translation) *Suppose that (x, y, z) with integer $z \geq y \geq x > 0$ satisfies Markoff's equation $x^2 + y^2 + z^2 = 3xyz$, and let*

$$E_0 = \begin{pmatrix} 0 & -1/z \\ z & 0 \end{pmatrix}, \quad E_1 = \begin{pmatrix} x/z & * \\ y & -x/z \end{pmatrix}, \quad E_2 = \begin{pmatrix} 3x - y/z & * \\ x & -3x + y/z \end{pmatrix},$$

where the $*$-entries are determined by the determinant condition for elements of $SL(2,\mathbb{R})$. Then there is a rational translation, $M \in SL(2,\mathbb{Q})$ such that

(1) *The image under M of the axis of $E_0S^3E_0S^{-3}$ is a highest lift of a PSSI;*
(2) *The image under M of the axis of $S^3E_0 = -E_2E_1$ is a highest lift of the paired simple closed geodesic;*
(3) *With e_i the fixed point of the corresponding E_i, the h-polygon of vertices ∞, Me_0, Me_1, ME_1e_0, Me_2, MS^3e_0 is a fundamental domain for Γ^3 acting on $\mathcal{H}$.*

Proof. Note that each of the E_i is of order two, and that $E_2E_1E_0 = S^3$. One easily checks that $E_1e_0 = E_2S^3e_0$; furthermore, the real parts of the e_i increase with i. Thus, the h-polygon of vertices ∞, e_0, e_1, E_1e_0, e_2, S^3e_0 has sides paired by S^3, E_1 and E_2. Therefore, this h-polygon is a fundamental domain for the action of the Fuchsian group of signature $(0; 2, 2, 2, \infty)$ generated by the E_i.

To finish the proof of (3), it is sufficient to show that there is a translation M conjugating each of the E_i into $\mathrm{PSL}(2,\mathbb{Z})$, thus to show that the conjugates have integer entries. This is a matter of using well-know properties of Markoff triples, as we now sketch.

The integers x, y, z are pair-wise relatively prime. There is thus an integer k solving $ky + x \equiv 0 \bmod z$. Again by relative primality, and Markoff's equation, one finds that $k^2 + 1 \equiv 0 \bmod z$. Consider the translation $M : w \mapsto w + k/z$. It is trivial to check that M conjugates E_0 into $\mathrm{PSL}(2,\mathbb{Z})$.

Conjugating E_1 by M, one immediately sees that all but possibly the resulting $(1,2)$-entry are integral. However, this remaining entry can be expressed as $-[(x+ky)^2 - z^2]/yz^2$; its numerator is certainly divisible by z^2. An application of the Markoff equation shows that the numerator is also divisible by y. By relative primality, this entry too is integral. Thus, both E_0 and E_1 are conjugated by M into $\mathrm{PSL}(2,\mathbb{Z})$; since S^3 certainly is also so conjugated, we deduce that E_2 is as well.

For item (1), note first that the axis of $MES^3E_0S^{-3}M^{-1}$ clearly passes through Me_0 and $3 + Me_0$. Since the polygon is indeed a fundamental domain, one sees that the h-line segment joining these points projects to a PSSI. Furthermore, the h-polygon has its single cusp at infinity, from this one sees that our h-line segment is indeed a highest lift.

Similar to the above, the h-line segment joining Me_1 and Me_2 gives a highest lift of a simple closed geodesic that does not meet the PSSI. Thus, (2) is also true. □

Theorem 3. *If γ is a PSSI, then there is a Markoff triple (x, y, z) such that γ is, up to isometry of $\Gamma^3\backslash\mathcal{H}$, the PSSI of Theorem 2.*

Proof. Let e denote an elliptic fixed point such that the h-line joining e and $3+e$ gives a highest lift of γ. Then the vertical line segments from each of e and $3+e$ to ∞ project to a single simple cusped geodesic.

Since γ and its paired simple closed geodesic do not intersect, we can draw a simple curve, our marking curve, on $\Gamma^3\backslash\mathcal{H}$ that joins their highest points but intersects them nowhere else. We can then join e to each endpoint of the paired simple closed geodesic with a simple curve, unique up to homotopy, that does not meet the marking curve. There are simple closed geodesics in each of these classes, the companions. We now can cut open $\Gamma^3\backslash\mathcal{H}$ along the cusped geodesic, the paired simple closed geodesic, and the companions. We find two components, with the compact component being a triangle of sides the paired simple closed geodesic and its companions.

We can lift the non-compact component to an h-polygon, of vertices (listed counter-clockwise) ∞, e, f, g and $e+3$, where f and g are endpoints of a highest lift of the paired simple closed geodesic. (See (the left part of) Figure 6 of § 6, or Figure 1 of [SS2].) The other non-vertical sides are lifts of each of the companion closed geodesics. There are elements order two elements $E, F, G \in \Gamma^3$ fixing respectively e, f and g.

Not only is the h-line segment joining e to f a lift of one of the companion closed geodesics, but so also is the h-line segment joining f and Fe. Similarly, both $(g, 3+e)$ and (GS^3e, g) are pairs of endpoints of h-line segments projecting to the second companion. But, the compact triangle lifts to an h-triangle with one side being the h-line segment of endpoints (f, g). Its other two sides are lifts of the companions. We conclude that $Fe = GS^3e$. Since the only non-trivial element of Γ^3 that fixes e is E, we find that $GF = S^3E$. We now have that the h-polygon of vertices ∞, e, Fe, g, $3+e$ is a fundamental domain for Γ', with side pairings by S^3, F and G.

It now suffices to prove that (x, y, z) solves Markoff's equation and $x \le y \le z$. After possibly applying an isometry of the form $w \mapsto w + 3n$, we may assume that the real part of each of e, f, g is positive. With our chosen labeling, we can write $e = (a+i)/z$, $f = (a'+i)/y$ and $g = (a''+i)/x$, with $0 < a/z < a'/y < a''/x$. We further normalize by an application of the isometry $w \mapsto -\bar{w}$ (and a translation) as necessary so that $y \ge x$.

Since each of E, F and G is of order two in $\mathrm{PSL}(2,\mathbb{Z})$, each is of trace zero. Expressing E as $\begin{pmatrix} a & * \\ z & -a \end{pmatrix}$ and F and G similarly, the equation (as elements of $\mathrm{SL}(2,\mathbb{R})$!) $GF = -S^3E$ leads to

$$\begin{aligned} a'x - a''y &= -z \\ x(a'^2+1) - a'a''y &= ay \\ a'a''x - (a''^2+1)y &= -3xz - ax\ . \end{aligned}$$

Elementary manipulations then show that x, y, z satisfy Markoff's equation. (And, we are done if $x = y = z = 1$.)

Now, for f, g to be terminal endpoints of a highest lift, they must lie on opposite sides of the high point of the axis of GF. In particular, we find $3/2 < 3 - y/xz$ (compare this latter with the fixed point of E_2). Thus, $2y < 3xz$. But, for a Markoff triple with $\max\{x, y, z\} > 1$, the quadratic formula shows that twice this maximum is greater than three times the product of the other two values. We conclude that our triple is in standard form: $x \le y \le z$. $\square$

Definition 2. We refer to a hexagonal fundamental domain of vertices ∞, $e, f, F(e), g, e+3$, as above as a *Markoff-indexed* fundamental domain. The simple closed geodesic paired with the PSSI is called the *parameter simple closed geodesic*. The Markoff-indexed fundamental domain is the union of: the *high triangle*, of vertices ∞, e and $3+e$; the *basic quadrangle*, of vertices e, f, g and $e+3$; and the *low triangle*, of vertices f, g and $F(e)$. The projections of the sides joining respectively e to f and g to $3+e$ are the *companion* simple closed geodesics.

4.2. **Inflation.** To further simplify calculations and to standardize arguments, we can also inflate. Although this inflation changes heights, it preserves relative sizes. The following lemma is easily verified. Recall that $T : w \mapsto -1/w$.

Lemma 2. *Let (x, y, z) with integer $z \ge y \ge x > 0$ be a solution to Markoff's equation $x^2 + y^2 + z^2 = 3xyz$, the matrices E_i be as above, and let Θ_z be the multiplication-by-z map on $\mathcal{H}$: $\Theta_z(w) = zw$. Then Θ_z conjugates the linear fractional transformations E_0 to T and S^3 to S^{3z}. Furthermore, if the conjugates by Θ_z of E_1 and E_2 are respectively denoted by $\tilde{E}_1$ and $\tilde{E}_2$, then $\tilde{E}_2\tilde{E}_1 = S^{3z}T$.*

An advantage of the use of the inflated-and-translated version of our fundamental domains is the simple form of the element $S^{3z}T$. Indeed, $S^{3z}T = \begin{pmatrix} 3z & -1 \\ 1 & 0 \end{pmatrix}$. The following is also trivially checked.

Lemma 3. *With notation as above, let $U = S^{3z}T$ and $Ax(U)$ denote the axis of U. Let u_- be the left foot of $Ax(U)$, and u_+ be the right foot. Then U is repulsive from u_- and attractive to u_+. Furthermore, the parameter simple closed geodesic of the Markoff-indexed fundamental domain of $\Gamma^3\backslash\mathcal{H}$ associated to the Markoff triple (x, y, z) is the projection of the image of $Ax(U)$ under $M^{-1}_{(x,y,z)} \circ \Theta_z^{-1}$, where $M_{(x,y,z)}$ is the Euclidean translation of Theorem 2.*

4.3. **Rays on the Surface.** The use of our Markoff-indexed fundamental domains allows us to identify a simple subray of any ray of $\Gamma^3\backslash\mathcal{H}$ that spins about a simple closed geodesic. Recall that the *basic quadrangle* is defined in Definition 2.

Lemma 4. *Fix some Markoff-indexed fundamental domain $\mathcal{F}$. Suppose that ℓ is any h-line that: (1) shares left foot with the h-line passing through the vertices f and g, where these vertices are expressed in our standard notation; and, (2) meets $\mathcal{F}$ non-trivially, with intersection contained in the basic quadrangle of $\mathcal{F}$. Orient ℓ such that its left foot is its positive end, and let p be the first point of intersection of ℓ with the closure of the basic quadrangle. Let σ be the projection* *of the subray of ℓ beginning at p. Then σ is a simple geodesic ray of $\Gamma^3\backslash\mathcal{H}$ that spins about the parameter simple closed geodesic for $\mathcal{F}$.*

Proof. Since the h-line passing through f and g projects to the parameter simple closed geodesic of $\mathcal{F}$, it is clear that σ spins about this geodesic of $\Gamma^3\backslash\mathcal{H}$. It remains to show that σ is itself simple. For simplicity of discussion, we translate and inflate; let (x, y, z) be the Markoff triple associated to the Markoff-indexed $\mathcal{F}$, translate by $M_{(x,y,z)}$ and apply the map Θ_z. We use a tilde to denote images under this translation-and-inflation process.

We find that $\tilde{\ell}$ has left foot u_- . The right foot of $\tilde{\ell}$ is greater than u_+. Therefore, $U(\tilde{\ell})$ has left foot u_- , and the right foot of $U(\tilde{\ell})$ lies closer to u_+ than does that of $\tilde{\ell}$. Since $\tilde{\ell}$ lies above $\mathrm{Ax}(U)$, and of course U acts so as to fix its own axis, we conclude that $U(\tilde{\ell})$ lies strictly between $\tilde{\ell}$ and $\mathrm{Ax}(U)$. Iterate: $U^n(\tilde{\ell})$ with $n \geq 0$ is a nested sequence of h-lines, converging from above to $\mathrm{Ax}(U)$; pairs of elements of this sequence are mutually disjoint in $\mathcal{H}$. In particular the intersection of this sequence with the inflated basic quadrangle is certainly simple.

Now, $\tilde{E}_1$ fixes the axis of U, interchanges u_- and u_+, and sends the side of the inflated basic quadrangle through which $\tilde{\ell}$ exits to the side of vertices $\tilde{f}$ and $\tilde{E}_1(i) = \tilde{F(e)}$. We conclude that $\tilde{E}_1 U^m(\tilde{\ell})$ is also a sequence of mutually disjoint h-lines, converging to $\mathrm{Ax}(U)$ from below and such that each $\tilde{E}_1 U^m(\tilde{\ell})$ meets $\mathcal{F}$ non-trivially, with intersection contained in the translated-and-inflated low triangle.

Since $U = \tilde{E}_2\tilde{E}_1$, we have accounted for all lifts to the inflated fundamental domain of the h-ray beginning at $\tilde{p}$ and lying on $\tilde{\ell}$. We deflate and translate to conclude that σ projects to a simple geodesic ray of $\Gamma^3\backslash\mathcal{H}$. □

Remark 1. Of course, the above lemma holds upon replacement of the word *left* by the word *right*. We apply the lemma in both situations, without further ado.

We now show that the above construction identifies explicit ends for any simple ray spinning about a simple closed geodesic.

Lemma 5. *Suppose that ρ is a simple geodesic ray of $\Gamma^3\backslash\mathcal{H}$ that spins about a simple closed geodesic, and that $\mathcal{F}$ is the Markoff-indexed fundamental domain indexed by the PSSI paired with this simple closed geodesic. Then, ρ admits a lift ℓ to $\mathcal{H}$ satisfying the hypotheses of Lemma 4.*

Proof. By applying an appropriate element of Γ^3, there is a lift of ρ whose positive end is one of the feet of the h-line passing through the points f and g of $\mathcal{F}$, with our standard notation. For ease of discussion, we apply our process of inflation and translation. By symmetry of argument to follow, we assume that the image of this lift, say $\tilde{\ell}$, has positive end at u_+.

We claim that there is some $m \in \mathbb{Z}$ such that $U^m(\,\tilde{\ell}\,)$ meets the inflated, translated, fundamental domain only in its basic quadrangle. For this, let α denote the remaining foot of $\tilde{\ell}$. If $\alpha < u_-$, then the orbit of α under negative powers of U converges to u_- . Now, let w be the left foot of the h-line passing through i and u_+; clearly, $w < 0 < u_-$. There is thus some $m < 0$ such that $w < U^m(\alpha) < u_-$. But, $U^m(\,\tilde{\ell}\,)$ then meets the fundamental domain in exactly its basic quadrangle.

If $\alpha > u_+$, we again apply negative powers of U! If in fact $\alpha \geq 3z$, then $U^{-1}(\alpha) \in [\,-\infty, 0\,)$; if $\alpha \in (\,u_+, 3z)$, then $U^{-1}(\alpha) \in (\,u_+, \infty\,)$. Since there is no fixed point of U^{-1} greater than u_+, there is some $n \in \mathbb{N}$ such that $U^{-n}(\alpha) < u_-$. But, we have already seen that our conclusion holds in this situation.

Finally, if $\alpha \in (\,u_-\,, u_+\,)$, then $\tilde{E}_1(\,\tilde{\ell}\,)$ is an h-line with positive end at u_- and remaining foot, $\tilde{E}_1(\alpha) \notin (\,u_-, u_+\,)$. By symmetry of argument of the previous two paragraphs, we conclude that there is some power of U effecting the required transformation. □

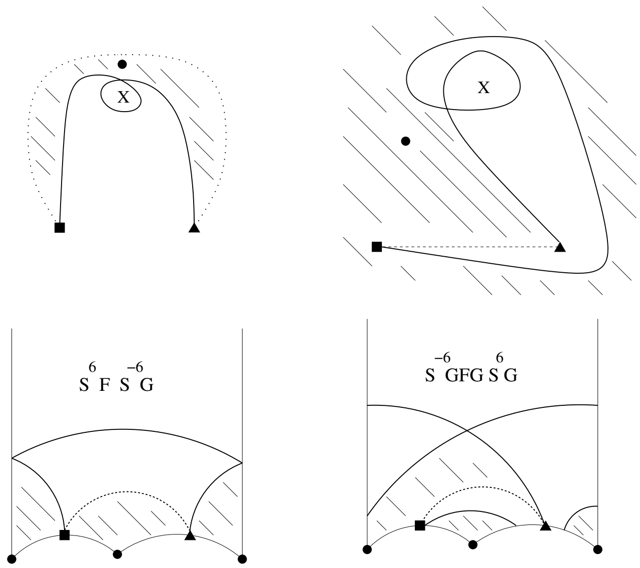

FIGURE 2. Two of three closed Type 2 geodesics with same parameter simple closed geodesic.

5. Type 2 Geodesics

5.1. **Type 2 Closed Geodesics.** Given a closed Type 2 geodesic γ, one can easily draw a simple curve connecting its terminal elliptic points without meeting γ elsewhere. In fact, up to homotopy (as always, relative to the three elliptic points), there are two classes of such curves. In the next definition, we specify one of these choices. Recall from [SS3] (Definition 1) that when a geodesic includes a monogon about the cusp, then we call the rays emanating away from the monogon along the path of the geodesic its *arms*.

Definition 3. If γ is a Type 2 closed geodesic then we define its *parameter simple closed geodesic* to be the simple closed geodesic which joins the end points of γ without further intersection and such that the (compact) triangle bounded by the arms of γ and this new geodesic contains the third elliptic point.

Proposition 1. *If γ is a closed Type 2 geodesic of height less than h=4.5, then there is a unique Markoff triple (x, y, z) such that, up to an application of an isometry, γ is the projection of the axis of*

$$M_{(x,y,z)}S^{-6}E_2S^6E_1M^{-1}_{(x,y,z)}\,,$$

where $M_{(x,y,z)}$ effects the rational translation of Theorem 2. The height of γ is the Euclidean diameter of this axis,

$$\frac{\sqrt{81(4xy-z)^2-4}}{6xy-z}\,.$$

Proof. By definition, (1) γ meets its parameter simple closed geodesic only in its elliptic points, and (2) the arms of γ enclose a compact triangle that contains the third elliptic point. We claim that there are exactly three closed Type 2 geodesics satisfying these criteria for any fixed simple closed geodesic as parameter. We will show that exactly one of these can possibly be of height less than 4.5, and that it does arise as claimed.

Indeed, to each isometry class of closed Type 2 geodesic γ and parameter simple closed geodesic (and thus paired PSSI) there is an associated Markoff triple (x, y, z) and its Markoff-indexed fundamental domain, $\mathcal{F}$, such that highest lifts of the parameter simple closed geodesic and its paired PSSI are (given in terms of the notation of Definition 2) by h-line segments joining f to g and e to $3 + e$.

Now, a case-by-case study of sequences of segments in $\mathcal{F}$ (with the sides of $\mathcal{F}$ identified as usual) joining g to f while satisfying (1) and (2) is simplified by the fact that the initial and ultimate segments cannot cross. Indeed, such a crossing completes a triangle with the parameter simple closed geodesic that does not contain the third elliptic point. Due to symmetry, one can first consider cases where the initial segment does not enter the lower triangle of $\mathcal{F}$. Case-by-case arguments show that exactly two possibilities are so found: the projections of the axes of $S^6FS^{-6}G$ and $S^{-6}GFGS^6F$, depicted in Figure 2. By symmetry, we also have that the axis of $S^6FGFS^{-6}F$ projects to a Type 2 closed geodesic satisfying our criteria. Note that each of these is a highest lift of its projected geodesic.

By use of translation, we have straightforward evaluations of matrix products, from which the formula for the Euclidean diameter of the axis of a hyperbolic matrix, and applications of the Markoff equation give formulas for the heights of these geodesics. In particular, the axis of $S^6FS^{-6}G$ has Euclidean diameter given in the statement of the Theorem. Finally, it is easily shown that the axis of each of $S^{-6}GFGS^6F$ and $S^6FGFS^{-6}F$, for any Markoff triple, is of Euclidean diameter greater than 4.5 . $\square$

Remark 2. In fact, all closed Type 2 geodesics are in the same orbit under the automorphisms of $\Gamma^3\backslash\mathcal{H}$ (see Proposition 7.5 of [CrEtAl] for this). Thus, already in this setting, we have explicit examples underlying the fact (in terms used by [CM]) that although the Markoff triples enumerate coset representatives of automorphisms modulo isometries, they do not list all such cosets — only (exactly) enough to enumerate the isometry classes of simple closed geodesics (and therefore of the paired PSSI).

5.2. **Type 2 Open Pointed Geodesics.** As in [SS3], call a geodesic *pointed* if it meets an elliptic point. A pointed open Type 2 geodesic must meet exactly one elliptic point, as any geodesic meeting two elliptic points is closed. Thus of its two arms emanating (as usual, up to orientation) from its single point of self-intersection, one has its path terminating at an elliptic point, the other is an infinite simple ray. Let us refer to any geodesic of height at most 4.5 as *low*.

Lemma 6. *If γ is a low pointed open Type 2 geodesic whose infinite arm spins about a simple closed geodesic, then for some Markoff triple (x, y, z) the height of γ is*

$$\frac{9}{2}\,\frac{36z^2 + 4 - 4z\sqrt{9z^2 - 4}}{36z^2 + 2}\,.$$

Proof. Since γ forms a monogon about the cusp, a consideration of lifts shows that it has a unique highest point. Consider the two rays of opposite direction along the path of γ that emanate from this point. Each ray is simple, let ρ denote the infinite simple ray. By assumption, ρ spins about some simple closed geodesic. Now, the monogon and the finite arm of γ form a rho. Just as a Type 3 arc has its parameter simple geodesic, so does this rho have a unique simple closed geodesic which does not meet it. But, if ρ spun about any other simple closed geodesic, it would meet this finite rho configuration (infinitely many times). Therefore, ρ does spin about this parameter simple closed geodesic. We fix $\mathcal{F}$, the Markoff-indexed fundamental domain associated to the parameter simple closed geodesic. By Lemma 5 there is a first lift, say λ_1, of ρ that is completely contained in the fundamental quadrangle of $\mathcal{F}$. Let p be the initial point of the intersection of this lift with the closure of the basic quadrangle; let σ be the subray of γ that lifts to begin at p.

Parametrize the path of γ as emanating from its terminal elliptic point; we follow the corresponding sequence of lifts to $\mathcal{F}$. Note that γ avoids the simple closed geodesic about which ρ spins, as otherwise γ has many more than just one point of self-intersection; thus, all lifts of γ avoid the corresponding side of the low triangle of $\mathcal{F}$. In particular, the initial lift of γ to $\mathcal{F}$ cannot begin at the vertex of the low triangle lifting the elliptic point of γ. For ease of discussion, we treat only the case of this first lift beginning at, in our usual notation, e; the other case is symmetric.

There are two initial possibilities: (1) the first lift lies in the high triangle of $\mathcal{F}$, or (2) this first lift lies in the basic quadrangle of $\mathcal{F}$. In case (2), this lift then ends on the side of vertices g and $e+3$; the second lift lies in the low triangle. Our parametrization of γ is such that upon reaching the point of self–intersection, a cusped disk is formed. If the third lift meets the first lift, we have a contradiction: a point of self-intersection is found before a cusped disk is formed. Otherwise, a full wrap about the lift of the simple closed geodesic is formed; but, some further lift must intersect this, or else there can be no cusped disk form. Again, a self-intersection occurs too early. We conclude that case (2) is void.

In case (1), the first lift ends on the vertical side given by $x = 3 + \Re(e)$, the second lift begins on the opposite vertical side. Were this second lift to continue to the opposite vertical side, then some further lift would cross both of these first two lifts, creating too many self-intersections. Thus, this second intersects the first. Since γ is a low geodesic, this second lift cannot intersect the mid-line of $\mathcal{F}$; nor can it exit on the side of vertices f and $F(e)$, for then the next left will cause a new point of intersection. Thus, it must meet the first lift and proceed so as to end on the edge of vertices e and f. The third lift lies in the low triangle. If the fourth lift exits the basic quadrangle, then it ends at one of the vertical sides of the high triangle. However, this would entail a new point of intersection being created. Since γ has only one point of self-intersection, the fourth lift lies completely in the basic quadrangle. We conclude that this fourth lift is λ_1. We have completely determined the sequence of lifts of γ; note that the highest point lifts to lie on the first or second lift.

For ease of computation, we now apply our translation-and-inflation process. In the case that we are explicitly treating, the highest lift then gives the h-line passing through i and $3z+u_-$. The height is then easily checked. Analogous arguments apply to the case not explicitly treated. $\square$

5.3. **Unpointed Type 2 Open Geodesics.**

Lemma 7. *If γ is an unpointed open Type 2 geodesic whose infinite arms spin about simple closed geodesics, then γ is of height 6. More precisely, there is some Markoff triple (x, y, z) such that, up to isometry, the geodesic γ is the projection of the translate given by $M^{-1}_{(x,y,z)}$ of the h-line of feet $(-3 - \sqrt{9 - 4/z^2})/2$ and $(9 - \sqrt{9 - 4/z^2})/2$.*

Proof. Let c be the highest point of γ, we consider the two rays emanating from c that γ defines. By hypothesis, each spins about some simple closed geodesic. But, if these simple rays spin about distinct simple closed geodesics, then it is easily seen that they intersect infinitely many times. Thus, they must spin about the same simple closed geodesic.

We fix $\mathcal{F}$, the Markoff-indexed fundamental domain associated to the simple closed geodesic about which the rays spin. Each ray admits a first, *distinguished*, lift contained in the basic quadrangle and lying on an h-line ending in a foot of the h-line passing through e and f, in our standard notation. If these end in distinct feet, then the spinning of the rays is with opposite

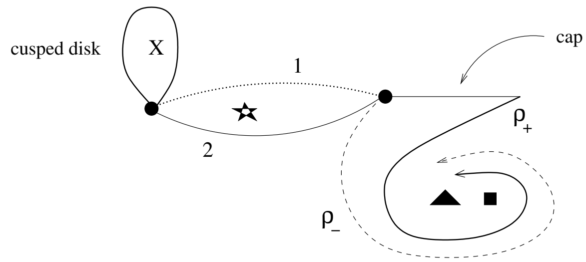

FIGURE 3. Joining cusped disk to capped rays.

orientation: that is, with oppositely signed winding number about (any fixed point on) the simple closed geodesic. However, two oppositely oriented rays spinning about a common simple closed geodesic intersect infinitely many times. Thus, these lifts share one foot.

We now show that only one Type 2 geodesic can give two rays with distinguished lifts of the same foot as above. Consider a cusped disk, with marked node point and two rays spinning about a simple closed geodesic. We may temporarily 'cap' the rays to join them so that they meet at a non-special point and without having any other point of intersection, see Figure 3. Up to homotopy fixing endpoints, there are exactly two distinct curves joining the node of the cusped disk to the temporary node of capped rays while avoiding the rays and the cusped disk; we make take non-self-intersecting representatives that do not intersect other than at their end points. Hence, up to homotopy, there are also two ways of completing our (uncapped) rays to meet the disk at the marked point. But, it is easily checked that one of these actually results in a replacement of γ by a simple geodesic. Therefore, there is exactly one free homotopy class of our Type 2 description.

We next explicitly give the rays. We again employ the translated-and-inflated model. Let $\alpha = U(\,6z + u_-\,)$ and $\beta = U(\,-(6z + u_-)\,)$. Let ℓ be the h-line of left foot u_- and right foot α; let m be the h-line of left foot u_- and right foot β. One checks that

$$u_+ < \alpha < 3z < \beta < 3z + u_- \,.$$

Thus, for all $n \in \mathbb{N}$, $u_+ < U^n(\,\alpha\,) < U^n(\,\beta\,) < 3z + u_-$; since U^n fixes u_-, we conclude that ℓ and m meet the translated-and-inflated $\mathcal{F}$ to give distinguished initial lifts of spinning for their respective rays. Applying U^{-1} to each, we find h-lines of foot u_-, but $U^{-1}(\,\beta\,) < 0$; $U^{-1}(\,\alpha\,) > 3z + u_-$. Thus, corresponding lifts of the rays enter the domain from the left vertical side, or from the right vertical side, respectively. Indeed, this previous pair of lifts is given by $S^{3z}U^{-1}(\,m\,)$ and $S^{-3z}U^{-1}(\,\ell\,)$. But, $S^{3z}U^{-1}(\,\beta\,) = -3z + u_-$ and $S^{3z}U^{-1}(\,u_-\,) = 3z + u_-$; whereas $S^{-3z}U^{-1}(\,\alpha\,) = 3z + u_-$ and $S^{-3z}U^{-1}(\,u_-\,) = -3z + u_-$. We have found the same h-line, but with orientation reversed! See Figure 4. One now easily checks that this common h-line gives the highest (translated-and-inflated) lift of our Type 2 unpointed geodesic. Checking feet and height, one finds the first case of our Lemma verified. The other case is that of translated-and-inflated rays having common foot u_+; analogous arguments apply. But, one finds that this gives a lift that is the image of the first under a composition of $w \mapsto -\bar{w}$ and $w \mapsto w + 3$. $\square$

6. Type 3 Geodesics

Type 3 geodesics are characterized by a description of a finite length arc. Thus, this threatens to be a very large class of geodesics. We were surprised to see that in fact, all Type 3 geodesics

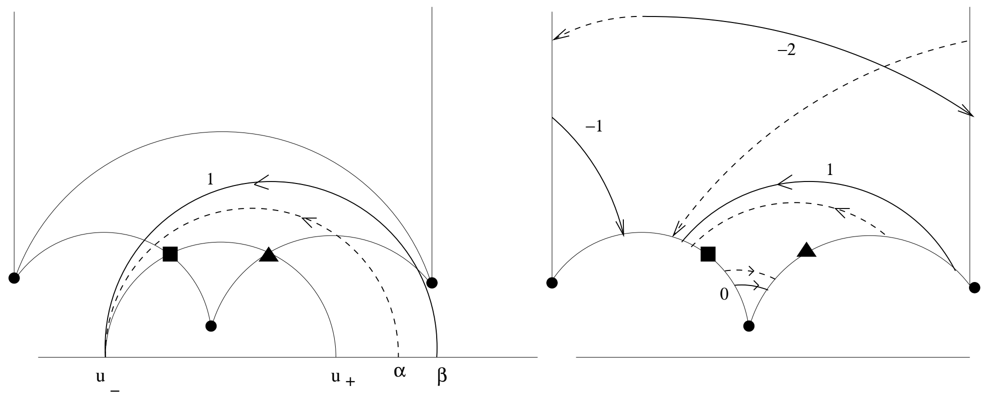

FIGURE 4. Opposite oriented rays of unpointed Type 2 geodesic. Working backwards to find common lift.

can be presented in a compact manner, given in Proposition 2. This and Lemma 9 imply Theorem 1.

We first treat arbitrary arcs that are of Type 3.

Lemma 8. *If ρ is a Type 3 geodesic arc, then there is a Markoff-indexed fundamental region of $\Gamma^3\backslash\mathcal{H}$ such that a highest lift of ρ lies in the basic quadrangle of this fundamental region.*

Proof. By definition, ρ emanates from an elliptic point and forms a high arc followed by a monogon about the remaining two elliptic points. The parameter simple closed geodesic of ρ, as given in Definition 6 of [SS3], joins these elliptic points, without intersecting ρ. This simple closed geodesic has a unique paired PSSI, and thus we find a uniquely associated Markoff-indexed fundamental region, $\mathcal{F}$, of $\Gamma^3\backslash\mathcal{H}$. We fix this region, and label the vertices in our standard manner; again, see Figure 6. All lifts will now be considered as lying in $\mathcal{F}$.

There is a lift of ρ emanating from one of e, $F(e)$, or $e+3$. But, as in the since ρ does not meet its parameter simple closed geodesic, there is no lift of ρ emanating from $F(e)$. The cases of an initial lift emanating from e or from $e+3$ are symmetric with respect to the following arguments; we thus suppose that an initial lift emanates from $e+3$.

We claim that our lift of ρ lies below the h-line arc joining $e+3$ and e. For if not, then the subsequent lift of ρ lies on the h-line that is the translation by 3 of the first lift; the two lifts intersect in $\mathcal{F}$, above the h-line arc joining $e+3$ and e. In particular, the projection of these lifts form a monogon about the cusp; this is not the geometry of ρ.

Similarly, since ρ does not meet the parameter simple closed geodesic, our initial lift of ρ lies above the h-line arc joining f and g. Thus, this lift lies in the basic quadrangle. It exits on the edge of vertices e and f.

The subsequent lift of ρ thus begins on the edge of vertices f and $F(e)$. By arguments as above, this lift exits $\mathcal{F}$ through the edge of vertices $F(e)$ and g. The third lift thus begins on the side of vertices g and $e+3$. If this third lift intersects the initial lift, then a lift of ρ is completed; clearly, the initial lift is then the highest lift of ρ. If there is no intersection of the initial lift and this third lift, then subsequent lifts must continue to encircle the h-line arc joining f and g — all of these lying lower than the initial lift —, until some lift (beginning on one of the sides of the basic quandrangle) meets an earlier lift, so that the projection completes ρ. In all cases, the initial lift is indeed the highest lift of ρ. □

Proposition 2. *A geodesic γ is of Type 3 if and only if there is some $n \in \mathbb{N}$ and some Markoff triple (x, y, z) such that γ is the projection of the axis of*

$$G_n := G_n(x, y, z) = M_{(x,y,z)}(E_1E_2)^n E_0 (E_2E_1)^n S^3 E_0 S^{-3} M^{-1}_{x,y,z}\,,$$

where $M_{(x,y,z)}$ and the E_i are as in Theorem 2.

The proof of Proposition 2 invokes the following two lemmas.

Lemma 9. *Let $U = S^{3z}T$ and let $\tilde{G}_n = U^{-n}TU^nS^{3z}TS^{-3z}$. Then the axis of $\tilde{G}_n$ is of diameter*

$$h(\tilde{G}_n) = \sqrt{9z^2 + 4/a_n^2}\,,$$

where $a_n = a_n(z)$ is as in Theorem 1. Furthermore, the image of this axis under the action of U^n is an h-line of the same diameter passing through i.

Proof. We have $U = \begin{pmatrix} 3z & -1 \\ 1 & 0 \end{pmatrix}$; it easily follows that $U^n = \begin{pmatrix} a_n & -a_{n-1} \\ a_{n-1} & -a_{n-2} \end{pmatrix}$. Applying the defining recurrence relation to the determinant of U^n, one finds that $1 + 3za_{n-1}a_{n-2} = a_{n-1}^2 + a_{n-2}^2$.

Note that if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R})$ then

$$A^{-1}TA = \begin{pmatrix} -(ab+cd) & -(b^2+d^2) \\ a^2+c^2 & ab+cd \end{pmatrix}.$$

Thus, by using the above identities, one finds

$$U^{-n}TU^n = \begin{pmatrix} 3za_{n-1}^2 & -(1+3za_{n-1}a_{n-2}) \\ 1+3za_na_{n-1} & -3za_{n-1}^2 \end{pmatrix}.$$

Also,

$$S^{3z}TS^{-3z} = \begin{pmatrix} 3z & -(1+9z^2) \\ 1 & -3z \end{pmatrix}.$$

Therefore, one finds the 2,1-element of $\tilde{G}_n$ to be

$$c(\tilde{G}_n) = 3za_n^2$$

and the trace to be

$$\begin{aligned} \mathrm{tr}(\tilde{G}_n) &= -2 - 9z^2 + 18z^2a_{n-1}^2 - 27z^3a_na_{n-1} - 3z(a_na_{n-1} + a_{n-1}a_{n-2}) \\ &= -2 - 9z^2a_n^2\,. \end{aligned}$$

Recall that if $A \in \mathrm{SL}(2, \mathbb{R})$ is hyperbolic, then the diameter of the axis of A is the height, $\frac{\sqrt{\mathrm{tr}(A)^2-4}}{|c(A)|}$, where $c(A)$ and $\mathrm{tr}(A)$ are the $2,1$-element and trace of A, respectively. The diameter of the axis of $\tilde{G}_n$ is thus easily computed to be as announced.

Now consider $M = U^n\tilde{G}_nU^{-n}$. The trace of M is obviously equal to the trace of $\tilde{G}_n$. Expanding, simplifying and then replacing T by the equivalent T^3, one finds $M = TU^{n+1}TU^{-(n+1)}$. That is,

$$M = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_na_{n-1} + a_na_{n+1} & * \\ * & * \end{pmatrix}.$$

We find that the 2,1-element is in fact $c(M) = 3za_n^2$.

Of course, U^n sends the axis of $\tilde{G}_n$ to that of M. These axes have the same diameters, as the traces and $2,1$-elements of the corresponding matrices are equal. From $M = TU^{n+1}TU^{-(n+1)}$, it follows that the axis of M passes through the fixed point of the elliptic of order two $U^{n+1}TU^{-(n+1)}$, as well as through the fixed point of T: the point i. $\square$

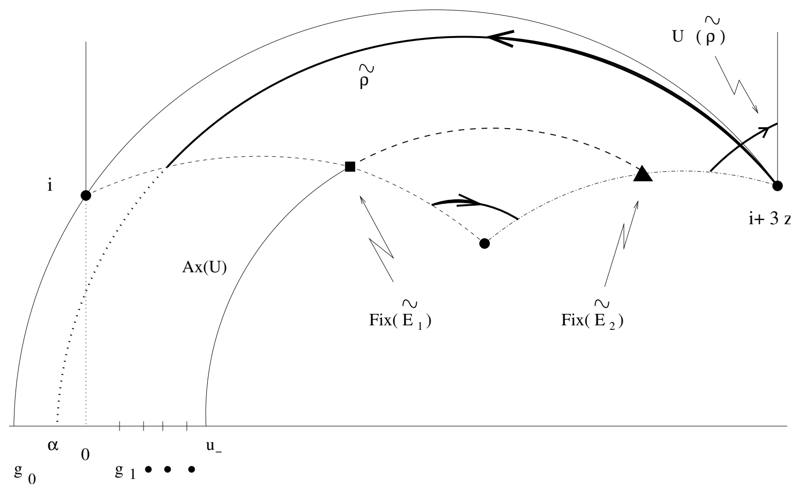

FIGURE 5. A Type 3 arc, translated and inflated: $\tilde{\rho}$.

Lemma 10. *Let ℓ be the h-line of feet α and β. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$. Then the image of ℓ under A is of larger Euclidean diameter than ℓ itself if and only if*

$$|\, d^2 + c^2\alpha\beta + cd(\alpha+\beta)\,| < 1$$

.

Proof. The elements of $\mathrm{SL}(2,\mathbb{R})$ fixing ℓ are

$$V_{\alpha,\beta,t} = \frac{1}{\alpha-\beta}\begin{pmatrix} \alpha/t - \beta t & \alpha\beta(t-1/t) \\ -t+1/t & \alpha t - \beta/t \end{pmatrix},$$

for t any nonzero real. Thus, each $AV_{\alpha,\beta,t}A^{-1}$ fixes the image of ℓ by A.

Since the trace is invariant under conjugation, we need only compare the $2,1$-element of $V_{\alpha,\beta,t}$ with that of $AV_{\alpha,\beta,t}A^{-1}$. This latter is $d^2 + c^2\alpha\beta + cd(\alpha+\beta)$ times the former. The result follows. □

Proof of Proposition 2 (stated on page 14): Just as in the proof of Lemma 8, there is a symmetry between the cases of γ lifting to begin at an elliptic point e or $e+3$ of its associated Markoff-indexed fundamental domain, $\mathcal{F}$. We thus let ρ be a Type 3 geodesic subarc of γ that exactly realizes its type, and assume that we are in the case treated explicitly in the proof of Lemma 8. Let (x,y,z) be the Markoff triple, and $E_i = E_i(x,y,z)$ for $i \in \{0,1,2\}$ the elliptic elements of order two, associated to $\mathcal{F}$. We show that $(E_2E_1)^n$ sends the initial lift of ρ to a higher lift unless this initial lift lies on the axis of some $G_n(x,y,z)$.

We apply the translation-and-inflation process. Denote the image of the initial lift of ρ as $\tilde{\rho}$; this h-line arc thus emanates from $i+3z$, see Figure 5. Call the left foot of the h-line it defines α and the right foot β.

The axis of U contains the highest lift of the (translated and inflated) parameter simple geodesic. The geometric action of $U = \tilde{E}_2\tilde{E}_1$ is to undo counter-clockwise wraps, as these take place about the h-line arc joining the fixed points of the elliptic elements of order two $\tilde{E}_2$ and $\tilde{E}_1$. The action of U induces an increasing, injective, function on the reals, with unique pole at $x=0$.

As in the proof of Lemma 8, the consecutive lifts of ρ wrap about the lift of its associated simple closed geodesic some number of times, until the monogon about this simple geodesic is finally formed. Let n be the number of these lifts that lie under the lift of the simple closed geodesic. We determine, in terms of n, the interval in which α lies.

For each non-negative integer m, the axis of $\tilde{G}_m$, $\mathrm{Ax}(\,\tilde{G}_m\,)$, also passes through $i + 3z$; let g_m be the left foot of this axis. By Lemma 9, the image under U^m of $\mathrm{Ax}(\,\tilde{G}_m\,)$ passes through i. Now, U^m has an attractive fixed point at u_+, and since the right foot of the axis of $\tilde{G}_m$ is greater than u_+, its image under U^m also remains greater. Therefore, the other foot of the image under U^m of $\mathrm{Ax}(\,\tilde{G}_m\,)$ is negative: $U^m(\,g_m\,) < 0$. The axis of $\tilde{G}_0$ is the h-line connecting i and $i + 3z$; as $\tilde{\rho}$ lies below this axis, $\alpha > g_0$.

Fix $m \geq 1$. Since all negative powers of U are attractive to u_-, we have $U^{-m+1}(0) \in [\,0, u_-\,)$. But, U^m has its only pole at $U^{-m+1}(0)$, so the image of the interval $(\,U^{-m+1}(0), u_-\,]$ under the (increasing, injective) U^m is connected; this image must clearly contain all of the negative real numbers. Since this is so, and $U^m(g_m) < 0$, we have that $g_m \in (\,U^{-m+1}(0), u_-\,)$.

Now, if $\alpha \in (\,U^{-m+1}(0), g_m\,)$, then the image of α under U^m is less than that of g_m. On the other hand, the image under U^m of $\tilde{\rho}$ shares $U^m(i + 3z)$ with the image of $\mathrm{Ax}(\,\tilde{G}_m\,)$; hence the image under U^m of $\tilde{\rho}$ lies above the image of $\mathrm{Ax}(\,\tilde{G}_m\,)$. We conclude that U^m sends the h-line containing $\tilde{\rho}$ to an h-line meeting the imaginary axis above i. The subsequent lift cannot enter the low triangle. Since the action of U "undoes" wraps, we conclude that the number of wraps of ρ is at most m.

Similarly to the above, if $g_{m-1} < \alpha < U^{-m+1}(0)$, then the image of α under U^{m-1} is negative and U^{m-1} sends $\tilde{\rho}$ to an h-line meeting the imaginary axis below i. But, then $U^m = S^{3z}T \cdot U^{m-1}$ sends $\tilde{\rho}$ to an h-line meeting the vertical h-line $x = 3z$ above $i + 3z$. Therefore, $g_{m-1} < \alpha < U^{-m+1}(0)$ implies that ρ has at most m wraps. We have found that for ρ to have n wraps, α must lie in the interval $(g_{n-1}, g_n]$. (The right foot β is in (u_+, ∞).)

We wish to show that if α is not some g_n, then there is a power of U sending the h-line containing $\tilde{\rho}$ to an h-line of greater diameter. Note first that there is a simple relation between the feet of any h-line passing through the point $i+3z$, in particular: $\beta = 3z + 1/(3z - \alpha)$. Using this relation, we of course have $\alpha\beta = (-3z\alpha^2)/(3z-\alpha)$ and $\alpha+\beta = (-\alpha^2)/(3z-\alpha)$, and Lemma 10 (upon clearing denominators) thus shows that any fixed matrix A increases the diameter of h-line passing through the point $i + 3z$ if and only if a certain quadratic inequality, in terms of a foot α and explicit in terms of the entries of A and the parameter z, holds.

We now show that if $\alpha \leq 0$ then the image under $U = S^{3z}T$ of the h-line containing $\tilde{\rho}$ has increased diameter. Since S^{3z} is a translation, it does not change diameter. Therefore, we consider the action of T. As discussed in the previous paragraph, Lemma 10 shows that T acts quadratically with respect to these diameters; the condition $\alpha < 0$, leads to an explicit quadratic inequality in α identifying the region of increased diameters. The boundary of this region is identified by the side condition that $\alpha < 0$ and by the solutions of the associated quadratic *equality.* Naturally, g_0 is the smaller of these solutions; the right foot of $\tilde{G}_0$ is the other solution — thus T either increases or decreases diameters of *all* h-lines passing through $i + 3z$ and of left foot $\alpha \in (g_0, 0)$. But, if α is sufficiently close to $x = 0$, then $T(\alpha)$ can be made arbitrarily large. On the other hand, β is bounded away from $x = 0$ and hence has bounded image under T; thus α sufficiently close to $x = 0$ is the foot of an h-line of increased diameter. We conclude that for all $\alpha \in (g_0, 0)$, the h-line containing the corresponding $\tilde{\rho}$ has increased height under T, and hence also U.

If now $\alpha > 0$, then we find a corresponding quadratic inequality for the locus of points forming feet of h-lines passing through the point $i + 3z$ for which U increases diameters. By Lemma 9, g_1 and the other foot of $\tilde{G}_1$ partition the real line into the components of constant increase

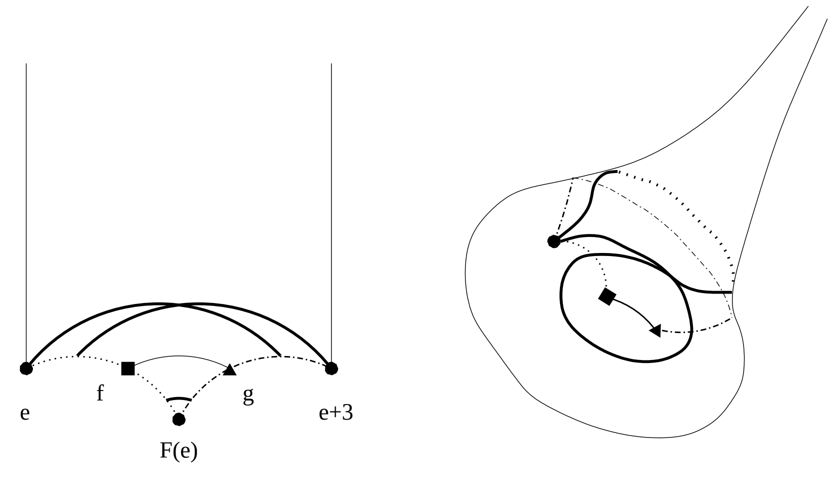

FIGURE 6. The axis of $G_1(1,1,1)$.

or decrease. Since U sends the interval $(0, 1/3z]$ to $(-\infty, 0]$ (in an orientation preserving, continuous manner), we can make $|\,U(\alpha)\,|$ arbitrarily large by choosing α sufficiently close to 0. But, the image of β is attracted to u_+; that is, this image is certainly bounded. We thus conclude that U increases diameters for all $\alpha \in (0, g_1)$.

We have shown that if the initial subarc of the Type 3 γ has $n = 1$ wrap (in the counter clock-wise direction), then γ must be the projection of the axis of $\tilde{G}_1$. For $n > 1$, U^n preserves the height of the h-line passing through $i + 3z$ and of left foot g_n, thus the axis of $\tilde{G}_n$. The quadratic nature of the increase of heights then can be invoked, and arguments as for $n = 1$ go through.

Finally, note that reflection about the line $\Re w = 3z/2$ sends the w-Poincaré upper half-plane to itself while acting as a transposition on each of the ordered pairs (u_-, u_+) and $(i, i+3z)$. This reflection thus also induces a transposition on each of the pairs (U, U^{-1}) and $(T, S^{3z}TS^{-3z})$. Thus, the image of $\mathrm{Ax}(\tilde{G}_n)$ under this reflection is the axis of $U^nS^{3z}TS^{-3z}U^{-n}T$. On the other hand, the final lift of $\mathrm{Ax}(\tilde{G}_n)$ in the process above is $U^n(\mathrm{Ax}(\tilde{G}_n))$; this is the axis of $TU^nS^{3z}TS^{-3z}U^{-n}$. That is, the reflection of the axis and its image under U^n are the same, up to orientation. Of course, the reflection gives an h-line projecting to a geodesic of n-wraps, but this h-line begins at e instead of $e + 3z$; confer Figure 6. The rest of the argument now follows by symmetry. $\square$

Remark 3. The sequences of the a_n are directly related to elliptic elements of order three in the modular group. These sequences also arose in recent work on the Markoff spectrum by Burger et al [BuEtAl]. We believe this to underline the importance in the Markoff theory of the phenomenon of wrapping about elliptic points. See also [V] for certain general techniques for height spectra that emphasize elements of finite order.

7. Examples of Other Types

Various subcases are easily shown to be of large height. Let us say that a geodesic is *high* if it has height at least 4.5.

Example 1. High Type 4 Geodesics Consider the word $S^3E_2S^{-3}E_2S^3E_2S^{-3}E_0$. The corresponding geodesic as seen in the (translated) Markoff-indexed fundamental domain: begins at the fixed point e_0 of E_0; proceeds to the right, crosses the fundamental domain; a second lift then begins at the imaginary axis, intersects the first lift, and descends to exit the domain to the right of the fixed point of E_2 — this is thus indeed a high geodesic —; the next lift begins from the low triangle and rises to the left to exit along the imaginary axis — it thus crosses the initial lift, and in so doing completes the subsequent disk that contains the elliptic point corresponding to E_2; we see that this is indeed a Type 4 geodesic —; finally, a lift begins at the translate of the imaginary axis by 3 and ends at the fixed point of E_2 . When $(x, y, z) = (1, 1, 1)$, this has height 5.49044.

Example 2. High Type 5 Geodesics By use of winding numbers, Type 5 geodesics can be assigned to two subcases. Given γ of Type 5, consider a parametrization of a subray of γ that includes the defining monogons about the cusp and about two elliptic points. The winding numbers of the monogon about the cusp and of the monogon about the remaining elliptic points are each ± 1.

Consider the subcase of the winding numbers being of opposite sign. Arguing as above, it is fairly straightforward to show that γ is then of height at least $h = 4.5$.

Example 3. Earlier Example of Type 6 Following a suggestion of W. Moran, Crisp et al. [CrEtAl] classify the closed curves of two self-intersections on the punctured torus. As an application of this, they find the automorphism class of twice self-intersecting closed geodesics of heights between 3 and 6 on $\Gamma'\backslash\mathcal{H}$. By applying the automorphism $(A, B) \mapsto (B^{-1}, A^{-1})$ to $A^2BA^{-2}B^{-1}$, one finds $B^{-2}A^{-1}B^2A$. In terms of Γ^3, this is $B^{-2}A^{-1}B^2A = (T_0T_1S^3)^2$. It is easily seen that the *primitive* $\Gamma^3\backslash\mathcal{H}$ geodesic is of our Type 6 — it is in fact a closed figure eight, with the cusp in one loop and two elliptic points in the second.

It is straightforward to give a formula for the height of each of geodesic of this type, using Markoff-indexed regions. However, Proposition 7.4 of [CrEtAl] already shows that these geodesics have relatively large heights. Here, we must point out that their *proof* is flawed. The final line of that proof would lead us to believe that these Type 6 geodesics have heights exactly 3 more than heights of simple closed geodesics. One can easily solve to show that this is false.

In their proof, [CrEtAl] correctly show that each isometry class of the Type 6 geodesics (in their setting as geodesics on the punctured torus) can be represented by a purely periodic continued fraction, whose period has the form $[5, 2, b_2, b_3, \dots, b_n]$. Furthermore, they show that $[2, 2, b_2, b_3, \dots, b_n]$ is the period of a purely periodic continued fraction representing a simple closed geodesic.

But, the Markoff value of such a period depends on the bi-infinite sequence given by repeating the period; clearly, the bi-infinite sequence formed by subtracting 3 from the initial value of the bi-infinite sequence of period $(5, 2, b_2, b_3, \dots, b_n)$ differs infinitely often with that of period $(2, 2, b_2, b_3, \dots, b_n)$. However, it is easily seen (see the first two lemmas in [CF]) that as the index n tends to infinity — thus as the length of the periods tends to infinity— the Markoff values do indeed tend to values that differ by 3.

Example 4. Remaining High Type 6 It is easy to show that any Type 6 geodesic that follows the path of the closed figure eights also has large height. That is, any Type 6 geodesic whose arms, continuing its bigon containing the two elliptic points, remain *exterior* to the bigon is indeed of large height.

Example 5. High Type 7 Geodesics Similar reasoning as for the above example of a high Type 4 shows that $S^3E_2S^3E_2S^{-3}E_2S^{-3}E_0$ leads to a high Type 7 geodesic. Here one arm terminates at e, the other passes underneath the fixed point of E_2 so as to then enter into the cusped disk and then terminate at E_2. With $(x, y, z) = (1, 1, 1)$, one finds a height of 4.91471.

There are indeed low height geodesics in each of these remaining cases. We give matrices, in terms of generators, whose axes — by tracing lifts in a (translated) Markoff-indexed domain — are easily checked to be of the case indicated. Height calculations are also easily performed. We have chosen Markoff triples so as to give decreasing heights for these selected examples; for the first triple $(x, y, z) = (1, 1, 1)$, all of these examples have fairly high lifts.

Example 6. Low Geodesics For each of the following, a representative of a geodesic of the indicated case is given by the projection to $\Gamma^3\backslash\mathcal{H}$ of the axis of a rational translation of the indicated matrix; an approximation to the height of this geodesic is as indicated.

$$\begin{aligned}
&\text{Type } 4: M = S^3E_1S^{-3}E_1S^{-3}E_1, && (x,y,z) = (1,2,5);\, h(\gamma_M) = 3.171;\\
&\text{Type } 5: M = S^3(E_1E_2)^2E_0(E_2E_1)^2S^{-3}E_0, && (x,y,z) = (1,1,2);\, h(\gamma_M) = 3.167;\\
&\text{Type } 6: M = E_1E_0E_1S^{-3}E_0E_1E_0S^3, && (x,y,z) = (1,13,34);\, h(\gamma_M) = 3.03;\\
&\text{Type } 7: M = S^3E_1S^3E_2S^{-3}E_1S^{-3}E_0, && (x,y,z) = (1,5,13);\, h(\gamma_M) = 3.01\;.
\end{aligned}$$

8. Limits

Recall that Crisp and Moran showed that PSSI are of height $\sqrt{9 + 4/z^2}$, where z can take on the value of the largest element of any Markoff triple. Obviously, one thus has the following.

Lemma 11. *The only limit of the set of heights of Type 1 geodesics is* $h = 3$.

8.1. **Limits of Type 2 Geodesics.**

Proposition 3. *A limit of a sequence of heights of distinct low Type 2 closed geodesics is either* 3*, or else is the height of some open pointed Type 2 geodesic. In particular, no such limit is the height of a closed geodesic of* $\Gamma^3\backslash\mathcal{H}$ *.*

Proof. Suppose that some h is the limit of a sequence of heights of Type 2 closed geodesics. Choose a convergent subsequence of these heights, and for each element of this subsequence fix a Type 2 closed geodesic realizing this height. From Proposition 1, there is a corresponding sequence of Markoff triples.

If the x-values of these triples are bounded, then we can choose a subsequence for which x is constant. From the quadratic equation and the Markoff equation, we then find that z/y tends to $(3x + \sqrt{9x^2 - 4}/2$. We thus find that the heights of the Type 2 geodesics converge to the height of the pointed open Type 2 geodesic as given in Lemma 6. upon replacing z there by x.

If the x values are unbounded, then z/y tends to $3x + 1/3x$, and one finds a limit that clearly tends to 3.

But, the height of a closed geodesic is the Euclidean diameter of the axis of some hyperbolic matrix with integer entries, and in particular is an irrational square root of a rational number. Clearly, our limit values are not of this form. □

Lemma 12. *A limit of a sequence of heights of distinct low Type 2 pointed geodesics, each with infinite arm spinning about some simple closed geodesic, is* 3.

Proof. This follows easily from the height formula given in Lemma 6. □

8.2. **Limits of Type 3 Geodesics.**

Lemma 13. *Every convergent sequence of heights of Type 3 height-achieving geodesics converges to $h = 3$.*

Proof. We have shown that to each solution (x, y, z) of Markoff's equation and to each n there is one n-wrap Type 3 geodesic, whose height is as in Theorem 1. From the formula for these heights, if either $a_n(z)$ or z tends to infinity, then the heights converge to $h = 3$. But, it is easily checked that as n tends to infinity so does $a_n(z)$, for z fixed. Since there are at most finitely many solutions to Markoff's equation with any given value of z, one now has that any infinite sequence of Type 3 geodesics admits a subsequence whose heights converge to $h = 3$ and that this is the only limit value of these sequences. □

8.3. **Limits of Certain Type 4 Geodesics; Limits and Spinning.** We now give an explicit computation of Type 4 geodesics with low height limits.

Recall that the standard generators of Γ^3 are T_j, with $j \in \{0, 1, 2\}$. Let $A = T_1T_2T_1T_0$ and $B = T_0T_1$; then A and B are the standard generating pair for Γ'. The fact that AB and A^2B^2 give rise to a simple closed geodesic and related PSSI of $\Gamma'\backslash\mathcal{H}$ is fundamental to the work of Crisp and Moran.

Lemma 14. *Let A and B be the standard generators of Γ'. Let γ_n be the projection to $\Gamma^3\backslash\mathcal{H}$ of the axis of A^nB^n. Then γ_1 is a simple closed geodesic, and γ_2 is its paired PSSI; for each $n > 2$, γ_n is a Type 4 geodesic. The limit of the heights of the γ_n exists and is equal to $L = 2 + \sqrt{5} = 4.236\ldots$.*

Proof. Each γ_n is also the projection of the axis of $B^{-n}A^{-n}$. We have $B^{-1}A^{-1} = S^3E$ with $E = T_0$; as well, $B^{-2}A^{-2} = T_1 \cdot ES^3ES^{-3} \cdot T_1$. Since the projections to $\Gamma^3\backslash\mathcal{H}$ of the axis of the conjugate of $B^{-2}A^{-2}$ by T_1 is also γ_2, we find that γ_2 is indeed the PSSI associated to the simple closed geodesic γ_1. Denote the vertices of $\mathcal{F}$ the Markoff-indexed fundamental domain in the usual manner — in particular, E, F, G fix the vertices e, f and g respectively. It is now an exercise emphasizing the order two nature of the elements E, F and G to find that γ_n is the projection of the h-line ℓ_n given by

$$\ell_n \text{ passes through } \begin{cases} (S^{-3}G)^m(f) \text{ and } (S^3F)^m(g) & \text{if } n = 2m+1\,; \\ \\ (S^{-3}G)^{m-1}(e) \text{ and } (S^3F)^{m-1}(g) & \text{if } n = 2m\,. \end{cases}$$

See Figure 7. Of course, $T_1A^{-1}T_1 = S^3F$ and $T_1B^{-1}T_1 = EF$; thus there is no surprise in the above formulas. Note that $S^{-3}G = EF$, and $S^3F = S^3ES^{-3}G$; the h-lines $\text{Ax}(S^{-3}G)$ and $\text{Ax}(HG)$ thus each meet $\mathcal{F}$ in two equivalent lifts of a companion simple closed geodesic. Therefore, each γ_n is, up to orientation, the union of two simple rays emanating from its high point. For $n \geq 2$, these rays first meet again as each begins its spinning about its respective 'guiding' simple closed geodesic. But, this gives a cusped disk followed by a subsequent bigon about the projection of e. That is, each of these γ_n is indeed of Type 4.

The limit as n goes to infinity is the projection of the h-line joining the attractive fixed point of $S^{-3}G$, say v_+, to that of S^3F, say w_+. These are $(1-\sqrt{5})/2$ and $(5+\sqrt{5})/2$, respectively. Thus, the limit height is $2 + \sqrt{5}$. □

We easily generalize the above.

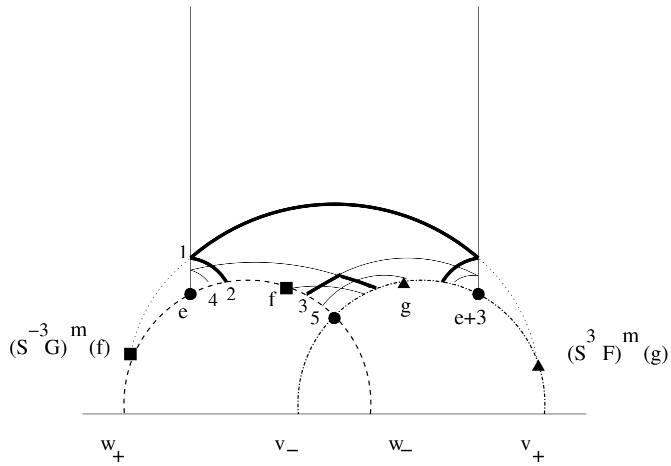

FIGURE 7. Highlighted arcs identify Type 4 aspect of Toral A^nB^n. Pictured: $n = 3$; lifts of one ray enumerated, odd numbers at beginning of arc, even at end.

Lemma 15. *Let (x, y, z) be a Markoff triple. Let $S^3E = GF$ be the factorization of the matrix whose axis projects to the simple closed geodesic corresponding as in Definition 2 to this triple. Let γ_n be the projection to $\Gamma^3\backslash\mathcal{H}$ of the axis of $(GS^{-3})^n(S^3F)^n$. Then γ_2 is the PSSI associated to this triple; and for each $n > 2$, γ_n is a closed geodesic of Type 4. There is a limit of these geodesics, it is the open geodesic that is the projection of the h-line joining the attractive fixed points of S^3F and $S^{-3}G$. If $(x, y, z) \neq (1, 1, 1)$, this geodesic is of height*

$$\frac{3 + \sqrt{9 - 4/x^2} + \sqrt{9 - 4/y^2} - \sqrt{9 - 4(\,1/x^2 + 1/y^2)}}{2}\,;$$

in no case is this the height of any closed geodesic of $\Gamma^3\backslash\mathcal{H}$. Upon fixing x and allowing y to tend to infinity, the limit of the heights of the corresponding open geodesics is 3.

Proof. The core of the proof is as for the lemma. Since the axes of S^3F and $S^{-3}G$ project to distinct simple closed geodesics of $\Gamma^3\backslash\mathcal{H}$, their attractive fixed points are each real quadratic numbers, but they are not conjugate. Therefore, their difference cannot give the height of a closed geodesic.

Now, with (x, y, z) fixed, the limit of the h-lines as n tends to infinity is the projection of the h-line joining the attractive fixed point of $S^{-3}G$, say v_+, to that of S^3F, say w_+; see Figure 7. The diameter of this h-line is exactly that of the h-line joining the attractive fixed point of $S^{-3}E_2$ to that of S^3E_1, where the E_i are as in Theorem 2. This diameter is easily found to be

$$\frac{2x/yz + 3 + \sqrt{9 - 4/y^2}}{2} - \Big(\frac{-2y/xz + 3 - \sqrt{9 - 4/x^2}}{2}\Big)\,,$$

But, by the quadratic equation applied to Markoff's Equation,

$$z = \frac{3xy \pm \sqrt{9x^2y^2 - 4(x^2 + y^2)}}{2}\,,$$

where whenever $(x, y, z) \neq (1, 1, 1)$, the larger root is taken. Hence, the diameter of this limit h-line is

$$L = \frac{3 + \sqrt{9 - 4/x^2} + \sqrt{9 - 4/y^2} - \sqrt{9 - 4(1/x^2 + 1/y^2)}}{2}\,.$$

Now, we indeed find a limit of 3 as y tends to infinity. □

The reader is encouraged to see the bibliography of [SS3] as well as below.

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