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NATURAL EXTENSIONS AND ENTROPY OF α-CONTINUED FRACTIONS

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Abstract. We construct a natural extension for each of Nakada's α-continued fractions and show the continuity as a function of α of both the entropy and the measure of the natural extension domain with respect to the density function $(1+xy)^{-2}$. In particular, we show that, for all $0 < \alpha \leq 1$, the product of the entropy with the measure of the domain equals $\pi^2/6$. As a key step, we give the explicit relationship between the α-expansion of $\alpha - 1$ and of α.

1. Introduction

The α-continued fractions, introduced by Nakada [Nak81], form a one dimensional family of interval maps, T_α with $\alpha \in [0,1]$. Luzzi and Marmi [LM08] gave a proof that the entropy of these maps is a continuous function of α whose limit at $\alpha = 0$ is zero. Unfortunately, their proof of continuity turned out to be incorrect; however, Tiozzo [Tio] was able to salvage the result for $\alpha > 0.056\dots$. Luzzi and Marmi conjectured that, for non-zero α, the product of the entropy and the area of the standard number theoretic planar extension for T_α is constant.

Let $h(T_\alpha)$ denote the entropy of the interval map T_α. Nakada and Natsui [NN08] gave specific regions on which $\alpha \mapsto h(T_\alpha)$ is respectively constant, increasing, decreasing. Indeed, they showed this by exhibiting intervals of α such that $T_\alpha^k(\alpha) = T_\alpha^{k'}(\alpha - 1)$ for pairs of positive integers (k, k') and showed that the entropy is constant (resp. increasing, decreasing) on such an interval if $k = k'$ (resp. $k > k'$, $k < k'$). They also conjectured that there is an open dense set of $\alpha \in [0,1]$ for which the T_α-orbits of $\alpha - 1$ and α synchronize. (Carminati and Tiozzo [CT] confirm this last and also identify maximal intervals where T_α-orbits synchronize.)

We confirm these conjectures (including reproving some of the results of [CT]). Let $\mu(\Omega_\alpha)$ denote the measure of the region of the standard number theoretic natural extension, as defined below.

Main Theorem. *The set of points $\alpha \in (0,1]$ such that the T_α-orbits of α and $\alpha - 1$ meet is the union of a countable number of intervals, contains all rational numbers in the open*

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unit interval, and is of full measure. Furthermore, both functions

$$\alpha \mapsto h(T_\alpha) \quad \text{and} \quad \alpha \mapsto \mu(\Omega_\alpha)$$

are continuous on $(0,1]$*, and* $h(T_\alpha)\,\mu(\Omega_\alpha) = \pi^2/6$.

Our results follow from giving an explicit description of a planar natural extension for every $\alpha \in (0,1]$, see Section 6, and this by way of giving details of the relationship between the α-expansions of $\alpha - 1$ and α, see Theorem 5.1.

Outline. The sections of the paper are increasingly technical, with the exception of the final section. After establishing notation in the following section, we first relate the regular continued fraction and the general α-expansion of a real number. This then allows a proof that the natural extension for T_α is given by our $\mathcal{T}_\alpha$ on the closure of the orbits of $(x,0)$. In order to reach the deeper results, we then give the explicit relationship between the α expansions of $\alpha - 1$ and of α. This is then applied in Section 6 to give a detailed description of the natural extension domain, as the union of fibers that are constant on intervals void of the T_α-orbits of $\alpha - 1$ and α. Independent of that section, in Section 7 the relationship between the α-expansions of the endpoints is used to describe the (maximal in an appropriate sense) intervals for synchronizing orbits. Relying on the previous two sections, in Section 8 we prove the main result of continuity. In the final section, we give further results on the set of synchronizing orbits, in particular showing the transcendence of limits under a natural folding operation on the set of intervals of synchronizing orbits.

2. Basic Notions and Notation

2.1. **One dimensional maps.** For $\alpha \in [0,1]$, we let $\mathbb{I}_\alpha := [\alpha - 1, \alpha]$ and define the map $T_\alpha : \mathbb{I}_\alpha \to [\alpha - 1, \alpha)$ by

$$T_\alpha(x) := \left|\frac{1}{x}\right| - \left\lfloor \left|\frac{1}{x}\right| + 1 - \alpha \right\rfloor \qquad (x \neq 0),$$

$T_\alpha(0) := 0$. For $x \in \mathbb{I}_\alpha$, put

$$\varepsilon(x) := \begin{cases} 1 & \text{if } x \geq 0\,, \\ -1 & \text{if } x < 0\,, \end{cases} \qquad \text{and} \quad d_\alpha(x) := \left\lfloor \left|\frac{1}{x}\right| + 1 - \alpha \right\rfloor,$$

with $d_\alpha(0) = \infty$. Furthermore, let

$$\varepsilon_n = \varepsilon_{\alpha,n}(x) := \varepsilon(T_\alpha^{n-1}(x)) \quad \text{and} \quad d_n = d_{\alpha,n}(x) := d_\alpha(T_\alpha^{n-1}(x)) \qquad (n \geq 1).$$

This yields the *α-continued fraction* expansion of $x \in \mathbb{R}$:

$$x = d_0 + \cfrac{\varepsilon_1}{d_1 + \cfrac{\varepsilon_2}{d_2 + \cdots}} =: [\![\, d_0;\, \varepsilon_1 : d_1,\, \varepsilon_2 : d_2,\, \ldots]\!]\,,$$

where $d_0 \in \mathbb{Z}$ is such that $x - d_0 \in [\alpha - 1, \alpha)$. (Standard convergence arguments justify equality of x and its expansion.) These include the regular continued fractions, given by $\alpha = 1$, and the nearest integer continued fractions, given by $\alpha = 1/2$.

The point α is included in the domain of T_α because its T_α-orbit plays an important role, along with that of $\alpha - 1$. We thus define

$$\underline{b}_n^\alpha = (\varepsilon_{\alpha,n}(\alpha-1) : d_{\alpha,n}(\alpha-1)) \quad \text{and} \quad \overline{b}_n^\alpha = (\varepsilon_{\alpha,n}(\alpha) : d_{\alpha,n}(\alpha)) \qquad (n \geq 1),$$

and informally refer to these sequences as the α-expansions of $\alpha - 1$ and α.

We usually write $[\![(\varepsilon_1 : d_1)(\varepsilon_2 : d_2)\cdots]\!]$ instead of $[\![0; \varepsilon_1 : d_1, \varepsilon_2 : d_2, \ldots]\!]$, allowing equalities of the type $[\![\, \underline{b}_1^\alpha \underline{b}_2^\alpha \cdots]\!] = \alpha - 1$ and $[\![\, \overline{b}_1^\alpha \overline{b}_2^\alpha \cdots]\!] = \alpha$. We also let

$$[\![(\varepsilon_1 : d_1)\cdots(\varepsilon_n : d_n),\, y\,]\!] := \cfrac{\varepsilon_1}{d_1 + \cdots + \cfrac{\varepsilon_n}{d_n + y}} \qquad (y \in \mathbb{R}).$$

The rank-one *cylinders* of T_α are defined as

$$\Delta_\alpha(\varepsilon : d) = \{x \in \mathbb{I}_\alpha \mid \varepsilon(x) = \varepsilon,\, d_\alpha(x) = d\}\,,$$

where $\varepsilon = -1$, $d \geq d_\alpha(\alpha - 1)$, or $\varepsilon = 1$, $d \geq d_\alpha(\alpha)$, or $(\varepsilon : d) = (1 : \infty)$. Of these cylinders, all are *full*, that is their image under T_α is the interval $[\alpha - 1, \alpha)$, except for

$$T_\alpha(\Delta_\alpha(\underline{b}_1^\alpha)) = [\,T_\alpha(\alpha-1), \alpha), \quad T_\alpha(\Delta_\alpha(\underline{b}_1^\alpha)) = [\,T_\alpha(\alpha), \alpha), \quad T_\alpha(\Delta_\alpha(1:\infty)) = \{0\}.$$

Remark 2.1. The so-called by-excess map, T_0, gives infinite expansions for all $x \in [-1, 0)$; each expansion has all signs $\varepsilon_n = -1$, and digits $d_n \in \{2, 3, \ldots\}$. (Rational numbers in this range have eventually periodic expansions of period $(\varepsilon : d) = (-1 : 2)$.) The map T_0 is well-known to be ergodic with respect to an infinite invariant measure, as recalled in [LM08]. In fact, the map T_α is ergodic with respect to an invariant measure ν_α for all $\alpha \in [0, 1]$, the proof for (beyond the classical cases of) $\alpha \in [1/2, 1]$ was given by Nakada [Nak81], and [LM08] adjust the proof to also include the remaining values.

2.2. **Two-dimensional maps, matrix formulation.** The standard number theoretic planar map associated to continued fractions is defined by

$$\mathcal{T}_\alpha(x, y) = \left(T_\alpha(x), \frac{1}{d_\alpha(x) + \varepsilon(x)\, y}\right) \quad (x \in \mathbb{I}_\alpha,\ y \in [0,1])\,; \tag{2.1}$$

it is easily checked that this map has invariant measure μ, given by

$$d\mu = \frac{dx\, dy}{(1 + xy)^2}\,.$$

Indeed, for any $x \in \Delta_\alpha(\varepsilon : d)$, with $\varepsilon = \pm 1$, $d \geq 1$, we have

$$\mathcal{T}_\alpha(x, y) = \big(M_{(\varepsilon:d)} \cdot x, N_{(\varepsilon:d)} \cdot y\big), \tag{2.2}$$

where

$$M_{(\varepsilon:d)} := \begin{pmatrix} -d & \varepsilon \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad N_{(\varepsilon:d)} := {}^t M_{(\varepsilon:d)}^{-1} = \varepsilon \begin{pmatrix} 0 & 1 \\ \varepsilon & d \end{pmatrix}.$$

As usual, the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts on real numbers by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x = \frac{ax + b}{cx + d}$, and ${}^t M$ denotes the transpose of M. Note that $M \cdot x$ is a projective action, in the sequel we often do not differentiate between a matrix and any non-zero constant multiple of it. An elementary "change of variables / Jacobian" calculation shows that for any rectangle $[x_1, x_2] \times [y_1, y_2] \subset \mathbb{R}^2$ and any unimodular matrix M,

$$\mu\big(M \cdot [x_1, x_2] \times {}^tM^{-1} \cdot [y_1, y_2]\big) = \mu\big([x_1, x_2] \times [y_1, y_2]\big). \tag{2.3}$$

2.3. **Symbolic notation, digit sequences to matrices.** For any non-empty set V, the Kleene star $V^* = \bigcup_{n\ge 0} V^n$ denotes the set of concatenations of a finite number of elements in V, and V^ω denotes the set of (right) infinite concatenations of elements in V. If $V = \{v\}$, then we write v^* and v^ω instead of V^* and V^ω. We will also use the abbreviations $v_{[m,n]} = v_m v_{m+1} \cdots v_n$, $v_{[m,n)} = v_m v_{m+1} \cdots v_{n-1}$, where $v_{[m,m)}$ is the empty word, and $v_{[m,\infty)} = v_m v_{m+1} \cdots$. The length of a finite word v is denoted by $|v|$.

We will usually consider words on one of the alphabets

$$\mathscr{A}_- = \{\, (-1 : d) \mid d \in \mathbb{Z},\, d \ge 2 \,\}, \quad \mathscr{A}_+ = \{\, (1 : d) \mid d \in \mathbb{Z},\, d \ge 1 \,\}, \quad \mathscr{A} = \mathscr{A}_- \cup \mathscr{A}_+,$$

and $\mathscr{A}_0 = \mathscr{A} \cup \{(1 : \infty)\}$. In light of (2.2), we set $M_v = M_{v_n} \cdots M_{v_1}$ for $v = v_1 \cdots v_n \in \mathscr{A}^*$. Thus, for example, if $v = \underline{b}^\alpha_{[1,n]}$, then $M_v \cdot (\alpha - 1) = T^n_\alpha(\alpha - 1)$.

3. Relation between α-expansions and RCF

We first show how the regular continued fraction expansion (RCF), thus the 1-expansion, of $x \in (0, \alpha]$ can be constructed from the α-expansion of x. This is a key argument in the following section.

In later sections, our use of formal language notation and vocabulary will facilitate the treatment of complicated matters; here, we prove fairly straightforward results in these terms, hoping to accustom the reader to our use of this approach.

Lemma 3.1. *Let $\alpha \in (0, 1)$. For any $x \in (0, \alpha]$, the 1-expansion of x is obtained from the α-expansion of x by repeating the following procedure: Replace the first occurrence of*

$$(1 : d)(-1 : d') \quad \text{by} \quad (1 : d - 1)(1 : 1)(1 : d' - 1),$$

if any such pattern (with $d \ge 1$, $d' \ge 2$) exists. In case $d = 1$, replace the appearing pattern

$$(1 : d'')(1 : 0)(1 : 1) \quad \text{by} \quad (1 : d'' + 1) \quad (\text{with } d'' \ge 1).$$

If applicable, replace $(1 : d)(1 : 1)(1 : \infty)^\omega$ (with $d \ge 1$) by $(1 : d + 1)(1 : \infty)^\omega$ at the end.

Proof. Let $v_{[1,\infty)} = (\varepsilon_{\alpha,1}(x) : d_{\alpha,1}(x))(\varepsilon_{\alpha,2}(x) : d_{\alpha,2}(x)) \cdots$ be the α-expansion of $x \in (0, \alpha]$. One easily checks that

$$M_{(-1:d')} M_{(1:d)} = M_{(1:d'-1)} M_{(1:1)} M_{(1:d-1)}, \quad M_{(1:1)} M_{(1:0)} M_{(1:d'')} = M_{(1:d''+1)},$$

and that $[\![(1 : d)(1 : 1)(1 : \infty)^\omega]\!] = 1/(d+1) = [\![(1 : d+1)(1 : \infty)^\omega]\!]$, thus any sequence $v'_{[1,\infty)}$ which is obtained from $v_{[1,\infty)}$ by the given replacements satisfies $[\![v'_{[1,\infty)}]\!] = [\![v_{[1,\infty)}]\!] = x$.

For any $x' \in [\alpha-1,0)$, we certainly have that $d_\alpha(x') \geq d_\alpha(\alpha-1) \geq 2$. Similarly, for $x' \in (0,\alpha]$, we have $d_\alpha(x') \geq d_\alpha(\alpha) \geq 1$. Thus, $v_{[1,\infty)}$ is in $\mathscr{A}_0^\omega$, and $x \in (0,\alpha]$ implies $v_1 \in \mathscr{A}_+$. Moreover, if $v_1 = (1:1)$, then $T_\alpha(x) = 1/x - 1 > 0$ implies that $v_2 \in \mathscr{A}_+$.

Now, if the α-expansion of x is infinite, that is $v_{[1,\infty)} \in \mathscr{A}^\omega$, then the described procedure yields a sequence $v'_{[1,\infty)} \in \mathscr{A}_+^\omega$ with $[\![v'_{[1,\infty)}]\!] = x$, thus $v'_{[1,\infty)}$ is the 1-expansion of x. If the α-expansion of x is finite, that is $v_{[1,\infty)} \in \mathscr{A}^*(1:\infty)^\omega$, then we obtain a sequence $v'_{[1,\infty)} \in \mathscr{A}_+^*(1:\infty)^\omega \setminus \mathscr{A}_+^*(1:1)(1:\infty)^\omega$, which is again the 1-expansion of x. □

Remark 3.2. The above can be compared with the conversions from α-expansions to RCF given in [NN02] and [NN08].

Lemma 3.3. *Let $\alpha \in (0,1)$, $x \in (0,\alpha]$ and suppose that there exists $m \geq 1$ such that $T_\alpha^m(x) > 0$. Then there is some $n \geq 1$ such that $T_\alpha^m(x) = T_1^n(x)$ and $\mathcal{T}_\alpha^m(x,y) = \mathcal{T}_1^n(x,y)$ for all $y \in [0,1]$.*

Proof. Let $v_{[1,\infty)} = (\varepsilon_{\alpha,1}(x) : d_{\alpha,1}(x))(\varepsilon_{\alpha,2}(x) : d_{\alpha,2}(x)) \cdots$ be the α-expansion of $x \in (0,\alpha]$. If $T_\alpha^m(x) > 0$, then let $v'_{[1,n]} \in \mathscr{A}_+^*$ be the sequence obtained from $v_{[1,m]}$ by the procedure described in Lemma 3.1. Since $[\![v'_{[1,n]}, T_\alpha^m(x)]\!] = x$, we obtain that $T_1^n(x) = T_\alpha^m(x)$. We have $M_{v_{[1,m]}} = M_{v'_{[1,n]}}$, thus $\mathcal{T}_\alpha^m(x,y) = \mathcal{T}_1^n(x,y)$ for all $y \in [0,1]$. □

Lemma 3.4. *Let $\alpha \in (0,1]$ and $x \in \mathbb{I}_\alpha$. The α-expansion of x contains no sequence of $d_\alpha(\alpha)$ consecutive digits $(-1:2)$.*

Proof. The α-expansion of x contains a sequence of $d_\alpha(\alpha)$ consecutive digits $(-1:2)$ if and only if the α-expansion of $T_\alpha^m(x)$ starts with $(-1:2)^{d_\alpha(\alpha)}$ for some $m \geq 0$. Therefore, it suffices to show that $(-1:2)^{d_\alpha(\alpha)}$ cannot be a prefix of an α-expansion.

Suppose on the contrary that the α-expansion of x starts with $(-1:2)^{d_\alpha(\alpha)}$. In particular, this means that $T_\alpha^n(x) = M_{(-1:2)}^n \cdot x < 0$ for all $0 \leq n < d_\alpha(\alpha)$. We have

$$M_{(-1:2)}^n = (-1)^n \begin{pmatrix} n+1 & n \\ -n & 1-n \end{pmatrix},$$

thus $M_{(-1:2)}^n \cdot x \geq 0$ for all $x \in \left[\frac{1}{n+1}-1, \frac{1}{n}-1\right)$. Since $x < 0$, it follows that $x < \frac{1}{d_\alpha(\alpha)} - 1$, and

$$\alpha > T_\alpha^{d_\alpha(\alpha)}(x) = M_{(-1:2)}^{d_\alpha(\alpha)} \cdot x \geq M_{(-1:2)}^{d_\alpha(\alpha)} \cdot (\alpha-1) = \frac{d_\alpha(\alpha)\alpha + \alpha - 1}{1 - d_\alpha(\alpha)\alpha},$$

where we have used that $x \geq \alpha - 1$ and that the action of $M_{(-1:2)}$ is order preserving on the negative numbers. Since $\alpha \leq x + 1 < \frac{1}{d_\alpha(\alpha)}$, we obtain that $d_\alpha(\alpha)\alpha^2 + d_\alpha(\alpha)\alpha - 1 < 0$, but we must also have $\alpha > T_\alpha(\alpha) = \frac{1}{\alpha} - d_\alpha(\alpha)$, thus $\alpha^2 + d_\alpha(\alpha)\alpha - 1 > 0$. Since $d_\alpha(\alpha) \geq 1$, this is impossible. □

Lemma 3.5. *Let $\alpha \in (0,1)$, $x \in (0,\alpha]$ and suppose that $T_\alpha^m(x) < 0$ for all $m \geq 1$. Then, for any $n \geq 1$, we cannot have both $d_{1,n}(x) > d_\alpha(\alpha)$ and $d_{1,n+1}(x) > d_\alpha(\alpha)$.*

Proof. In Lemma 3.1, we replace any pattern

$$(1:d)(-1:2)^{d'}(-1:d'') \quad \text{by} \quad (1:d-1)(1:d'+1)(1:d''-1),$$

$(d \geq 2,\ d' \geq 0,\ d'' \geq 3)$. Since $\varepsilon_{\alpha,m}(x) = -1$ for all $m \geq 2$, all digits in the 1-expansion of x result from replacements of this form, in particular at least every second digit is of the form $(1 : d' + 1)$, where d' is the length of a maximal block of consecutive digits equal to $(-1 : 2)$. By Lemma 3.4, we have $d' < d_\alpha(\alpha)$, hence we cannot have two consecutive "large" digits. □

Lemma 3.6. *Let $\alpha \in (0,1)$, $x \in (0,\alpha]$, and $T_1^{n-1}(x) \in \big(0, \frac{1}{d_\alpha(\alpha)+1}\big]$, $T_1^n(x) \in \big(0, \frac{1}{d_\alpha(\alpha)+1}\big]$ for some $n \geq 1$. Then there is some $m \geq 1$ such that $T_1^n(x) = T_\alpha^m(x)$ and $\mathcal{T}_1^n(x,y) = \mathcal{T}_\alpha^m(x,y)$ for all $y \in [0,1]$.*

Proof. Let $v_{[1,\infty)} = (\varepsilon_{\alpha,1}(x) : d_{\alpha,1}(x))(\varepsilon_{\alpha,2}(x) : d_{\alpha,2}(x))\cdots$ be the α-expansion of $x \in (0,\alpha]$, and $v'_{[1,\infty)} = (\varepsilon_{1,1}(x) : d_{1,1}(x))(\varepsilon_{1,2}(x) : d_{1,2}(x))\cdots$ its 1-expansion. If $T_1^{n-1}(x) \in \big(0, \frac{1}{d_\alpha(\alpha)+1}\big]$, $T_1^n(x) \in \big(0, \frac{1}{d_\alpha(\alpha)+1}\big]$, then $v'_n = (1 : d)$, $v'_{n+1} = (1 : d')$ with $d, d' > d_\alpha(\alpha)$. By Lemma 3.5 and its proof, we see that v'_{n+1} cannot come from a digit in $\mathscr{A}_-$. Thus, there is some m such that $v'_{[1,n]}$ is given by replacements from $v_{[1,m]}$, and $T_\alpha^m(x) > 0$; from this, $T_1^n(x) = T_\alpha^m(x)$ and $\mathcal{T}_1^n(x,y) = \mathcal{T}_\alpha^m(x,y)$ for all $y \in [0,1]$. □

4. Natural extensions and entropy

The advantage for number theoretic usage of the natural extension map in the form $\mathcal{T}_\alpha$ is that the Diophantine approximation to an $x \in [\alpha-1,\alpha)$ by the finite steps in its α-expansion is directly related to the $\mathcal{T}_\alpha$-orbit of $(x,0)$, see [Kra91]. We define our natural extension domain in terms of these orbits. We show moreover that the entropy of T_α is directly related to the measure of the natural extension domain.

Definition 4.1. For α in $(0,1]$, the *natural extension domain* is

$$\Omega_\alpha = \overline{\big\{\mathcal{T}_\alpha^n(x,0) \mid x \in [\alpha-1,\alpha),\ n \geq 0\big\}}\,.$$

We use the following lemmas to show that $\mathcal{T}_\alpha$ and Ω_α give indeed a natural extension of T_α.

Lemma 4.2. *Let $\alpha \in (0,1]$. We have*

$$\big[0, \tfrac{1}{d_\alpha(\alpha)+1}\big]^2 \subset \Omega_\alpha \subseteq \mathbb{I}_\alpha \times [0,1],$$

thus $0 < \mu(\Omega_\alpha) < \infty$.

Proof. The inclusion $\Omega_\alpha \subseteq \mathbb{I}_\alpha \times [0,1]$ follows from the inclusion $N_a \cdot [0,1] \subset [0,1]$, which holds for every $a \in \mathscr{A}$. Therefore, Ω_α is bounded away from $y = -1/x$, and its compactness yields $\mu(\Omega_\alpha) < \infty$.

It remains to show that Ω_α contains the square $\big[0, \frac{1}{d_\alpha(\alpha)+1}\big]^2$, which implies $\mu(\Omega_\alpha) > 0$. Every point in $\big[0, \frac{1}{d_\alpha(\alpha)+1}\big]^2$ can be approximated by points $\mathcal{T}_1^n(x_n, 0)$, $n \geq 1$, with $x_n \in [0, \alpha]$, $T_1^{n-1}(x_n) \leq \frac{1}{d_\alpha(\alpha)+1}$, $T_1^n(x_n) \leq \frac{1}{d_\alpha(\alpha)+1}$. By Lemma 3.6, there exist numbers $m_n \geq 1$ such that $\mathcal{T}_1^n(x_n, 0) = \mathcal{T}_\alpha^{m_n}(x_n, 0)$, from which we conclude that $\big[0, \frac{1}{d_\alpha(\alpha)+1}\big]^2 \subset \Omega_\alpha$. □

Lemma 4.3. *Let $\alpha \in (0, 1]$. Up to a set of μ-measure zero, $\mathcal{T}_\alpha$ is a bijective map from Ω_α to Ω_α.*

Proof. For $a \in \mathscr{A}$, let $D_\alpha(a) = \{(x, y) \in \Omega_\alpha \mid x \in \Delta_\alpha(a)\}$. The map $\mathcal{T}_\alpha$ is one-to-one, continuous and μ-preserving on each $D_\alpha(a)$. Now, as the $\Delta_\alpha(a)$ partition $\mathbb{I}_\alpha \setminus \{0\}$, $\mathcal{T}_\alpha$ is continuous on Ω_α except for its intersection with a countable number of vertical lines. Since Ω_α is compact and bounded away from $y = -1/x$, these lines are of μ-measure zero. Thus, we find that

$$\mathcal{T}_\alpha(\Omega_\alpha) = \overline{\big\{\mathcal{T}_\alpha^{n+1}(x, 0) \mid x \in [\alpha - 1, \alpha),\, n \geq 0\big\}}\,, \tag{4.1}$$

up to a μ-measure zero set, hence $\mathcal{T}_\alpha(\Omega_\alpha) = \Omega_\alpha$. This implies that

$$\sum_{a \in \mathscr{A}} \mu\big(\mathcal{T}_\alpha(D_\alpha(a))\big) = \sum_{a \in \mathscr{A}} \mu\big(D_\alpha(a)\big) = \mu\big(\Omega_\alpha\big) = \mu\big(\mathcal{T}_\alpha(\Omega_\alpha)\big) = \mu\bigg(\bigcup_{a \in \mathscr{A}} \mathcal{T}_\alpha(D_\alpha(a))\bigg)\,,$$

and thus

$$\mu\big(\mathcal{T}_\alpha(D_\alpha(a)) \cap \mathcal{T}_\alpha(D_\alpha(a'))\big) = 0 \quad \text{for all } a, a' \in \mathscr{A} \text{ with } a \neq a'\,.$$

From its map injectivity on the $D_\alpha(a)$, we conclude that $\mathcal{T}_\alpha$ is bijective on Ω_α up to a set of measure zero. □

Theorem 4.4. *Let $\alpha \in (0, 1]$, μ_α be the probability measure given by normalizing μ on Ω_α, ν_α denote the marginal measure obtained by integrating μ_α over the fibers $\{x\} \times \Phi_\alpha(x)$, where $\Phi_\alpha(x) = \{y \mid (x, y) \in \Omega_\alpha\}$, $\mathscr{B}_\alpha$ be the Borel σ-algebra of $\mathbb{I}_\alpha$, $\mathscr{B}'_\alpha$ be the Borel σ-algebra of Ω_α. Then $(\Omega_\alpha, \mathcal{T}_\alpha, \mathscr{B}'_\alpha, \mu_\alpha)$ is a natural extension of $(\mathbb{I}_\alpha, T_\alpha, \mathscr{B}_\alpha, \nu_\alpha)$.*

Proof. Since $\mathcal{T}_\alpha$ is invertible, with μ_α as an invariant probability measure, we must only show that $\mathscr{B}'_\alpha = \bigvee_{n \geq 0} \mathcal{T}_\alpha^n \pi^{-1} \mathscr{B}_\alpha$, where π is the projection map to the first coordinate. As usual, we define rank n cylinders as $\Delta_\alpha(v_{[1,n]}) = \bigcap_{j=1}^n T_\alpha^{-j+1}(\Delta_\alpha(v_j))$. Since T_α is expanding, for any $v_{[1,\infty)} \in \mathscr{A}_0^\omega$ the Lebesgue measure of $\Delta_\alpha(v_{[1,n]})$ tends to zero as n goes to infinity. Thus P_α, the collection of all of these cylinders, generates $\mathscr{B}_\alpha$. Let $\mathcal{P}_\alpha = \pi^{-1} P_\alpha$, it suffices to show that $\bigvee_{n \in \mathbb{Z}} \mathcal{T}_\alpha^n \mathcal{P}_\alpha$ separates points of Ω_α. We know that $\bigvee_{n \geq 0} T_\alpha^n P_\alpha$ separates points of $\mathbb{I}_\alpha$, thus $\bigvee_{n \geq 0} \mathcal{T}_\alpha^n \mathcal{P}_\alpha$ separates points of the form $(x, y), (x', y')$ with $x \neq x'$. It now suffices to show that powers of $\mathcal{T}_\alpha^{-1}$ on $\mathcal{P}_\alpha$ can separate points sharing the same x-value. Now, on some neighborhood of μ_α-almost any point of Ω_α, there is $a \in \mathscr{A}$ such that $\mathcal{T}_\alpha^{-1}$ is given locally by $(x, y) \mapsto (M_a^{-1} \cdot x, N_a^{-1} \cdot y)$. But, $N_a^{-1} \cdot y$ is an expanding map. Since $\mathcal{T}_\alpha^{-1}$ takes horizontal strips to vertical strips, one can separate points. □

With the help of the following lemma and Abramov's formula, we will show that the product of the entropy and the measure of the natural extension domain is constant.

Lemma 4.5. *Let $\alpha \in (0,1]$, $\mathcal{T}_{1,\alpha}$ be the first return map of $\mathcal{T}_1$ on $\Omega_1 \cap \Omega_\alpha$, and $\mathcal{T}_{\alpha,1}$ be the first return map of $\mathcal{T}_\alpha$ on $\Omega_1 \cap \Omega_\alpha$. For μ-almost all $(x,y) \in \Omega_1 \cap \Omega_\alpha$, these two maps are defined and $\mathcal{T}_{1,\alpha}(x,y) = \mathcal{T}_{\alpha,1}(x,y)$.*

Proof. Note first that $\Omega_1 \cap \Omega_\alpha = \{(x,y) \in \Omega_\alpha \mid x \geq 0\}$ since $\Omega_1 = [0,1]^2$. The ergodicity of T_α yields that, for ν_α-almost every $x \in [0,\alpha]$, there exists some $m \geq 0$ such that $T_\alpha^m(x) \geq 0$, and thus there exists some $n \geq 0$ such that $\mathcal{T}_{\alpha,1}(x,y) = \mathcal{T}_1^n(x,y)$ by Lemma 3.3. Then we have $\mathcal{T}_{1,\alpha}(x,y) = \mathcal{T}_1^{n'}(x,y)$ with $1 \leq n' \leq n$, thus $\mathcal{T}_{1,\alpha}$ and $\mathcal{T}_{\alpha,1}$ are defined for ν_α-almost all $x \in [0,\alpha]$, hence for μ-almost all $(x,y) \in \Omega_1 \cap \Omega_\alpha$.

Suppose that $\mathcal{T}_{1,\alpha}(x,y) \neq \mathcal{T}_{\alpha,1}(x,y)$, that is $n' < n$. The ergodicity of T_1 yields that, for ν_1-almost every $x \in [0,\alpha]$, there exists some $n'' \geq 1$ such that $T_1^{n''-1}(x) \leq 1/(d_\alpha(\alpha)+1)$ and $T_1^{n''}(x) \leq 1/(d_\alpha(\alpha)+1)$. By Lemma 3.6, we obtain that $\mathcal{T}_1^{n''}(x,y) = \mathcal{T}_\alpha^{m'}(x,y)$ for some $m' \geq 1$. Furthermore, Lemma 3.6 implies that $\mathcal{T}_1^{n''}(x,y) = \mathcal{T}_1^{n''-n'}\mathcal{T}_1^{n'}(x,y) = \mathcal{T}_\alpha^{m''}\mathcal{T}_1^{n'}(x,y)$ for some $m'' \geq 1$, with $m'' < m'$. Since $\mathcal{T}_\alpha$ is bijective μ-almost everywhere, we obtain that $\mathcal{T}_1^{n'}(x,y) = \mathcal{T}_\alpha^{-m''}\mathcal{T}_1^{n''}(x,y) = \mathcal{T}_\alpha^{m'-m''}(x,y)$ for μ-almost all $(x,y) \in \Omega_1 \cap \Omega_\alpha$, contradicting that $\mathcal{T}_{\alpha,1}(x,y) = \mathcal{T}_1^n(x,y)$. □

Theorem 4.6. *For every $\alpha \in (0,1]$, we have $h(T_\alpha)\,\mu(\Omega_\alpha) = \pi^2/6$.*

Proof. It is well known that $h(T_1) = \pi^2/(6\log 2)$, see [DK02], and $\mu(\Omega_1) = \mu([0,1]^2) = \log 2$, thus $h(T_1)\,\mu(\Omega_1) = \pi^2/6$. With the definitions of Lemma 4.5, Abramov's formula [Abr59] yields that

$$h(\mathcal{T}_{1,\alpha}) = \frac{\mu(\Omega_1)}{\mu(\Omega_1 \cap \Omega_\alpha)}\, h(\mathcal{T}_1) \quad \text{and} \quad h(\mathcal{T}_{\alpha,1}) = \frac{\mu(\Omega_\alpha)}{\mu(\Omega_1 \cap \Omega_\alpha)}\, h(\mathcal{T}_\alpha)\,.$$

Since $\mathcal{T}_{1,\alpha}$ and $\mathcal{T}_{\alpha,1}$ are equal (up to a set of measure zero), and a system and its natural extension have the same entropy [Roh61], we obtain that $h(T_\alpha)\,\mu(\Omega_\alpha) = h(T_1)\,\mu(\Omega_1)$. □

5. Relation between α-expansions of $\alpha - 1$ and of α

Luzzi and Marmi indicate in [LM08, Remark 3] that the natural extension can be described in a more explicit way when one has an explicit relation between the α-expansions of $\alpha - 1$ and of α. Such a relation is easily found for $\alpha \geq \sqrt{2} - 1$. Nakada and Natsui [NN08] find the relation on some subintervals of $(0, \sqrt{2} - 1)$, showing that it can be rather complicated. The aim of this section is to provide a relation for every $\alpha \in (0,1]$.

For all $\alpha \in (0,1)$, both $T_\alpha(\alpha)$ and $\alpha - 1 = T_\alpha^0(\alpha - 1)$ are negative; it turns out that if either of these T_α-orbits thereafter contains a positive real number then the T_α-orbits of α and $\alpha - 1$ synchronize. (We show that such synchronization occurs on intervals; other than for isolated synchronization for certain quadratic numbers, the converse holds as well.) Thus, the relation between the α-expansions of $\alpha - 1$ and of α is particularly interesting for α in

$$\Gamma := \left\{\alpha \in (0,1] \mid T_\alpha^n(\alpha - 1) \geq 0 \text{ or } T_\alpha^n(\alpha) \geq 0 \text{ for some } n \geq 1\right\},$$

see also Corollary 5.10. Proposition 9.1 shows that Γ contains almost all $\alpha \in (0,1]$.

Theorem 5.1. *Let $\alpha \in \Gamma$. Then there exist (unique) integers $k \geq 1$, $k' \geq 2$ such that*

$$T_\alpha^k(\alpha-1) = T_\alpha^{k'}(\alpha)\,,$$

$$T_\alpha^{j-1}(\alpha-1) < 0 \quad \text{for all} \quad 1 \leq j < k\,, \qquad T_\alpha^{j-1}(\alpha) < 0 \quad \text{for all} \quad 2 \leq j < k'\,,$$

$$\operatorname{sgn}\big(T_\alpha^{k-1}(\alpha-1)\big) = -\operatorname{sgn}\big(T_\alpha^{k'-1}(\alpha)\big)\,, \qquad M_{\overline{b}^\alpha_{k'}} = M_{\underline{b}^\alpha_k} W \quad \text{if } T_\alpha^{k-1}(\alpha-1) \neq 0\,,$$

$$M_{\overline{b}^\alpha_{k'-1}} \cdots M_{\overline{b}^\alpha_1} = \pm W M_{\underline{b}^\alpha_{k-1}} \cdots M_{\underline{b}^\alpha_1} E\,. \tag{5.1}$$

5.1. **Preliminaries to proof.** The proof of the theorem mainly consists in manipulating words on the alphabet $\mathscr{A}_-$ defined in Section 2.3. Indeed, key to finding the relationship between $\underline{b}^\alpha$ and $\overline{b}^\alpha$ is the transformation $v \mapsto \widehat{v}$, with $\widehat{v}$ defined as follows.

Definition 5.2. For any word $v \in \mathscr{A}_-^*$, we set

$$\widehat{v} := (-1:3+c_0)(-1:2)^{c_1}(-1:3+c_2)\cdots(-1:2)^{c_{2\ell-1}}(-1:3+c_{2\ell})\,,$$

where $\ell \geq 0$ and the $c_j \geq 0$, for $0 \leq j \leq 2\ell$, are defined by

$$v = (-1:2)^{c_0}(-1:3+c_1)(-1:2)^{c_2}\cdots(-1:3+c_{2\ell-1})(-1:2)^{c_{2\ell}}\,. \tag{5.2}$$

Similarly, for any infinite word $v \in \mathscr{A}_-^\omega \setminus \mathscr{A}_-^*(-1:2)^\omega$, we set

$$\widehat{v} := (-1:3+c_0)(-1:2)^{c_1}(-1:3+c_2)(-1:2)^{c_3}\cdots\,,$$

where the $c_j \geq 0$, $j \geq 0$, are defined by

$$v = (-1:2)^{c_0}(-1:3+c_1)(-1:2)^{c_2}(-1:3+c_3)\cdots\,.$$

In what follows, we consider *projective* identities of matrices.

Lemma 5.3. *For every $v \in \mathscr{A}_-^*$, we have $M_{\widehat{v}} = EWM_vEW$.*

Proof. Let $F = EW = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$, then the identities

$$FM_{(-1:2)} = EF\,, \quad F^2 = M_{(-1:3)}\,, \quad EM_{(-1:d)} = M_{(-1:d+1)}\,,$$

give

$$FM_{(-1:2)}^d F = \big(FM_{(-1:2)}F^{-1}\big)^d F^2 = E^d M_{(-1:3)} = M_{(-1:d+3)}\,.$$

This proves the lemma for $v = (-1:2)^{c_0}$, i.e., $\ell = 0$. If $\ell \geq 1$, then

$$\begin{aligned} FM_vF &= FM_{(-1:2)}^{c_{2\ell}} M_{(-1:3+c_{2\ell-1})} \cdots M_{(-1:2)}^{c_2} M_{(-1:3+c_1)} M_{(-1:2)}^{c_0} F \\ &= FM_{(-1:2)}^{c_{2\ell}} M_{(-1:3+c_{2\ell-1})} \cdots M_{(-1:2)}^{c_2} M_{(-1:3+c_1)} F^{-1} M_{(-1:2+c_0)} \\ &= FM_{(-1:2)}^{c_{2\ell}} M_{(-1:3+c_{2\ell-1})} \cdots M_{(-1:2)}^{c_2} F M_{(-1:2)}^{c_1} M_{(-1:2+c_0)} \\ &= \cdots = M_{(-1:3+c_{2\ell})} M_{(-1:2)}^{c_{2\ell-1}} \cdots M_{(-1:3+c_2)} M_{(-1:2)}^{c_1} M_{(-1:2+c_0)} = M_{\widehat{v}}\,. \end{aligned}$$

□

Note both that W^2 is the identity, and also that

$$M_{(\varepsilon:d)}W = M_{(-\varepsilon:d+\varepsilon)}\,. \tag{5.3}$$

The action of E is $E\cdot x = x-1$, and

$$E^{\pm 1}M_{(\varepsilon:d)} = M_{(\varepsilon:d\pm 1)}\,.$$

This motivates the following.

Definition 5.4. Let

$$^{(W)}(\varepsilon:d) = \begin{cases} (-\varepsilon:d+\varepsilon) & \text{if } d<\infty\,,\\ (1:\infty) & \text{if } d=\infty\,,\end{cases} \qquad (\varepsilon:d)^{(\pm 1)} = \begin{cases} (\varepsilon:d\pm 1) & \text{if } d<\infty\,,\\ (1:\infty) & \text{if } d=\infty\,.\end{cases}$$

We extend this definition in the natural way to words $v = v_{[1,n]} \in \mathscr{A}_0^*$, $n\ge 2$, by ${}^{(W)}v = {}^{(W)}v_1 v_{[2,n]}$ and $v^{(\pm 1)} = v_{[1,n)}v_n{}^{(\pm 1)}$. Similarly, we set ${}^{(W)}v = {}^{(W)}v_1 v_{[2,\infty)}$ for $v = v_{[1,\infty)} \in \mathscr{A}_0^\omega$.

By Lemma 5.3, we have

$$M_v\cdot(\alpha-1) = M_v\,E\cdot\alpha = WE^{-1}M_{\widehat{v}}\,W\cdot\alpha\,,$$

which gives the following corollary.

Corollary 5.5. *If $\alpha-1 = [\![v_{[1,k)}\,v_{[k,\infty)}]\!]$ with $v_{[1,k)}\in\mathscr{A}_-^*$, $v_{[k,\infty)}\in\mathscr{A}_0^\omega$, then*

$$\alpha = [\![{}^{(W)}\widehat{v_{[1,k)}}^{(-1)}\,{}^{(W)}v_{[k,\infty)}]\!]\,.$$

If $\alpha-1 = [\![v_{[1,\infty)}]\!]$, with $v_{[1,\infty)}\in\mathscr{A}_-^\omega$, then $\alpha = [\![{}^{(W)}\widehat{v_{[1,\infty)}}]\!]$.

We also frequently use the following lemmas. Note that we have $|T_\alpha(x) - M_a\cdot x| < 1$ in particular when $M_a\cdot x\in[\alpha-1,\alpha)$.

Lemma 5.6. *Let $x\in\mathbb{I}_\alpha\setminus\{0\}$, and $a\in\mathscr{A}$. If $|T_\alpha(x)-M_a\cdot x|<1$, then $T_\alpha(x) = M_a\cdot x$ and $(\varepsilon(x):d_\alpha(x)) = a$.*

Proof. Let $a = (\varepsilon':d')\in\mathscr{A}$, then we have $M_a\cdot x = \varepsilon'/x - d'$ and $T_\alpha(x) = \varepsilon(x)/x - d_\alpha(x)$. We cannot have $\varepsilon'\neq\varepsilon(x)$ since this would imply $M_a\cdot x\le -2$, contradicting $|T_\alpha(x)-M_a\cdot x|<1$. Since d', $d_\alpha(x)$ are integers and $\varepsilon' = \varepsilon(x)$, $|T_\alpha(x)-M_a\cdot x|<1$ yields that $d' = d_\alpha(x)$. □

Definition 5.7. For $v_{[1,\infty)} = (-1:d_1)(-1:d_2)\cdots$, $v'_{[1,\infty)} = (-1:d'_1)(-1:d'_2)\cdots\in\mathscr{A}_-^\omega$, let

$$v_{[1,\infty)} \prec v'_{[1,\infty)} \ \text{ if } \ d_1d_2\cdots < d'_1d'_2\cdots \ \text{(lexicographically)},$$

and similarly let $v_{[1,n]}\prec v'_{[1,n]}$ if $d_1\cdots d_n < d'_1\cdots d'_n$.

Lemma 5.8. *Let $v_{[1,\infty)}, v'_{[1,\infty)}\in\mathscr{A}_-^\omega$. If $v_{[1,\infty)}\prec v'_{[1,\infty)}$, then $-1\le[\![v_{[1,\infty)}]\!] < [\![v'_{[1,\infty)}]\!] < 0$. If $v_{[1,n]}\prec v'_{[1,n]}$ and $-1<x\le x'+1$, then $-1 < [\![v_{[1,n]},\,x]\!] \le [\![v'_{[1,n]},\,x']\!] < 0$.*

Proof. The second statement is proved by induction on n. The first is simply the ordering of the T_0-expansions, and is easily verified. □

Lemma 5.9. *Let $x\in\mathbb{I}_\alpha$. If $W\cdot x\in\mathbb{I}_\alpha$, then $T_\alpha(x) = T_\alpha(W\cdot x)$.*

Proof. For $x = 0$, this is clear since $W \cdot 0 = 0$. For $x \neq 0$, we have $T_\alpha(x) = M_{(\varepsilon:d)} \cdot x$ for some $(\varepsilon : d) \in \mathscr{A}$, and $M_{(\varepsilon:d)} = M_{(-\varepsilon:d+\varepsilon)}W$ by (5.3), thus

$$T_\alpha(x) = M_{(\varepsilon:d)} \cdot x = M_{(-\varepsilon:d+\varepsilon)}W \cdot x = T_\alpha(W \cdot x)\,. \qquad \square$$

5.2. **Proof of Theorem 5.1.**

Proof of Theorem 5.1. Let $g = (-1+\sqrt{5})/2$, the small golden number. In case $\alpha > g$, we have $T_\alpha(\alpha) = (1-\alpha)/\alpha = W \cdot (\alpha - 1)$, thus Lemma 5.9 gives $T_\alpha^2(\alpha) = T_\alpha(\alpha - 1)$, and we find that the theorem holds with $k = 1$, $k' = 2$. We have $g \notin \Gamma$ since $T_g^n(g-1) = T_g^n(g) = g - 1 < 0$ for all $n \geq 1$.

In the following, we assume that $0 < \alpha < g$, which implies that $\underline{b}_1^\alpha = (-1 : 2)$ and that $d_{\alpha,n}(x) \geq 2$ for all $x \in \mathbb{I}_\alpha$, $n \geq 1$.

Suppose first that $T_\alpha^{n-1}(\alpha - 1) \geq 0$ for some $n \geq 2$, and let n be minimal with this property, thus $\underline{b}_{[1,n)}^\alpha \in \mathscr{A}_-^*$. Setting $w_{[1,m)} = \widehat{\underline{b}_{[1,n)}^\alpha}^{(-1)}$, by Corollary 5.5 we obtain

$$\alpha = [\![{}^{(W)}w_{[1,m)}\,{}^{(W)}\underline{b}_{[n,\infty)}^\alpha]\!]\,, \tag{5.4}$$

with $m \geq 2$. Since $\varepsilon_{\alpha,n}(\alpha - 1) = 1$, we have

$$[\![{}^{(W)}\underline{b}_{[n,\infty)}^\alpha]\!] = \frac{-1}{1 + d_{\alpha,n}(\alpha-1) + T_\alpha^n(\alpha-1)} \geq \frac{-1}{2+\alpha} > \alpha - 1\,, \tag{5.5}$$

and also $[\![{}^{(W)}\underline{b}_{[n,\infty)}^\alpha]\!] = [\![{}^{(W)}\underline{b}_n^\alpha, T_\alpha^n(\alpha-1)]\!] \leq 0$. Lemma 5.8 gives that $[\![w_{[j,m)}\,{}^{(W)}\underline{b}_{[n,\infty)}^\alpha]\!] = [\![w_{[j,m)}, [\![{}^{(W)}\underline{b}_{[n,\infty)}^\alpha]\!]]\!] \in (-1,0)$ for all $1 \leq j < m$, in particular

$$M_{{}^{(W)}w_1} \cdot \alpha = [\![w_{[2,m)}\,{}^{(W)}\underline{b}_{[n,\infty)}^\alpha]\!] \in (-1, 0]\,.$$

If $T_\alpha(\alpha) \leq 0$, then Lemma 5.6 yields that $\overline{b}_1^\alpha = {}^{(W)}w_1$. If furthermore $T_\alpha^j(\alpha) \leq 0$ for all $2 \leq j < m$, then we obtain inductively that $\overline{b}_{[1,m)}^\alpha = {}^{(W)}w_{[1,m)}$, and that $T_\alpha^m(\alpha) = [\![\,\underline{b}_{[n+1,\infty)}^\alpha]\!] = T_\alpha^n(\alpha - 1)$. The theorem then holds with $k = n$, $k' = m$.

Now, let $w_{[1,m)}$ be as in the preceding paragraph and suppose that $T_\alpha^{m'-1}(\alpha) > 0$ for some $2 \leq m' < m$. Let m' be minimal with this property. Arguing as above, if $m' \geq 3$, we find that $\overline{b}_{[1,m'-1)}^\alpha = {}^{(W)}w_{[1,m'-1)}$ and $\overline{b}_{m'-1}^\alpha = w_{m'-1}{}^{(-1)}$. If $m' = 2$, then $\overline{b}_{[1,m')}^\alpha = {}^{(W)}w_{[1,m')}{}^{(-1)}$ holds as well. This implies that $w_{m'-1} = (-1 : d)$ with $d \geq 3$, and it follows that $w_{[1,m')} = \widehat{\underline{b}_{[1,n')}^\alpha}$ for some $1 \leq n' < n$. By Corollary 5.5, we have $\alpha = [\![{}^{(W)}w_{[1,m')}{}^{(-1)}\,{}^{(W)}\underline{b}_{[n',\infty)}^\alpha]\!]$, thus $k = n'$, $k' = m'$.

Finally suppose that $T_\alpha^n(\alpha - 1) < 0$ for all $n \geq 1$, i.e., $\underline{b}_{[1,\infty)}^\alpha \in \mathscr{A}_-^\omega$. Let $w_{[1,\infty)} = \widehat{\underline{b}_{[1,\infty)}^\alpha}$, and $m' \geq 2$ be minimal such that $T_\alpha^{m'-1}(\alpha) \geq 0$. Then we have $w_{[1,m')} = \widehat{\underline{b}_{[1,n')}^\alpha}$ for some $n' \geq 1$, and $k = n'$, $k' = m'$, as in the preceding paragraph. $\square$

The proof of Theorem 5.1 yields the following corollary.

Corollary 5.10. *The α-expansion of α is obtained from that of $\alpha - 1$ in the following way:*

(i) *If* $\alpha \in (0,1] \setminus \Gamma$*, then* $\overline{b}^{\alpha}_{[1,\infty)} = {}^{(W)}\widehat{\underline{b}^{\alpha}_{[1,\infty)}}$.

(ii) *If* $\alpha \in \Gamma$ *and* k, k' *are as in Theorem 5.1, then*

$$\overline{b}^{\alpha}_{[1,\infty)} = {}^{(W)}\widehat{\underline{b}^{\alpha}_{[1,k)}}^{(-1)}\,{}^{(W)}\underline{b}^{\alpha}_{[k,\infty)}\,.$$

Let $n, m \geq 2$ *be minimal such that* $T_\alpha^{n-1}(\alpha-1) \geq 0$, $T_\alpha^{m-1}(\alpha) \geq 0$*. Then* $k = n$ *if both* $n < \infty$ *and* $m \geq \big|\widehat{\underline{b}^{\alpha}_{[1,n)}}\big|$, $k' = m$ *else. Moreover,* $k' = |\widehat{\underline{b}^{\alpha}_{[1,k)}}| + 1$.

We give examples realizing some of the various cases that arise in the theorem. Further examples can be adapted from Corollaries 7.7 and 7.6.

Example 5.11. If $\alpha = 1/r$ for some positive integer r, then $\alpha - 1 = [\![(-1:2)^{r-1}]\!]$, thus $k = r$, $k' = 2$.

6. Structure of the natural extension domains

For an explicit description of Ω_α, we require detailed knowledge of the effects of $\mathcal{T}_\alpha$ on the regions fibered above non-full cylinders determined by the T_α-orbits of $\alpha - 1$ and α. To this end, we use the languages introduced below.

Extend the order $\preceq$ to $\mathscr{A}_0$ by setting

$$(\varepsilon : d) \preceq (\varepsilon' : d') \quad \text{if} \quad \varepsilon/d \leq \varepsilon'/d'\,.$$

Definition 6.1. For $\alpha \in (0,1] \setminus \Gamma$, we define the following:

$$U_{\alpha,1} = \big\{\underline{b}^{\alpha}_{[1,j)} \mid j \geq 1\big\}\,, \quad U_{\alpha,3} = \big\{\underline{b}^{\alpha}_{[1,j)}\, a \mid j \geq 1,\, a \in \mathscr{A},\, \underline{b}^{\alpha}_{j} \prec a \prec \overline{b}^{\alpha}_{1}\big\}\,,$$
$$U_{\alpha,2} = \big\{\overline{b}^{\alpha}_{[1,j)} \mid j \geq 2\big\}\,, \quad U_{\alpha,4} = \big\{\overline{b}^{\alpha}_{[1,j)}\, a \mid j \geq 2,\, a \in \mathscr{A},\, \overline{b}^{\alpha}_{j} \prec a \prec \overline{b}^{\alpha}_{1}\big\}\,.$$

For $\alpha \in \Gamma$, let k, k' be as in Theorem 5.1, and define the following:

$$U_{\alpha,1} = \big\{\underline{b}^{\alpha}_{[1,j)} \mid 1 \leq j \leq k\big\}\,,$$
$$U_{\alpha,2} = \big\{\overline{b}^{\alpha}_{[1,j)} \mid 2 \leq j \leq k'\big\}\,,$$
$$U_{\alpha,3} = \big\{\underline{b}^{\alpha}_{[1,j)}\, a \mid 1 \leq j < k,\, a \in \mathscr{A},\, \underline{b}^{\alpha}_{j} \prec a \prec \overline{b}^{\alpha}_{1}\big\} \cup \big\{\underline{b}^{\alpha}_{[1,k)}\, a \mid a \in \mathscr{A}_+,\, a \prec \overline{b}^{\alpha}_{1}\big\}\,,$$
$$U_{\alpha,4} = \big\{\overline{b}^{\alpha}_{[1,j)}\, a \mid 2 \leq j < k',\, a \in \mathscr{A},\, \overline{b}^{\alpha}_{j} \prec a \prec \overline{b}^{\alpha}_{1}\big\} \cup \big\{\overline{b}^{\alpha}_{[1,k')}\, a \mid a \in \mathscr{A}_+,\, a \prec \overline{b}^{\alpha}_{1}\big\}\,.$$

In both cases, let

$$\mathscr{L}_\alpha = (U_{\alpha,3} \cup U_{\alpha,1}\, U^*_{\alpha,2}\, U_{\alpha,4})^*\,, \quad \mathscr{L}'_\alpha = \mathscr{L}_\alpha\, U_{\alpha,1}\, U^*_{\alpha,2}\,,$$
$$\Psi_\alpha = \big\{N_w \cdot 0 \mid w \in \mathscr{L}_\alpha\big\}\,, \qquad \Psi'_\alpha = \big\{N_w \cdot 0 \mid w \in \mathscr{L}'_\alpha\big\}\,.$$

The languages introduced above allow us to view the region Ω_α as being the union of pieces, each of which fibers over a subinterval whose left endpoint is in the T_α-orbit of α or of $\alpha - 1$. We will see in Lemma 6.5 that $\mathscr{L}'_\alpha$ is the language of the α-expansions avoiding $(1:\infty)$ if either $\alpha \in (0,1] \setminus \Gamma$ or $T_\alpha^{k-1}(\alpha-1) = T_\alpha^{k'-1}(\alpha) = 0$. For other α, $\mathscr{L}'_\alpha$ is slightly different from the language of the α-expansions.

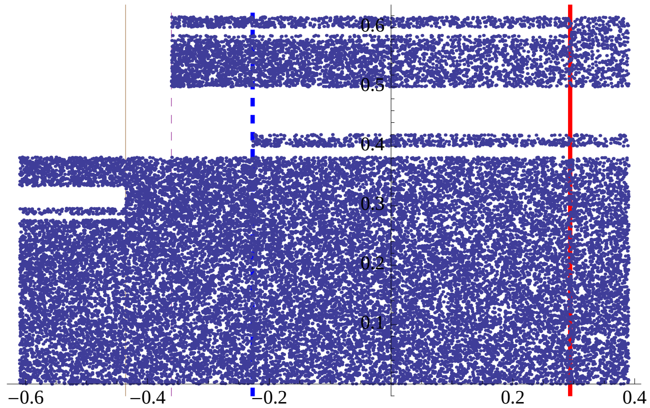

FIGURE 1. Plotting 30,000 points of the $\mathcal{T}_\alpha$- orbit of $(\pi/10, 0)$ gives an indication of Ω_α, here $\alpha = 0.39$. Vertical solid lines: $T_\alpha(\alpha)$ (brown), $T_\alpha^2(\alpha)$ (thick red); dashed: $T_\alpha(\alpha-1)$ (purple), $T_\alpha^2(\alpha-1)$ (thick blue).

Proposition 6.2. *Let $\alpha \in (0,1]$. Then we have*

$$(6.1)\quad \bigcup_{n\ge 0} \mathcal{T}_\alpha^n\big([\alpha-1,\alpha)\times\{0\}\big)$$
$$= [\alpha-1,\alpha)\times\Psi_\alpha \ \cup \bigcup_{1\le j<k} \big[T_\alpha^j(\alpha-1),\alpha\big)\times N_{\underline{b}^\alpha_{[1,j]}}\cdot\Psi_\alpha \ \cup \bigcup_{1\le j<k'} \big(T_\alpha^j(\alpha),\alpha\big)\times N_{\overline{b}^\alpha_{[1,j]}}\cdot\Psi'_\alpha\,,$$

where k, k' is as in Theorem 5.1 if $\alpha\in\Gamma$, $k=k'=\infty$ otherwise.

Here, (a,b) always denotes the open interval between a and b (and not a point in $\mathbb{R}^2$), and the map $\mathcal{T}_\alpha$ acts on products of two sets in $\mathbb{R}$.

The following lemmas are used in the proof of the proposition.

Lemma 6.3. *For any $\alpha\in(0,1]$, $\mathscr{L}'_\alpha$ admits the partition*

$$\mathscr{L}'_\alpha = \mathscr{L}_\alpha \ \cup \bigcup_{1\le j<k} \mathscr{L}_\alpha\,\underline{b}^\alpha_{[1,j]} \ \cup \bigcup_{1\le j<k'} \mathscr{L}'_\alpha\,\overline{b}^\alpha_{[1,j]}\,.$$

Proof. In the factorization $\mathscr{L}'_\alpha = \mathscr{L}_\alpha\,U_{\alpha,1}\,U^*_{\alpha,2}$, there are two cases: the exponent of $U_{\alpha,2}$ being zero or not. In the first case, the element of $U_{\alpha,1}$ can be the empty word $\underline{b}^\alpha_{[1,1)}$, which gives $\mathscr{L}_\alpha$, or a word $\underline{b}^\alpha_{[1,j]}$, $1\le j<k$. In the second case, we can factor exactly one power of $U_{\alpha,2}$ to the right. Since the decomposition of every $w\in\mathscr{L}'_\alpha$ into factors in $\mathscr{L}_\alpha, U_{\alpha,1}, U^*_{\alpha,2}$ (in this order) is unique, this proves the lemma. □

By Lemma 6.3, we can write (6.1) as

$$(6.2)\qquad \bigcup_{n\ge 0} \mathcal{T}_\alpha^n\big([\alpha-1,\alpha)\times\{0\}\big) = \bigcup_{w\in\mathscr{L}'_\alpha} J^\alpha_w \times \{N_w\cdot 0\}\,,$$

where

$$J_w^\alpha = \begin{cases} [\alpha-1,\alpha) & \text{if } w \in \mathscr{L}_\alpha, \\ \big[T_\alpha^j(\alpha-1),\alpha\big) & \text{if } w \in \mathscr{L}_\alpha\, \underline{b}^\alpha_{[1,j]},\ 1 \le j < k\,, \\ \big(T_\alpha^j(\alpha),\alpha\big) & \text{if } w \in \mathscr{L}'_\alpha\, \overline{b}^\alpha_{[1,j]},\ 1 \le j < k'\,. \end{cases}$$

From now on, denote by $\Delta_\alpha(w)$, $w \in \mathscr{A}^*$, the set of numbers $x \in [\alpha-1,\alpha)$ with α-expansion starting with w. This only differs from previous definitions in that $\Delta_\alpha(w)$ never contains the point α.

Lemma 6.4. *Let $\alpha \in (0,1] \setminus \Gamma$ or $\alpha \in \Gamma$, $T_\alpha^{k-1}(\alpha-1) = T_\alpha^{k'-1}(\alpha) = 0$. For any $w \in \mathscr{L}'_\alpha$,*

$$J_w^\alpha = T_\alpha^{|w|}\big(\Delta_\alpha(w)\big) = M_w \cdot \Delta_\alpha(w)\,.$$

Proof. The second equality follows immediately from the definitions.

The first equality clearly holds if w is the empty word. We proceed by induction on $|w|$. The definition of $\mathscr{L}'_\alpha$ implies that every $w' \in \mathscr{L}'_\alpha$ with $|w'| \ge 1$ can be written as $w' = wa$ with $w \in \mathscr{L}'_\alpha$, $a \in \mathscr{A}$. Let first $w \in \mathscr{L}_\alpha\, \underline{b}^\alpha_{[1,j)}$, $1 \le j < k$, which implies $\underline{b}^\alpha_j \preceq a \preceq \overline{b}^\alpha_1$. By the conditions on α, we have $T_\alpha^{j-1}(\alpha-1) < 0$, thus

$$J_w^\alpha = \big[T_\alpha^{j-1}(\alpha-1),\alpha\big) = \big[T_\alpha^{j-1}(\alpha-1), \tfrac{-1}{d_{\alpha,j}(\alpha-1)+\alpha}\big) \ \cup \bigcup_{\substack{a\in\mathscr{A} \\ \underline{b}^\alpha_j \prec a \preceq \overline{b}^\alpha_1}} \Delta_\alpha(a)\ \cup \{0\}\,.$$

Then, $J_w^\alpha = T_\alpha^{|w|}(\Delta_\alpha(w))$ implies that

$$\begin{aligned} T_\alpha^{|w|+1}\big(\Delta_\alpha(w\underline{b}^\alpha_j)\big) &= T_\alpha\big(J_w^\alpha \cap \Delta_\alpha(\underline{b}^\alpha_j)\big) = \big[T_\alpha^j(\alpha-1),\alpha\big) = J_{w\underline{b}^\alpha_j}\,, \\ T_\alpha^{|w|+1}\big(\Delta_\alpha(wa)\big) &= [\alpha-1,\alpha) = J_{wa} \quad (\underline{b}^\alpha_j \prec a \prec \overline{b}^\alpha_1)\,, \\ T_\alpha^{|w|+1}\big(\Delta_\alpha(w\overline{b}^\alpha_1)\big) &= \big(T_\alpha(\alpha),\alpha\big) = J_{w\overline{b}^\alpha_1}\,. \end{aligned}$$

If $w \in \mathscr{L}'_\alpha\, \overline{b}^\alpha_{[1,j)}$, $2 \le j < k'$, then similar arguments yield that $T_\alpha^{|w|+1}(\Delta_\alpha(wa)) = J_{wa}$ for $\overline{b}^\alpha_j \preceq a \preceq \overline{b}^\alpha_1$. Finally, if $w \in \mathscr{L}_\alpha\, \underline{b}^\alpha_{[1,k)}$ and $w \in \mathscr{L}'_\alpha\, \overline{b}^\alpha_{[1,k')}$ respectively (and thus $\alpha \in \Gamma$), then our hypothesis gives $J_w^\alpha = [0,\alpha)$ and $J_w^\alpha = (0,\alpha)$ respectively. Here, $wa \in \mathscr{L}'_\alpha$ is equivalent with $a \in \mathscr{A}_+$, $a \preceq \overline{b}^\alpha_1$, and we obtain again that $T_\alpha^{|w|+1}(\Delta_\alpha(wa)) = J_{wa}$. □

Lemma 6.5. *Let $\alpha \in (0,1] \setminus \Gamma$ or $\alpha \in \Gamma$, $T_\alpha^{k-1}(\alpha-1) = T_\alpha^{k'-1}(\alpha) = 0$. For any $n \ge 1$,*

$$[\alpha-1,\alpha) = \bigcup_{w\in\mathscr{L}'_\alpha:\,|w|=n} \Delta_\alpha(w)\ \cup\ \{x \in [\alpha-1,\alpha) \mid T_\alpha^{n-1}(x) = 0\}\,, \tag{6.3}$$

i.e., $\mathscr{L}'_\alpha$ is the language of the α-expansions of $x \in [\alpha-1,\alpha)$ avoiding $(1:\infty)$.

Proof. We have $\mathscr{L}'_\alpha \cap \mathscr{A} = \{a \in \mathscr{A} \mid \underline{b}^\alpha_1 \preceq a \preceq \overline{b}^\alpha_1\}$, thus (6.3) holds for $n = 1$. In the proof of Lemma 6.4, we have seen that

$$T_\alpha^{|w|}\big(\Delta_\alpha(w)\big) \setminus \{0\} = \bigcup_{a\in\mathscr{A}:\,wa\in\mathscr{L}'_\alpha} T_\alpha^{|w|}\big(\Delta_\alpha(wa)\big)$$

for all $w \in \mathscr{L}'_\alpha$. By applying M_w^{-1}, we obtain the corresponding subdivision of $\Delta_\alpha(w)$, which yields inductively (6.3) for all $n \geq 1$. □

Lemmas 6.4 and 6.5 show that (6.2) and thus (6.1) hold if $\alpha \in (0,1] \setminus \Gamma$ or $\alpha \in \Gamma$, $T_\alpha^{k-1}(\alpha-1) = T_\alpha^{k'-1}(\alpha) = 0$.

The proof of Proposition 6.2 for general $\alpha \in \Gamma$ is slightly more complicated. Note that

$$\mathcal{T}_\alpha\big(J_w^\alpha \times \{N_w \cdot 0\}\big) = \{0\} \times \{0\} \ \cup \bigcup_{a \in \mathscr{A}:\, wa \in \mathscr{L}'_\alpha} J_{wa}^\alpha \times \{N_{wa} \cdot 0\} \tag{6.4}$$

holds for all $w \in \mathscr{L}'_\alpha \setminus \big(\mathscr{L}_\alpha\, \underline{b}^\alpha_{[1,k)} \cup \mathscr{L}'_\alpha\, \overline{b}^\alpha_{[1,k')}\big)$, by arguments similar to the proof of Lemma 6.4. For $w \in \mathscr{L}_\alpha\, \underline{b}^\alpha_{[1,k)} \cup \mathscr{L}'_\alpha\, \overline{b}^\alpha_{[1,k')}$, we use the following lemmas.

Lemma 6.6. *Let $\alpha \in (0,1]$, $v \in \mathscr{A}^*$ with $|v| \geq 1$. Then $v \in \mathscr{L}_\alpha$ if and only if $v^{(-1)} \in \mathscr{L}'_\alpha$.*

Proof. This follows directly from the definition of $\mathscr{L}_\alpha$ and $\mathscr{L}'_\alpha$. □

Lemma 6.7. *Let $\alpha \in \Gamma$ and $w = \underline{b}^\alpha_{[1,k)}$, $w' = \overline{b}^\alpha_{[1,k')}$, or $w = v\, \underline{b}^\alpha_{[1,k)}$, $w' = v^{(-1)}\, \overline{b}^\alpha_{[1,k')}$ with $v \in \mathscr{L}_\alpha$, $|v| \geq 1$. Then $w, w' \in \mathscr{L}'_\alpha$ and*

$$\begin{aligned} &\mathcal{T}_\alpha\big(J_w^\alpha \times \{N_w \cdot 0\}\big) \ \cup\ \mathcal{T}_\alpha\big(J_{w'}^\alpha \times \{N_{w'} \cdot 0\}\big) \\ &\quad = \{0\} \times \{0\} \ \cup \bigcup_{a \in \mathscr{A}:\, wa \in \mathscr{L}'_\alpha} J_{wa}^\alpha \times \{N_{wa} \cdot 0\} \ \cup \bigcup_{a \in \mathscr{A}:\, w'a \in \mathscr{L}'_\alpha} J_{w'a}^\alpha \times \{N_{w'a} \cdot 0\}\,. \end{aligned} \tag{6.5}$$

Proof. Recall that $\operatorname{sgn}(T_\alpha^{k-1}(\alpha-1)) = -\operatorname{sgn}(T_\alpha^{k'-1}(\alpha))$ by Theorem 5.1. We assume that $T_\alpha^{k-1}(\alpha-1) < 0$, the case $T_\alpha^{k'-1}(\alpha) < 0$ being symmetric, and the case $T_\alpha^{k-1}(\alpha-1) = 0$ being trivial since (6.4) holds for w and w' in this case (except for the point $\{0\} \times \{0\}$ not belonging to $\mathcal{T}_\alpha(J_{w'}^\alpha \times \{N_{w'} \cdot 0\})$). This implies

$$\begin{aligned} \mathcal{T}_\alpha\big(J_w^\alpha \times \{N_w \cdot 0\}\big) &= \big[T_\alpha^k(\alpha-1), \alpha\big) \times \{N_{w\underline{b}_k^\alpha} \cdot 0\} \ \cup\ \{0\} \times \{0\} \\ &\quad \cup\ [\alpha-1, \alpha) \times \big\{N_{wa} \cdot 0 \mid \underline{b}_k^\alpha \prec a \prec \overline{b}_1^\alpha\big\} \ \cup\ \big(T_\alpha(\alpha), \alpha\big) \times \{N_{w\overline{b}_1^\alpha} \cdot 0\} \end{aligned}$$

and, if $\overline{b}_{k'}^\alpha \prec \overline{b}_1^\alpha$,

$$\begin{aligned} \mathcal{T}_\alpha\big(J_{w'}^\alpha \times \{N_{w'} \cdot 0\}\big) &= \big[\alpha-1, T_\alpha^{k'}(\alpha)\big) \times \{N_{w'\overline{b}_{k'}^\alpha} \cdot 0\} \\ &\quad \cup\ [\alpha-1, \alpha) \times \big\{N_{w'a} \cdot 0 \mid \overline{b}_{k'}^\alpha \prec a \prec \overline{b}_1^\alpha\big\} \ \cup\ \big(T_\alpha(\alpha), \alpha\big) \times \{N_{w'\overline{b}_1^\alpha} \cdot 0\}\,, \end{aligned}$$

whereas $\mathcal{T}_\alpha\big(J_{w'}^\alpha \times \{N_{w'} \cdot 0\}\big) = \big(T_\alpha(\alpha), T_\alpha^{k'}(\alpha)\big) \times \{N_{w'\overline{b}_1^\alpha} \cdot 0\}$ if $\overline{b}_{k'}^\alpha = \overline{b}_1^\alpha$.

Theorem 5.1 gives $T_\alpha^k(\alpha-1) = T_\alpha^{k'}(\alpha)$, $\underline{b}_k^\alpha = {}^{(W)}\overline{b}_{k'}^\alpha$, and $M_{w'{}^{(W)}a} = M_{wa}E$ for any $a \in \mathscr{A}$ if $w = \underline{b}^\alpha_{[1,k)}$, $w' = \overline{b}^\alpha_{[1,k')}$, whereas

$$M_{w'{}^{(W)}a} = M_{v^{(-1)}\, \overline{b}^\alpha_{[1,k')}{}^{(W)}a} = M_a W M_{\overline{b}^\alpha_{[1,k')}} E^{-1} M_v = M_a M_{\underline{b}^\alpha_{[1,k)}} M_v = M_{wa}$$

otherwise. In all cases, this yields that $N_{wa}\cdot 0 = N_{w'(W)_a}\cdot 0$. Applying this for $a\in\mathscr{A}_-$, we obtain that

$$\begin{aligned}\mathcal{T}_\alpha\big(J_w^\alpha\times\{N_w\cdot 0\}\big)\cup\mathcal{T}_\alpha\big(J_{w'}^\alpha\times\{N_{w'}\cdot 0\}\big) &= \big(T_\alpha(\alpha),\alpha\big)\times\{N_{w\overline{b}_1^\alpha}\cdot 0,\, N_{w'\overline{b}_1^\alpha}\cdot 0\}\cup\{0\}\times\{0\}\\ &\cup\ [\alpha-1,\alpha)\times\big\{N_{wa}\cdot 0\mid a\in\mathscr{A}_+,\, a\prec\overline{b}_1^\alpha\big\}\ \cup\ [\alpha-1,\alpha)\times\big\{N_{w'a}\cdot 0\mid a\in\mathscr{A}_+,\, a\prec\overline{b}_1^\alpha\big\},\end{aligned}$$

which is precisely (6.5). □

Proof of Proposition 6.2. We have already noted that (6.1) is equivalent to (6.2), and that (6.2) follows from Lemmas 6.4 and 6.5 for $\alpha\in(0,1]\setminus\Gamma$ or $\alpha\in\Gamma$, $T_\alpha^{k-1}(\alpha-1) = T_\alpha^{k'-1}(\alpha)=0$. For general $\alpha\in\Gamma$, we already know that (6.4) holds for $w\in\mathscr{L}'_\alpha\setminus\big(\mathscr{L}_\alpha\,\underline{b}^\alpha_{[1,k)}\cup\mathscr{L}'_\alpha\,\overline{b}^\alpha_{[1,k')}\big)$. Together with Lemma 6.7, this gives inductively that

$$\bigcup_{w\in\mathscr{L}'_\alpha:\,|w|\le nm/m'} J_w^\alpha\times\{N_w\cdot 0\}\subseteq\bigcup_{0\le j\le n}\mathcal{T}_\alpha^j\big([\alpha-1,\alpha)\times\{0\}\big)\subseteq\bigcup_{w\in\mathscr{L}'_\alpha:\,|w|\le nm'/m} J_w^\alpha\times\{N_w\cdot 0\}\,.$$

for every $n\ge 0$, where $m=\min(k,k')$ and $m'=\max(k,k')$. This shows again (6.2), hence the proposition. □

For $\alpha\in(0,1]$, $x\in\mathbb{I}_\alpha$, the *x-fiber* is $\Phi_\alpha(x)=\{y\mid(x,y)\in\Omega_\alpha\}$. The description of Ω_α as the union of pieces fibering above the various J_w shows both that fibers are constant between points in the union of the orbits of α and $\alpha-1$ and that a fiber contains every fiber to its left. The maximal fiber is therefore $\Phi_\alpha(\alpha)$, which is equal to $\overline{\Psi'_\alpha}$ by (6.1) and Lemma 6.3. To be precise, we state the following.

Corollary 6.8. *Let $\alpha\in(0,1]$. If $x,x'\in\mathbb{I}_\alpha$, $x\le x'$, then $\Phi_\alpha(x)\subseteq\Phi_\alpha(x')\subseteq\Phi_\alpha(\alpha)=\overline{\Psi'_\alpha}$. Let k,k' be as in Theorem 5.1 if $\alpha\in\Gamma$, $k=k'=\infty$ if $\alpha\notin\Gamma$. If*

$$(x,x']\cap\Big(\big\{T_\alpha^j(\alpha-1)\mid 0\le j<k\big\}\cup\big\{T_\alpha^j(\alpha)\mid 1\le j<k'\big\}\Big)=\emptyset\,,$$

then $\Phi_\alpha(x)=\Phi_\alpha(x')$.

The set $\mathscr{A}_\alpha:=\{(-1:d')\mid 2\le d'\le d_\alpha(\alpha)+1\}\cup\{(1:d_\alpha(\alpha))\}$ plays an important role in the following.

Lemma 6.9. *Let $\alpha\in(0,1]$, $w\in\mathscr{L}'_\alpha$. If the final letter of w is not in $\mathscr{A}_\alpha$, then $w\in\mathscr{L}_\alpha$.*

Proof. We only have to show that $\underline{b}^\alpha_{[1,k)}$ and $\overline{b}^\alpha_{[1,k')}$ are in $\mathscr{A}_\alpha^*$. If $k'=2$, then this is clearly true since $\overline{b}^\alpha_{[1,k')}=(1:d_\alpha(\alpha))$, and hence $\underline{b}^\alpha_{[1,k)}=(-1:2)^{d_\alpha(\alpha)-1}$. Suppose that $k'\ge 3$, and $\underline{b}^\alpha_{[1,k)}$ or $\overline{b}^\alpha_{[1,k')}$ contains some letter $(-1:d')$, $d'\ge d_\alpha(\alpha)+2$. Then $\underline{b}^\alpha_{[1,k)}$ or $\overline{b}^\alpha_{[1,k')}$ contains a factor $(-1:2)^{d_\alpha(\alpha)-1}$, but this is impossible since $(-1:2)^{d_\alpha(\alpha)-1}\prec\underline{b}^\alpha_{[1,d_\alpha(\alpha))}$. □

Proposition 6.10. *Let $\alpha\in(0,1]$. Then $\big[0,\frac{1}{d_\alpha(\alpha)+1}\big]\subset\overline{\Psi_\alpha}$, thus $\mathbb{I}_\alpha\times\big[0,\frac{1}{d_\alpha(\alpha)+1}\big]\subset\Omega_\alpha$ and*

$$\Omega_\alpha=\overline{\bigcup_{w\in\mathscr{L}'_\alpha\cap\mathscr{A}_\alpha^*} J_w^\alpha\times N_w\cdot\big[0,\tfrac{1}{d_\alpha(\alpha)+1}\big]}\,.$$

Proof. We already know from Lemma 4.2 and Corollary 6.8 that $\big[0, \frac{1}{d_\alpha(\alpha)+1}\big] \subset \overline{\Psi'_\alpha}$. The last letter of any $w \in \mathscr{L}'_\alpha$ with $0 < N_w \cdot 0 < \frac{1}{d_\alpha(\alpha)+1}$ is not in $\mathscr{A}_\alpha$, thus $w \in \mathscr{L}_\alpha$ by Lemma 6.9, and $\big[0, \frac{1}{d_\alpha(\alpha)+1}\big] \subset \overline{\Psi_\alpha}$. It follows from (6.1) that $\mathbb{I}_\alpha \times \big[0, \frac{1}{d_\alpha(\alpha)+1}\big] \subset \Omega_\alpha$.

By (6.2) and since $\mathscr{L}'_\alpha = \mathscr{L}_\alpha \mathscr{L}'_\alpha$, we have $\bigcup_{w\in\mathscr{L}'_\alpha} J^\alpha_w \times N_w \cdot \big[0, \frac{1}{d_\alpha(\alpha)+1}\big] \subseteq \Omega_\alpha$. For containment in the other direction, write any $w' \in \mathscr{L}'_\alpha$ as $w' = vw$, with $w \in \mathscr{A}^*_\alpha$ and v empty or ending with a letter in $\mathscr{A} \setminus \mathscr{A}_\alpha$. Then we have $v \in \mathscr{L}_\alpha$ by Lemma 6.9, $w \in \mathscr{L}'_\alpha$ and $J^\alpha_{w'} = J^\alpha_w$, thus

$$J^\alpha_{w'} \times N_{w'} \cdot 0 = J^\alpha_w \times N_w N_v \cdot 0 \subset J^\alpha_w \times N_w \cdot \big[0, \tfrac{1}{d_\alpha(\alpha)+1}\big].$$

Using again (6.2), this shows that $\Omega_\alpha \subseteq \overline{\bigcup_{w\in\mathscr{L}'_\alpha\cap\mathscr{A}^*_\alpha} J^\alpha_w \times N_w \cdot \big[0, \frac{1}{d_\alpha(\alpha)+1}\big]}$. □

7. Intervals of synchronizing orbits

Definition 7.1. The set of *labels of finitely synchronizing orbits* is

$$\mathscr{F} = \big\{\underline{b}^\alpha_{[1,k)} \mid \alpha \in \Gamma,\ k \text{ as in Theorem 5.1}\big\}.$$

Given $v \in \mathscr{F}$, set

$$\Gamma_v = \big\{\alpha \in \Gamma \mid \underline{b}^\alpha_{[1,k)} = v,\ k \text{ as in Theorem 5.1}\big\}.$$

Let $\zeta_v = g$, $\eta_v = 1$ if v is the empty word, and for nonempty $v \in \mathscr{F}$, define

$$\zeta_v = [\![(v\,\widehat{v}\,)^\omega]\!] + 1 \quad\text{and}\quad \eta_v = [\![(v^{(+1)})^\omega]\!] + 1\,.$$

Finally, for $v \in \mathscr{F}$ let

$$L_v = M_v E \quad\text{and}\quad R_v = E^{-1} M_{\widehat{v}} W\,.$$

See Figure 2.

We identify the elements of $\mathscr{F}$ and the intervals of common digits, Γ_v, for each $v \in \mathscr{F}$. To this end, we use the following order.

Definition 7.2. We use the *alternating (partial) order* on words of integers, i.e.,

$$c_{[0,n]} <_{\text{alt}} c'_{[0,n]} \quad\text{if}\quad c_{[0,j)} = c'_{[0,j)},\ (-1)^j c_j < (-1)^j c'_j \text{ for some } 0 \le j \le n\,.$$

Proposition 7.3. *Let $v \in \mathscr{A}^*_-$ and $c_{[0,2\ell]}$, $\ell \ge 0$, as in (5.2). Then $v \in \mathscr{F}$ if and only if*

$$c_{[2j,2\ell]} \le_{\text{alt}} c_{[0,2\ell-2j]} \text{ and } c_{[2j-1,2\ell]} <_{\text{alt}} c_{[0,2\ell-2j+1]} \quad \text{for all } 1 \le j \le \ell\,. \tag{7.1}$$

If v is the empty word, then $\Gamma_v = (g,1]$. For any other $v \in \mathscr{F}$, we have

$$\Gamma_v = (\zeta_v, \eta_v)\,.$$

Moreover, $M_v \cdot (\eta_v - 1) = \eta_v$ and $M_{v'} \cdot (\zeta_v - 1) = \zeta_v$, where $v' = {}^{(W)}\widehat{v}^{(-1)}$. For any $\alpha \in \Gamma_v$, $M_{v_{[1,n]}} \cdot (\alpha - 1) > \eta_v - 1$ for all $1 \le n \le |v|$, and $M_{v'_{[1,n]}} \cdot \alpha > \eta_v - 1$ for all $1 \le n \le |v'|$.

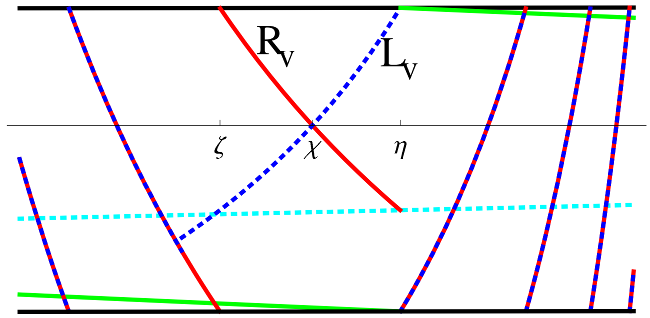

FIGURE 2. Graphs of $\alpha \mapsto T_\alpha^4(\alpha)$ in solid red, $\alpha \mapsto T_\alpha^4(\alpha-1)$ in dotted blue, $\alpha \mapsto T_\alpha^2(\alpha)$ in solid green, and $\alpha \mapsto T_\alpha^2(\alpha-1)$ in dotted cyan, near Γ_v for $v = (-1:2)(-1:3)(-1:4)(-1:2) = \Theta(\,(-1:2)(-1:3)\,)$. On $\Gamma_v = (0.3867\ldots, 0.3874\ldots)$, R_v and L_v agree with $\alpha \mapsto T_\alpha^4(\alpha)$ and $\alpha \mapsto T_\alpha^4(\alpha-1)$, respectively; have a common zero at $\chi = \chi_v$; and, meet the graph of the identity function at $\zeta = \zeta_v$ and $\eta = \eta_v$, respectively. For $\alpha \in \Gamma_{(-1:2)(-1:3)} = (0.3874\ldots, 0.4142\ldots)$, one has $T_\alpha^4(\alpha) = T_\alpha^4(\alpha-1)$, whereas to the left of $\zeta = \zeta_v$, one sees that there is a gap before once again these agree. The transcendental τ_v lies in this gap.

Remark 7.4. The proposition implies in particular that for each $v \in \mathscr{F}$, the graphs of $L_v \cdot x$ and $R_v \cdot x$ cross above (ζ_v, η_v), with common zero at $\chi_v = [\![v, 0]\!]$. See Figure 2.

In the proof of the proposition, we use the following lemma.

Lemma 7.5. *For $v = v_{[1,k)}$ as in (5.2), and $\widehat{v} = \widehat{v}_{[1,k')}$, let*

$$s_j = \sum_{i=0}^{j-1}(c_{2i}+1) \quad \text{and} \quad s'_j = 1 + \sum_{i=1}^{j-1}(c_{2i-1}+1)\,.$$

(i) *For all $1 \le j \le \ell$, we have*

$$c_{[2j,2\ell]} \le_{\mathrm{alt}} c_{[0,2\ell-2j]} \quad \text{if and only if} \quad v_{[s_j+1,k)} \succeq v_{[1,k-s_j)}\,.$$

(ii) *For all $1 \le j \le \ell$, we have*

$$c_{[2j-1,2\ell]} <_{\mathrm{alt}} c_{[0,2\ell-2j+1]} \quad \text{if and only if} \quad \widehat{v}_{[s'_j+1,\min(k',s'_j+k))} \succ v_{[1,\min(k'-s'_j,k))}\,.$$

(iii) *If (7.1) holds, then we have*

$$\begin{aligned} v_{[n,k)} &\succeq v_{[1,k-n]} && \text{for all } 2 \le n < k, \\ \widehat{v}_{[n,\min(k',n+k-2))} &\succ v_{[1,\min(k'-n+1,k-1))} && \text{for all } 1 \le n < k' \text{ (if } k \ge 3), \end{aligned}$$

and inverse relations hold upon exchanging v and $\widehat{v}$, i.e.,

$$\begin{aligned} \widehat{v}_{[n,k')} &\preceq \widehat{v}_{[1,k'-n]} && \text{for all } 2 \le n < k', \\ v_{[n,\min(k,n+k'-2))} &\prec \widehat{v}_{[1,\min(k-n+1,k'-1))} && \text{for all } 1 \le n < k \text{ (if } k' \ge 3). \end{aligned}$$

Proof. To prove (i), suppose first that $c_{[2j,2\ell]} <_{\mathrm{alt}} c_{[0,2\ell-2j]}$. If $c_{[2j,2i-1)} = c_{[0,2i-2j-1)}$ and $c_{2i-1} > c_{2i-2j-1}$ for some $j < i \le \ell$, then

$$v_{[s_j+1,s_i)} = (-1:2)^{c_{2j}}(-1:3+c_{2j+1})\cdots(-1:2)^{c_{2i-2}} = v_{[1,s_{i-j})}$$
$$\text{and}\quad v_{s_i} = (-1:3+c_{2i-1}) \succ (-1:3+c_{2i-2j-1}) = v_{s_{i-j}}$$

yield $v_{[s_j+1,k)} \succ v_{[1,k-s_j)}$. On the other hand, if $c_{[2j,2i)} = c_{[0,2i-2j)}$ and $c_{2i} < c_{2i-2j}$ for some $j \le i \le \ell$, then correspondingly

$$v_{[s_j+1,s_i]} = (-1:2)^{c_{2j}}(-1:3+c_{2j+1})\cdots(-1:2)^{c_{2i-2}}(-1:3+c_{2i-1}) = v_{[1,s_{i-j}]}$$

(if $j < i$) and, since $v_{[s_{i-j}+1,s_{i-j+1})} = (-1:2)^{c_{2i-2j}}$, we have

$$v_{[s_i+1,s_{i+1}]} = (-1:2)^{c_{2i}}(-1:3+c_{2i+1}) \succ (-1:2)^{c_{2i}+1} = v_{[s_{i-j}+1,s_{i-j}+c_{2i}+1]} \quad \text{if } i < \ell\,,$$
$$v_{[s_i+1,k)} = (-1:2)^{c_{2i}} = v_{[s_{i-j}+1,k-s_j)} \quad \text{if } i = \ell\,.$$

Clearly, $c_{[2j,2\ell]} = c_{[0,2\ell-2j]}$ implies $v_{[s_j+1,k)} = v_{[1,k-s_j)}$. Therefore, $c_{[2j,2\ell]} \le_{\mathrm{alt}} c_{[0,2\ell-2j]}$ yields $v_{[s_j+1,k)} \succeq v_{[1,k-s_j)}$, and the converse relation follows similarly.

To prove (ii), suppose now that $c_{[2j-1,2\ell]} <_{\mathrm{alt}} c_{[0,2\ell-2j+1]}$. If $c_{[2j-1,2i-1)} = c_{[0,2i-2j)}$ and $c_{2i-1} < c_{2i-2j}$ for some $j \le i \le \ell$, then

$$\widehat{v}_{[s'_j+1,s'_i]} = (-1:2)^{c_{2j-1}}(-1:3+c_{2j})\cdots(-1:2)^{c_{2i-3}}(-1:3+c_{2i-2}) = v_{[1,s_{i-j}]} \quad (\text{if } j < i)$$
$$\text{and}\quad \widehat{v}_{[s'_i+1,s'_{i+1}]} = (-1:2)^{c_{2i-1}}(-1:3+c_{2i}) \succ (-1:2)^{c_{2i-1}+1} = v_{[s_{i-j}+1,s_{i-j}+c_{2i-1}+1]}\,.$$

If $c_{[2j-1,2i)} = c_{[0,2i-2j]}$ and $c_{2i} > c_{2i-2j+1}$ for some $j \le i \le \ell$, then

$$\widehat{v}_{[s'_j+1,s'_{i+1})} = (-1:2)^{c_{2j-1}}(-1:3+c_{2j})\cdots(-1:2)^{c_{2i-1}} = v_{[1,s_{i-j+1})}$$
$$\text{and}\quad \widehat{v}_{s'_{i+1}} = (-1:3+c_{2i}) \succ (-1:3+c_{2i-2j+1}) = v_{s_{i-j+1}}\,.$$

Therefore, $c_{[2j-1,2\ell]} <_{\mathrm{alt}} c_{[0,2\ell-2j+1]}$ implies $\widehat{v}_{[s'_j+1,\min(k',s'_j+k))} \succ v_{[1,\min(k'-s'_j,k))}$. Again, the converse relation is shown similarly.

If (7.1) holds, then (i) gives that $v_{[n,k)} \succeq v_{[1,k-n]}$ for $n = s_j+1$, $0 \le j \le \ell$. For $s_j < n \le s_{j+1}$, we have $v_{[s_j+1,k)} = (-1:2)^{n-s_j-1}\, v_{[n,k)}$, thus $v_{[n,k)} \succeq v_{[s_j+1,k+s_j-n]} \succeq v_{[1,k-n]}$. We obtain similarly that $\widehat{v}_{[n,\min(k',n+k-1))} \succ v_{[1,\min(k'-n+1,k))}$. This implies $\widehat{v}_{[n,\min(k',n+k-2))} \succ v_{[1,\min(k'-n+1,k-1))}$, except if $n+k-2 < k'$ and $\widehat{v}_{[n,n+k-2)} = v_{[1,k-1)}$. Note that $n+k-2 < k'$, $n \ge 1$, $k \ge 3$, yield $k' \ge 3$, thus $\ell \ge 1$. Furthermore,

$$v_{[1,k-1)} = \begin{cases} (-1:2)^{c_0}(-1:3+c_1)\cdots(-1:2)^{c_{2\ell-2}}(-1:3+c_{2\ell-1})(-1:2)^{c_{2\ell}-1} & \text{if } c_{2\ell} \ge 1, \\ (-1:2)^{c_0}(-1:3+c_1)\cdots(-1:2)^{c_{2\ell-2}} & \text{if } c_{2\ell} = 0, \end{cases}$$

and $\widehat{v}_{[2,k'-1)}$ contains only $\ell-1$ elements which are different from $(-1:2)$. And, if $n = 1$, one finds that all of the c_j must vanish, contradicting (7.1). Therefore, $n \ge 2$ and $\widehat{v}_{[n,n+k-2)} = v_{[1,k-1)}$ with $n < n+k-2 < k'$, thus implying $c_{2\ell} = 0$. From this, $c_{[2,2\ell-2]} = c_{[1,2\ell-3]}$, $c_1 \ge c_0$ and $c_{2\ell-1} \ge c_{2\ell-2}$. Since $c_{[0,2\ell-1]} >_{\mathrm{alt}} c_{[1,2\ell]}$, we obtain $c_0 = c_1 = \cdots = c_{2\ell-1} < c_{2\ell} = 0$, which is impossible. Hence we have shown that $\widehat{v}_{[n,\min(k',n+k-2))} \succ v_{[1,\min(k'-n+1,k-1))}$.

As in the proof that $v_{[s_j+1,k)} \succeq v_{[1,k-s_j)}$, the inequality $c_{[2j,2\ell]} \le_{\mathrm{alt}} c_{[0,2\ell-2j]}$ implies $\widehat{v}_{[s'_{j+1},k')} \preceq \widehat{v}_{[1,k'-s'_{j+1}]}$. Since $\widehat{v}_n = (-1:2) \prec \widehat{v}_1$ for $n \ne s'_{j+1}$, $0 \le j \le \ell$, we have

$\widehat{v}_{[n,k')} \preceq \widehat{v}_{[1,k'-n']}$ for all $2 \le n < k'$. Similarly, $v_n = (-1:2) \prec \widehat{v}_1$ for $n \ne s_j$, $1 \le j \le \ell$, and $c_{[2j-1,2\ell]} <_{\mathrm{alt}} c_{[0,2\ell-2j+1]}$ yields $v_{[s_j,\min(k,s_j+k'-1))} \prec \widehat{v}_{[1,\min(k-s_j+1,k'))}$. Now, $v_{[s_j,s_j+k'-2)} = \widehat{v}_{[1,k'-1)}$ with $s_j < s_j+k'-2 < k$ implies $j=1$, $c_{[1,2\ell)} = c_{[0,2\ell-1)}$ and $c_{2\ell} > c_{2\ell-1}$, thus $c_0 = c_1 = \cdots = c_{2\ell-1} < c_{2\ell}$, contradicting that $c_{2\ell} \le c_0$. Hence we have shown that $v_{[s_j,\min(k,s_j+k'-2))} \prec \widehat{v}_{[1,\min(k-s_j+1,k'-1))}$, which concludes the proof of the lemma. □

Proof of Proposition 7.3. If $\ell = 0$, i.e., $v = (-1:2)^{c_0}$, then (7.1) holds trivially. We have $k = c_0+1$, $\widehat{v} = (-1:3+c_0)$, $v' = (1:1+c_0)$, $k'=2$,

$$\zeta_v - 1 = [\![((-1:2)^{c_0}\,(-1:3+c_0))^\omega]\!]\,, \quad \zeta_v = [\![(1:2+c_0)\,((-1:2)^{c_0}\,(-1:3+c_0))^\omega]\!]$$

and, for $c_0 \ge 1$,

$$\eta_v - 1 = [\![((-1:2)^{c_0-1}\,(-1:3))^\omega]\!]\,, \quad \eta_v = [\![(1:1+c_0)\,(-1:2+c_0)^\omega]\!]\,.$$

In this case, the proposition is easily verified.

Assume from now on that $\ell \ge 1$. Let first $v \in \mathscr{F}$, i.e., there exists some $\alpha \in (0,1]$ with $\underline{b}^\alpha_{[1,k)} = v$, $\overline{b}^\alpha_{[1,k')} = {}^{(W)}\widehat{v}^{(-1)}$. Write v as in (5.2), and use the notation of Lemma 7.5. For $1 \le j \le \ell$, the α-expansion of $T_\alpha^{s_j}(\alpha-1)$ starts with $v_{[s_j+1,k)}$. Since $T_\alpha^{s_j}(\alpha-1) \ge \alpha-1$, we have $v_{[s_j+1,k)} \succeq v_{[1,k-s_j)}$ by Lemma 5.8, thus $c_{[2j,2\ell]} \le_{\mathrm{alt}} c_{[0,2\ell-2j]}$ by Lemma 7.5 (i). Similarly, the α-expansion of $T_\alpha^{s'_j}(\alpha)$ starts with $\widehat{v}_{[s'_j+1,k')}{}^{(-1)}$, and $T_\alpha^{s'_j}(\alpha) \ge \alpha-1$ yields that $\widehat{v}_{[s'_j+1,k')}{}^{(-1)} \succeq v_{[1,k'-s'_j)}$ if $k'-s'_j \le k$, and $\widehat{v}_{[s'_j+1,s'_j+k)} \succeq v_{[1,k)}$ if $k'-s'_j > k$. We cannot have $\widehat{v}_{[s'_j+1,s'_j+k)} = v_{[1,k)}$ by the definition of v and $\widehat{v}$, thus $\widehat{v}_{[s'_j+1,\min(k',s'_j+k))} \succ v_{[1,\min(k'-s'_j,k))}$. Then Lemma 7.5 (ii) gives $c_{[2j-1,2\ell]} <_{\mathrm{alt}} c_{[0,2\ell-2j+1]}$, and (7.1) holds.

Suppose now that $v = v_{[1,k)}$ satisfies (7.1), let $v' = v'_{[1,k')} = {}^{(W)}\widehat{v}^{(-1)}$ and $\alpha \in (\zeta_v, \eta_v)$. We must show that $\underline{b}^\alpha_{[1,k)} = v$ and $\overline{b}^\alpha_{[1,k')} = v'$. By Lemma 5.6, this is equivalent to having both $M_{v_{[1,n)}} \cdot (\alpha-1) \in [\alpha-1,\alpha)$ for all $2 \le n \le k$, and $M_{v'_{[1,n)}} \cdot (\alpha) \in [\alpha-1,\alpha)$ for all $2 \le n \le k'$. To show these relations, we consider the trajectories of ζ_v and η_v.

Since $\eta_v - 1 = [\![(v^{(+1)})^\omega]\!]$, we have $M_{v_{[1,n)}} \cdot (\eta_v - 1) \in (-1,0)$ for all $2 \le n < k$, and

$$M_v \cdot (\eta_v - 1) = 1 + [\![(v^{(+1)})^\omega]\!] = \eta_v\,.$$

Now consider the left endpoint. Since $\zeta_v - 1 = [\![(v\,\widehat{v})^\omega]\!]$, we have $M_{v_{[1,n)}} \cdot (\zeta_v - 1) \in (-1,0)$ for all $2 \le n \le k$, and

$$M_{v_{[1,n)}} \cdot (\zeta_v - 1) = [\![v_{[n,k)},\, M_v \cdot (\zeta_v - 1)]\!] \ge [\![v_{[1,k-n]}\,(\widehat{v}\,v)^\omega]\!]\,.$$

If $n \ge k'$, then Lemma 7.5 (iii) shows that $\widehat{v}_{[1,k'-1)} \succ v_{[k-n+1,k-n+k'-1)}$ and

$$[\![v_{[1,k-n]}\,(\widehat{v}\,v)^\omega]\!] > [\![v_{[1,k-n]}\,\widehat{v}_{[1,k'-1)},\,-1]\!] \ge [\![v_{[1,k-n+k'-1)},\,0]\!] > [\![(v^{(+1)})^\omega]\!] = \eta_v - 1\,.$$

If $n < k'$, then Lemma 7.5 (iii) gives $\widehat{v}_{[1,n)} \succeq v_{[k-n+1,k)}{}^{(+1)}$, thus

$$[\![v_{[1,k-n]}\,(\widehat{v}\,v)^\omega]\!] \ge [\![v^{(+1)}\,\widehat{v}_{[n,k')}\,(v\,\widehat{v})^\omega]\!]\,.$$

Now, we have $\widehat{v}_{[n,\min(k',n+k-2))} \succ v_{[1,\min(k'-n+1,k-1))}$ by Lemma 7.5 (iii), hence

$$[\![v^{(+1)}\,\widehat{v}_{[n,k')}\,(v\,\widehat{v})^{\omega}]\!] > [\![v^{(+1)}\,v_{[1,\min(k'-n+1,k-1))},\,0]\!] > [\![(v^{(+1)})^{\omega}]\!] = \eta_v - 1\,.$$

This shows that $M_{v_{[1,n)}} \cdot (\zeta_v - 1) \in (\eta_v - 1, 0)$ for all $2 \le n \le k$.

The map $M_{v_1} \cdot x$ is monotonically increasing and expanding on $(-1, 0)$. Inductively, we obtain that, for all $2 \le n < k$, $M_{v_{[1,n)}} \cdot (\zeta_v - 1, \eta_v - 1) \subset (\eta_v - 1, 0)$ and $M_{v_{[1,n]}} \cdot x$ is a monotonically increasing, expanding map on $(\zeta_v - 1, \eta_v - 1)$. This shows that $M_{v_{[1,n)}} \cdot (\alpha - 1) \in (\alpha - 1, 0)$ for all $2 \le n < k$. Since $M_v \cdot (\zeta_v - 1, \eta_v - 1) \subset (\eta_v - 1, \eta_v)$, we obtain that $M_v \cdot (\alpha - 1) \in (\alpha - 1, \alpha)$, thus $\underline{b}^{\alpha}_{[1,k)} = v$ for all $\alpha \in (\zeta_v, \eta_v)$.

Consider now $\overline{b}^{\alpha}_{[1,k')}$. Since $\zeta_v - 1 = [\![(v\,\widehat{v})^{\omega}]\!]$, Corollary 5.5 gives that

$$\zeta_v = [\![^{(W)}\widehat{(v\,\widehat{v}\,)^{\omega}}]\!] = [\![^{(W)}(\widehat{v}\,v)^{\omega}]\!]\,,$$

hence $M_{v'_{[1,n)}} \cdot \zeta_v \in (-1, 0)$ for all $2 \le n < k'$, and

$$M_{v'} \cdot \zeta_v = 1 + [\![(v\,\widehat{v})^{\omega}]\!] = \zeta_v\,.$$

By Corollary 5.5 and by considering the cases $c_{2\ell} = 0$ and $c_{2\ell} \neq 0$, we obtain

$$\eta_v = [\![^{(W)}(\widehat{v^{(+1)}})^{\omega}]\!] = [\![^{(W)}(\widehat{v}^{(-1)})^{\omega}]\!]\,,$$

thus $M_{v'_{[1,n)}} \cdot \eta_v \in (-1, 0)$ for all $2 \le n \le k'$. If $n \le k' - k + 1$, then we obtain $M_{v'_{[1,n)}} \cdot \eta_v > \eta_v - 1$, as for $M_{v'_{[1,n)}} \cdot \zeta_v$. If $n > k' - k + 1$, then we get

$$M_{v'_{[1,n)}} \cdot \eta_v = [\![v'_{[n,k')},\, M_{v'} \cdot \eta_v]\!] \ge [\![v_{[1,k'-n]}\,(\widehat{v}^{(-1)})^{\omega}]\!]\,.$$

Now, we proceed as for $M_{[1,n)} \cdot (\zeta_v - 1)$. If $n \ge 2k' - k$, then $\widehat{v}_{[1,k'-1)} \succ v_{[k'-n+1,2k'-n-1)}$ and

$$[\![v_{[1,k'-n]}\,(\widehat{v}^{(-1)})^{\omega}]\!] > [\![v_{[1,k-n]}\,\widehat{v}_{[1,k'-1)},\,-1]\!] \ge [\![v_{[1,2k'-n-1)},\,0]\!] > [\![(v^{(+1)})^{\omega}]\!] = \eta_v - 1\,.$$

If $k' - k + 1 < n < 2k' - k$, then $\widehat{v}_{[1,k-n+k')} \succeq v_{[k'-n+1,k)}{}^{(+1)}$ and

$$[\![v_{[1,k'-n]}\,(\widehat{v}^{(-1)})^{\omega}]\!] \ge [\![v^{(+1)}\,\widehat{v}_{[k-n+k',k')}{}^{(-1)}\,(\widehat{v}^{(-1)})^{\omega}]\!] = [\![v^{(+1)},\, M_{v'_{[1,k-n+k')}} \cdot \eta_v]\!]\,.$$

Inductively, we obtain that $M_{v'_{[1,n)}} \cdot \eta_v \ge [\![(v^{(+1)})^{\omega}]\!] = \eta_v - 1$.

Now, the map $M_{v'_1} \cdot x$ is monotonically decreasing and expanding on $(0, 1)$. Inductively, we obtain that, for all $2 \le n < k'$, $M_{v'_{[1,n)}} \cdot (\zeta_v, \eta_v) \subset (\eta_v - 1, 0)$ and $M_{v'_{[1,n]}} \cdot x$ is a monotonically decreasing, expanding map on (ζ_v, η_v). This shows that $M_{v'_{[1,n)}} \cdot \alpha \in (\alpha - 1, 0)$ for all $2 \le n < k$. Since $M_{v'} \cdot (\zeta_v, \eta_v) \subset (\eta_v - 1, \zeta_v)$, we obtain that $M_{v'} \cdot \alpha \in (\alpha - 1, \alpha)$, thus $\overline{b}^{\alpha}_{[1,k')} = v'$ for all $\alpha \in (\zeta_v, \eta_v)$, and the proposition is proved. □

Corollary 9.5 shows that $v \in \mathscr{F}$ for which $|v| = |\widehat{v}\,|$ abound. In the following two results, we exhibit a family of words showing that strict inequality (in each direction) also arises infinitely often. Note that [NN08] also give infinite families realizing each of the three types of behavior.

Corollary 7.6. *Let m and ℓ be positive integers and set $v = (-1:2)^m(-1:3)^\ell(-1:2)$. Then $v \in \mathscr{F}$ and satisfies $\widehat{v} = (-1:3+m)(-1:3)^{\ell-1}(-1:4)$.*

Proof. We place our word in the form of (5.2), and find

$$v = (-1:2)^m(-1:3)^\ell(-1:2) = (-1:2)^{c_0}(-1:3+c_1)\cdots(-1:3+c_{2\ell-1})(-1:2)^{c_{2\ell}},$$

with $c_0 = m, c_{2\ell} = 1$ and $c_1 = c_2 = \cdots = c_{2\ell-1} = 0$. Now, since $00\cdots 01 <_{\text{alt}} m00\cdots 0$, Proposition 7.3 implies that $v_\ell \in \mathscr{F}$. A direct calculation verifies the formula for $\widehat{v}$. □

Corollary 7.7. *Let $m \geq 2$ and ℓ be positive integers and set $w = (-1:2)^m(-1:4)^\ell$. Then $w \in \mathscr{F}$. Furthermore, $\widehat{w} = (-1:3+m)\big((-1:2)(-1:3)\big)^{\ell-1}$.*

Proof. Here we find that the characteristic sequence is $c_0 ... c_{2\ell} = m(10)^\ell$. Again, membership in $\mathscr{F}$ follows after trivially checking the hypotheses of Proposition 7.3. Also direct calculation verifies the formula for $\widehat{w}$. □

8. Continuity of entropy and measure of the natural extension domain

Given $v \in \mathscr{F}$, both $\underline{b}^\alpha_{[1,k)}$ and $\overline{b}^\alpha_{[1,k')}$ are invariant within the interval Γ_v. The same is hence true for Ψ_α and Ψ'_α, which we accordingly denote by Ψ_v and Ψ'_v, respectively. The evolution of the natural extension domain, and of the entropy, is now straightforward to describe along such an interval. The following lemma is mainly a rewording of Proposition 6.2, but addresses the endpoints of Γ_v.

Lemma 8.1. *Let $v = v_{[1,k)} \in \mathscr{F}$, $v' = v'_{[1,k')} = {}^{(W)}\widehat{v}^{(-1)}$. For any $\alpha \in [\zeta_v, \eta_v]$, we have*

$$\Omega_\alpha = \mathbb{I}_\alpha \times \overline{\Psi_v} \cup \bigcup_{1\le j<k} \overline{\big[M_{v_{[1,j]}}\cdot(\alpha-1),\alpha\big)} \times N_{v_{[1,j]}}\cdot\overline{\Psi_v} \cup \bigcup_{1\le j<k'} \overline{\big(M_{v'_{[1,j]}}\cdot\alpha,\alpha\big)} \times N_{v'_{[1,j]}}\cdot\overline{\Psi'_v}. \tag{8.1}$$

Proof. Since $M_{v_{[1,j]}}\cdot(\alpha-1) \in [\alpha-1,\alpha]$ for all $1 \leq j < k$, and $M_{v'_{[1,j]}}\cdot\alpha \in [\alpha-1,\alpha]$ for all $1 \leq j < k'$, the equation follows from the proof of Proposition 6.2. □

Remark 8.2. Note that $(M_{v'}\cdot\zeta_v, \zeta_v)$ is the empty interval, therefore the contribution from $N_{v'}\cdot\overline{\Psi'_v}$ vanishes at $\alpha = \zeta_v$. Similarly, if v is not the empty word, then $[M_v\cdot(\eta_v-1), \eta_v)$ is the empty interval and there is no contribution from $N_v\cdot\overline{\Psi_v}$ at $\alpha = \eta_v$.

Lemma 8.4 shows that the union in (8.1) is disjoint in measure for any $\alpha \in [\zeta_v, \eta_v]$. We use the following lemma in the proof of Lemma 8.4.

Lemma 8.3. *Let $v = v_{[1,k)} \in \mathscr{F}$, $v' = v'_{[1,k')} = {}^{(W)}\widehat{v}^{(-1)}$, and set $\alpha = \chi_v$. We have*

$$\mathcal{T}_\alpha^{-n}\big(\Delta_\alpha(w) \times \overline{\Psi_v}\big) = \bigcup_{1\le j\le k} \bigcup_{\substack{w'\in\mathscr{A}^n\\ v_{[1,j)}w'\in\mathscr{L}_\alpha}} \Delta_\alpha(w'w) \times N_{v_{[1,j)}} \cdot \overline{\Psi_v}$$
$$\cup \bigcup_{2\le j\le k'} \bigcup_{\substack{w'\in\mathscr{A}^n\\ v'_{[1,j)}w'\in\mathscr{L}_\alpha}} \Delta_\alpha(w'w) \times N_{v'_{[1,j)}} \cdot \overline{\Psi'_v}\,,$$

for every $w \in \mathscr{L}'_\alpha$ and $n \ge 0$, up to a set of measure zero.

Proof. By Lemma 4.3, it suffices to show that the images under $\mathcal{T}_\alpha^n$ are equal. Note first that $T_\alpha^n(\Delta_\alpha(w'w)) = J_{w'}^\alpha \cap \Delta_\alpha(w) = \Delta_\alpha(w)$ for all $w' \in \mathscr{L}_\alpha$ since $J_{w'}^\alpha = [\alpha-1,\alpha)$. It remains to prove that

$$\bigcup_{1\le j\le k} \bigcup_{\substack{w'\in\mathscr{A}^n\\ v_{[1,j)}w'\in\mathscr{L}_\alpha}} N_{v_{[1,j)}w'} \cdot \overline{\Psi_v} \;\cup \bigcup_{2\le j\le k'} \bigcup_{\substack{w'\in\mathscr{A}^n\\ v'_{[1,j)}w'\in\mathscr{L}_\alpha}} N_{v'_{[1,j)}w'} \cdot \overline{\Psi'_v} = \overline{\Psi_v}\,. \tag{8.2}$$

We can write every $u \in \mathscr{L}_\alpha$ with $|u| \ge n$ as $u = u'w'$ with $w' \in \mathscr{A}^n$, but then u' is in $\mathscr{L}'_\alpha$ and can be decomposed according to Lemma 6.3 into $u' = \tilde{u}\,\tilde{v}$ with $\tilde{u} \in \mathscr{L}_\alpha$, $\tilde{v} = v_{[1,j)}$, $1 \le j \le k$, or $\tilde{u} \in \mathscr{L}'_\alpha$, $\tilde{v} = v'_{[1,j)}$, $2 \le j \le k'$. Finally, $\tilde{u}\,\tilde{v}\,w' \in \mathscr{L}_\alpha$ is equivalent to $\tilde{v}\,w' \in \mathscr{L}_\alpha$, which shows (8.2). ☐

Lemma 8.4. *Let $v = v_{[1,k)} \in \mathscr{F}$, $v' = v'_{[1,k')} = {}^{(W)}\widehat{v}^{(-1)}$. The sets $\overline{\Psi_v}$, $N_{v_{[1,j]}} \cdot \overline{\Psi_v}$, $1 \le j < k$, and $N_{v'_{[1,j]}} \cdot \overline{\Psi'_v}$, $1 \le j < k'$, are pairwise disjoint in (one-dimensional Lebesgue) measure.*

Proof. The disjointness of $\overline{\Psi_v}$, $N_{v_{[1,j]}} \cdot \overline{\Psi_v}$, $1 \le j < k$, and $N_{v'_{[1,j]}} \cdot \overline{\Psi'_v}$, $1 \le j < k'$, is equivalent to the disjointness of the union (8.1), for any $\alpha \in \Gamma_v$. We consider $\alpha = \chi_v$. Then, Lemma 6.4 and Lemma 4.3 yield that

$$\mathcal{T}_\alpha^{-j}\big(J^\alpha_{v_{[1,j]}} \times N_{v_{[1,j]}} \cdot \overline{\Psi_v}\big) = \Delta_\alpha(v_{[1,j]}) \times \overline{\Psi_v} \qquad (1 \le j < k)\,,$$
$$\mathcal{T}_\alpha^{-j}\big(J^\alpha_{v'_{[1,j]}} \times N_{v'_{[1,j]}} \cdot \overline{\Psi'_v}\big) = \Delta_\alpha(v'_{[1,j]}) \times \overline{\Psi'_v} \qquad (1 \le j < k')$$

(up to sets of measure zero).

If $1 \le j < i < k'$, then $J^\alpha_{v'_{[1,j]}} \times N_{v'_{[1,j]}} \cdot \overline{\Psi'_v}$ and $J^\alpha_{v'_{[1,i]}} \times N_{v'_{[1,i]}} \cdot \overline{\Psi'_v}$ are disjoint because $\Delta_\alpha(v'_{[1,j]}) \times \overline{\Psi'_v}$ and $\mathcal{T}_\alpha^{-j}\big(J^\alpha_{v'_{[1,i]}} \times \overline{\Psi'_v}\big) \subseteq \Delta_\alpha(v'_{[i-j+1,i]}) \times N_{v'_{[1,i-j]}} \cdot \overline{\Psi'_v}$ are disjoint. Similarly, $J^\alpha_{v'_{[1,j]}} \times N_{v'_{[1,j]}} \cdot \overline{\Psi'_v}$, $1 \le j < k'$, and $J^\alpha_{v_{[1,i]}} \times N_{v_{[1,i]}} \cdot \overline{\Psi_v}$, $1 \le i < k$, are disjoint if $j \le i$ because $\Delta_\alpha(v'_{[1,j]}) \times \overline{\Psi'_v}$ and $\Delta_\alpha(v_{[i-j+1,i]}) \times N_{v_{[1,i-j]}} \cdot \overline{\Psi_v}$ (where $v_{[1,0]}$ is the empty word) are disjoint.

If $0 \le i < j < k$, then $\Delta_\alpha(v_{[1,j]}) \times \overline{\Psi_v}$ and $\mathcal{T}_\alpha^{-j}\big(J^\alpha_{v_{[1,i]}} \times N_{v_{[1,i]}} \cdot \overline{\Psi_v}\big) = \mathcal{T}_\alpha^{i-j}\big(\Delta_\alpha(v_{[1,i]}) \times \overline{\Psi_v}\big)$ are disjoint by Lemma 8.3 because no word in $\mathcal{L}_\alpha$ ends with $v_{[1,j-i]}$, thus $J^\alpha_{v_{[1,j]}} \times N_{v_{[1,j]}} \cdot \overline{\Psi_v}$

and $J^\alpha_{v_{[1,i]}} \times N_{v_{[1,i]}} \cdot \overline{\Psi_v}$ are disjoint. Finally, similar arguments show that $J^\alpha_{v'_{[1,j]}} \times N_{v'_{[1,j]}} \cdot \overline{\Psi'_v}$, $1 \le j < k'$, and $J^\alpha_{v_{[1,i]}} \times N_{v_{[1,i]}} \cdot \overline{\Psi_v}$, $0 \le i < k$, are disjoint if $j > i$. □

Lemma 8.5. *Let $v = v_{[1,k)} \in \mathscr{F}$, and $v' = v'_{[1,k')} = {}^{(W)}\widehat{v}^{(-1)}$. For $\alpha \in \Gamma_v$, we have*

$$\mu(\Omega_\alpha) = \mu(\Omega_{\eta_v}) + (k-k')\,\mu([\alpha,\eta_v] \times \overline{\Psi'_v}) \tag{8.3}$$

and

$$\begin{aligned} \mu(\Omega_\alpha) &= \mu(\Omega_{\zeta_v}) + (k'-k)\,\mu([\zeta_v-1,\alpha-1] \times \overline{\Psi_v}) \\ &= \mu(\Omega_{\zeta_v})\big(1 + (k'-k)\,\nu_{\zeta_v}([\zeta_v-1,\alpha-1])\big)\,. \end{aligned} \tag{8.4}$$

Proof. Compare first Ω_α with Ω_{η_v}. By Lemmas 8.1 and 8.4, we have, up to sets of measure zero,

$$\Omega_\alpha \setminus \Omega_{\eta_v} = \bigcup_{1\le j\le k} M_{v_{[1,j)}} \cdot [\alpha-1,\eta_v-1] \times N_{v_{[1,j)}} \cdot \overline{\Psi_v} \ \setminus\ [\alpha,\eta_v] \times N_v \cdot \overline{\Psi_v}\,,$$
$$\Omega_{\eta_v} \setminus \Omega_\alpha = \bigcup_{1\le j\le k'} M_{v'_{[1,j)}} \cdot [\alpha,\eta_v] \times N_{v'_{[1,j)}} \cdot \overline{\Psi'_v} \ \setminus\ [\alpha,\eta_v] \times N_v \cdot \overline{\Psi_v}\,.$$

For the second statement, we have used that $[\alpha,\eta_v] \times (\Psi'_v \setminus N_v \cdot \Psi_v) \subset \Omega_{\eta_v}$ since Lemma 6.3 yields that

$$\Psi'_v = \bigcup_{1\le j\le k} N_{v_{[1,j)}} \cdot \Psi_v \ \cup \bigcup_{2\le j\le k'} N_{v'_{[1,j)}} \cdot \Psi'_v\,.$$

By Lemma 6.6, we have $\overline{\Psi_v} = {}^tE^{-1} \cdot \overline{\Psi'_v}$, thus

$$\mu\big([\alpha-1,\eta_v-1] \times \overline{\Psi_v}\big) = \mu\big(E \cdot [\alpha,\eta_v] \times {}^tE^{-1} \cdot \overline{\Psi'_v}\big) = \mu\big([\alpha,\eta_v] \times \overline{\Psi'_v}\big)$$

by (2.3). This proves (8.3).

Compare now Ω_α with Ω_{ζ_v}. Up to sets of measure zero, we have

$$\Omega_{\zeta_v} \setminus \Omega_\alpha = \bigcup_{1\le j\le k} M_{v_{[1,j)}} \cdot [\zeta_v-1,\alpha-1] \times N_{v_{[1,j)}} \cdot \overline{\Psi_v} \ \setminus\ [\zeta_v, M_v\cdot(\alpha-1)] \times N_v \cdot \overline{\Psi_v}\,,$$
$$\Omega_\alpha \setminus \Omega_{\zeta_v} = \bigcup_{1\le j\le k'} M_{v'_{[1,j)}} \cdot [\zeta_v,\alpha] \times N_{v'_{[1,j)}} \cdot \overline{\Psi'_v} \ \setminus\ [\zeta_v, M_v\cdot(\alpha-1)] \times N_v \cdot \overline{\Psi_v}\,,$$

where $[\zeta_v, M_v\cdot(\alpha-1)]$ is empty if $M_v\cdot(\alpha-1) < \zeta_v$. As above, we obtain (8.4).

By Proposition 7.3, we have $M_{v_{[1,j)}}\cdot(\zeta_v-1) \ge \eta_v-1$ for all $2 \le j \le k$ and $M_{v'_{[1,j)}}\cdot\zeta_v \ge \eta_v-1$ for all $2 \le j \le k'$, which gives that $[\zeta_v-1,\alpha-1] \times \overline{\Psi_v} = \{(x,y) \in \Omega_{\zeta_v} \mid x \le \alpha-1\}$, thus

$$\mu([\zeta_v-1,\alpha-1] \times \overline{\Psi_v}) = \mu(\Omega_{\zeta_v})\,\nu_{\zeta_v}([\zeta_v-1,\alpha-1])\,.$$

This concludes the proof of the lemma. □

This explicit description of the natural extension domains gives the following strengthening of a result of [NN08], see also [CT]. Compare [KSS10] for similar arguments.

Corollary 8.6. *For each $v \in \mathscr{F}$, the entropy $h(T_\alpha)$ is given for $\alpha \in \Gamma_v$ by*

$$h(T_\alpha) = h(T_{\zeta_v}) \left(1 + (|v| - |\widehat{v}|)\, \nu_{\zeta_v}([\zeta_v - 1, \alpha - 1])\right), \tag{8.5}$$

i.e., $h(T_\alpha)$ is: constant on Γ_v if $|v| = |\widehat{v}|$; increasing if $|v| > |\widehat{v}|$; decreasing if $|v| < |\widehat{v}|$. Furthermore, the function sending α to $h(T_\alpha)\mu(\Omega_\alpha)$ is constant on Γ_v.

The next result, along with Corollary 8.12, will show that $[g^2, g]$ is a maximal interval of constancy for the entropy function.

Corollary 8.7. *There exists a sequence converging to g^2 from below such that the entropy function is increasing at each element of this sequence.*

Proof. Consider the family of words $(1:2)(-1:3)^\ell(-1:2)$. From Corollary 7.6, we have both that for each v of this form, $v \in \mathscr{F}$ and $|v| = |\,\widehat{v}\,| + 1$; thus the entropy function is increasing along each Γ_v. Now for each, choose some $\alpha \in \Gamma_v$. Since $\underline{b}^{g^2}_{[1,\infty)} = (-1:2)\,(-1:3)^\omega$, we find that this sequence converges to g^2 from below. □

Definition 8.8. The set of *synchronizing interval separation points* is $Z = \{\zeta_v \mid v \in \mathscr{F}\}$.

The set of *labels of non-synchronizing orbits* is $\mathscr{N} = \{\underline{b}^\alpha_{[1,\infty)} \mid \alpha \in (0,1] \setminus (\Gamma \cup Z)\}$.

By Lemma 8.1, it is easy to see that, for every $v \in \mathscr{F}$, the normalizing constant $\mu(\Omega_\alpha)$ is continuous on $[\zeta_v, \eta_v]$. Since $\zeta_v = \eta_{\Theta(v)}$, it is continuous to the left of ζ_v as well. Therefore, it is sufficient to study $\alpha \in [\![\mathscr{N}]\!]$, the set of points with non-synchronizing orbits.

Lemma 8.9. *Let $\underline{b}^\alpha_{[1,\infty)} = (-1:2)^{c_0}(-1:3+c_1)(-1:2)^{c_2}(-1:3+c_3)\cdots \in \mathscr{N}$. For every $n \geq 1$, there exists some $\delta > 0$ such that*

$$\underline{b}^{\alpha'}_{[1,n]} = \underline{b}^{\alpha}_{[1,n]} \text{ and } \overline{b}^{\alpha'}_{[1,n]} = \overline{b}^{\alpha}_{[1,n]} \text{ for all } \alpha' \in \begin{cases} [\alpha, \alpha+\delta) & \text{if } \alpha = \eta_v \text{ for some } v \in \mathscr{F}, \\ (\alpha-\delta, \alpha+\delta) & \text{else.} \end{cases}$$

Proof. Recall that $\alpha \notin \Gamma$ implies that $T^n_\alpha(\alpha-1) < 0$ and $T^n_\alpha(\alpha) < 0$ for all $n \geq 1$. Therefore, the statement is true if $T^n_\alpha(\alpha-1) > \alpha - 1$ and $T^n_\alpha(\alpha) > \alpha - 1$ for all $n \geq 1$.

Suppose that $T^n_\alpha(\alpha-1) = \alpha - 1$ or $T^n_\alpha(\alpha) = \alpha - 1$ for some $n \geq 1$. Then $c_{[i,\infty)} = c_{[0,\infty)}$ for some $i \geq 1$, and let i be minimal with this property. As in Proposition 7.3, we obtain that $c_{[2j,\infty)} \leq_{\text{alt}} c_{[0,\infty)}$ and $c_{[2j-1,\infty)} \leq_{\text{alt}} c_{[0,\infty)}$ for all $j \geq 1$.

We show that i must be even. Suppose that $i = 2\ell + 1$ and set $v = (-1:2)^{c_0}(-1:3+c_1)\cdots(-1:2)^{c_{2\ell}}$, then $\underline{b}^\alpha_{[1,\infty)} = (v\,\widehat{v})^\omega$. Since $\alpha \notin Z$, v is not in $\mathscr{F}$, hence $c_{[2j-1,2\ell]} = c_{[0,2\ell-2j+1]}$ for some $1 \leq j \leq \ell$. This implies $c_{[2j-1,\infty)} = c_{[2j-1,2\ell]}\, c_{[0,\infty)} \geq_{\text{alt}} c_{[0,\infty)}$, and $c_{[2j-1,\infty)} \leq_{\text{alt}} c_{[0,\infty)}$, thus $c_{[2j-1,\infty)} = c_{[0,\infty)}$, contradicting the minimality of i.

Let now $i = 2\ell$ and set $v = (-1:2)^{c_0}(-1:3+c_1)\cdots(-1:3+c_{2\ell-1})$. Then we have $v^{(-1)} \in \mathscr{F}$ and $\underline{b}^\alpha_{[1,\infty)} = v^\omega$, thus $\alpha = \eta_{v^{(-1)}}$. Since $T^n_\alpha(\alpha) > \alpha - 1$ for all $n \geq 1$, but $T^{|v|}_\alpha(\alpha-1) = \alpha - 1$, $\underline{b}^{\alpha'}_{[1,n]} = \underline{b}^{\alpha}_{[1,n]}$ and $\overline{b}^{\alpha'}_{[1,n]} = \overline{b}^{\alpha}_{[1,n]}$ hold only for $\alpha' \in [\alpha, \alpha+\delta)$. □

Lemma 8.10. *Let $\alpha \in (0,1] \setminus \Gamma$ or $\alpha = \chi_v$ for some $v \in \mathscr{F}$. Then*

$$\mu\Big(\overline{\bigcup_{w\in\mathscr{L}'_\alpha\cap\mathscr{A}_\alpha^*:\,|w|\ge n} J_w^\alpha \times N_w \cdot \big[0, \tfrac{1}{d_\alpha(\alpha)+1}\big]}\Big) \le \big(\tfrac{d_\alpha(\alpha)}{d_\alpha(\alpha)+\alpha}\big)^n \mu(\Omega_\alpha) \quad \text{for all } n \ge 0\,.$$

Proof. Since $J_w^\alpha = T_\alpha^{|w|}(\Delta_\alpha(w)) = M_w \cdot \Delta_\alpha(w)$, we have

$$\mathcal{T}_\alpha^{-n}\Big(\overline{\bigcup_{w\in\mathscr{L}'_\alpha\cap\mathscr{A}_\alpha^*:\,|w|\ge n} J_w^\alpha \times N_w \cdot \big[0, \tfrac{1}{d_\alpha(\alpha)+1}\big]}\Big) = \bigcup_{w\in\mathscr{L}'_\alpha\cap\mathscr{A}_\alpha^n} \Delta_\alpha(w) \times \big[0, \tfrac{1}{d_\alpha(\alpha)+1}\big]$$
$$\cup\ \overline{\bigcup_{w\in\mathscr{L}'_\alpha\cap\mathscr{A}_\alpha^*:\,|w|>n} T_\alpha^{|w|-n}(\Delta_\alpha(w)) \times N_{w_{[1,|w|-n]}} \cdot \big[0, \tfrac{1}{d_\alpha(\alpha)+1}\big]}\,.$$

Note that $T_\alpha^{|w|-n}(\Delta_\alpha(w)) \subseteq \Delta_\alpha(w_{[|w|-n+1,|w|]})$ and that $w_{[|w|-n+1,|w|]} \in \mathscr{L}'_\alpha \cap \mathscr{A}_\alpha^n$, thus

$$\mathcal{T}_\alpha^{-n}\Big(\overline{\bigcup_{w\in\mathscr{L}'_\alpha\cap\mathscr{A}_\alpha^*:\,|w|\ge n} J_w^\alpha \times N_w \cdot \big[0, \tfrac{1}{d_\alpha(\alpha)+1}\big]}\Big) \subseteq \bigcup_{w\in\mathscr{L}'_\alpha\cap\mathscr{A}_\alpha^n} \Delta_\alpha(w) \times [0,1]\,.$$

Since, for any measurable set $X \subseteq \mathbb{I}_\alpha$,

$$\frac{\mu\big(X \times [0, \frac{1}{d_\alpha(\alpha)+1}]\big)}{\mu\big(X \times [0,1]\big)} \le \max_{x\in\mathbb{I}_\alpha} \frac{\int_0^{1/(d_\alpha(\alpha)+1)} \frac{1}{(1+xy)^2}\,dy}{\int_0^1 \frac{1}{(1+xy)^2}\,dy} = \max_{x\in\mathbb{I}_\alpha} \frac{d_\alpha(\alpha)}{d_\alpha(\alpha)+1+x} = \frac{d_\alpha(\alpha)}{d_\alpha(\alpha)+\alpha}\,,$$

we obtain that

$$\frac{\mu\big(\overline{\bigcup_{w\in\mathscr{L}'_\alpha\cap\mathscr{A}_\alpha^*:\,|w|>n} J_w^\alpha \times N_w \cdot [0, \frac{1}{d_\alpha(\alpha)+1}]}\big)}{\mu\big(\overline{\bigcup_{w\in\mathscr{L}'_\alpha\cap\mathscr{A}_\alpha^*:\,|w|\ge n} J_w^\alpha \times N_w \cdot [0, \frac{1}{d_\alpha(\alpha)+1}]}\big)} \le \frac{d_\alpha(\alpha)}{d_\alpha(\alpha)+\alpha}\,. \qquad \square$$

Theorem 8.11. *The measure of the natural extension domain $\mu(\Omega_\alpha)$ and the entropy $h(T_\alpha)$ are continuous on $(0,1]$.*

Proof. By the remarks preceding Lemma 8.9, we only have to consider the continuity around $\alpha \in [\![\mathscr{N}]\!]$. Moreover, we only have to show right continuity if $\alpha = \eta_v$ for some $v \in \mathscr{F}$, and it suffices to compare $\mu(\Omega_\alpha)$ with $\mu(\Omega_{\alpha'})$, $\alpha' \in (0,1] \setminus \Gamma$.

If $\mathscr{L}'_\alpha \cap \mathscr{A}_\alpha^n = \mathscr{L}'_{\alpha'} \cap \mathscr{A}_{\alpha'}^n$, $n \ge 1$, then $d_{\alpha'}(\alpha') = d_\alpha(\alpha)$, and Proposition 6.10 yields that

$$\big|\mu(\Omega_{\alpha'}) - \mu(\Omega_\alpha)\big| \le \mu\Big(\bigcup_{w\in\mathscr{L}'_\alpha\cap\mathscr{A}_\alpha^*:\,|w|\le n} (J_w^\alpha \setminus J_w^{\alpha'} \cup J_w^{\alpha'} \setminus J_w^\alpha) \times N_w \cdot \big[0, \tfrac{1}{d_\alpha(\alpha)+1}\big]\Big)$$
$$+\, \mu\Big(\overline{\bigcup_{w\in\mathscr{L}'_\alpha\cap\mathscr{A}_\alpha^*:\,|w|>n} J_w^\alpha \times N_w \cdot \big[0, \tfrac{1}{d_\alpha(\alpha)+1}\big]} \cup \overline{\bigcup_{w\in\mathscr{L}'_{\alpha'}\cap\mathscr{A}_\alpha^*:\,|w|>n} J_w^{\alpha'} \times N_w \cdot \big[0, \tfrac{1}{d_\alpha(\alpha)+1}\big]}\Big)\,.$$

Fix $\epsilon > 0$. By applying Lemma 8.10 to α and α', where we can use the trivial bound $\mu(\Omega_{\alpha'}) \le \mu(\mathbb{I}_{\alpha'} \times [0,1])$, we can choose an interval around α and some $n \ge 0$ such that

$$\mu\Big(\overline{\bigcup_{w\in\mathscr{L}'_\alpha\cap\mathscr{A}_\alpha^*:\,|w|>n} J_w^\alpha \times N_w \cdot \big[0, \tfrac{1}{d_\alpha(\alpha)+1}\big]}\Big) + \mu\Big(\overline{\bigcup_{w\in\mathscr{L}'_{\alpha'}\cap\mathscr{A}_\alpha^*:\,|w|>n} J_w^{\alpha'} \times N_w \cdot \big[0, \tfrac{1}{d_\alpha(\alpha)+1}\big]}\Big) < \frac{\epsilon}{2}\,,$$

Lemma 8.9 gives some $\delta > 0$ such that $\mathscr{L}'_\alpha \cap \mathscr{A}_\alpha^n = \mathscr{L}'_{\alpha'} \cap \mathscr{A}_{\alpha'}^n$ for all $\alpha' \in (\alpha-\delta, \alpha+\delta)$ and $\alpha' \in [\alpha, \alpha+\delta)$ respectively. Since $\mathscr{L}'_\alpha \cap \mathscr{A}_\alpha^*$ contains only finitely many words of length at most n, we have

$$\mu\Big(\bigcup_{w\in\mathscr{L}'_\alpha\cap\mathscr{A}_\alpha^*:\,|w|\le n} (J_w^\alpha \setminus J_w^{\alpha'} \cup J_w^{\alpha'} \setminus J_w^\alpha) \times N_w \cdot \big[0, \tfrac{1}{d_\alpha(\alpha)+1}\big]\Big) < \frac{\epsilon}{2}$$

for α' suficiently close to α. This shows the continuity of $\mu(\Omega_\alpha)$. By Theorem 4.6, this yields the continuity of $h(T_\alpha)$. $\square$

Corollary 8.12. *For any $\alpha \in [g^2, g]$, we have $\mu(\Omega_\alpha) = \log(1+g)$, $h(T_\alpha) = \pi^2/(6\log(1+g))$.*

Proof. By Corollary 8.6 and Theorem 8.11, it suffices to show that $|v| = |\widehat{v}|$ for all $v \in \mathscr{F}$ such that $\Gamma_v \subset [g^2, g]$. Let $\alpha \in \Gamma \cap (g^2, g)$. Since $\underline{b}^{g^2}_{[1,\infty)} = (-1:2)\,(-1:3)^\omega$, the words $v = \underline{b}^\alpha_{[1,k)}$ and $\overline{b}^\alpha_{[1,k')}$ contain no factor in $(-1:2)\,(-1:3)^*\,(-1:2)$. The same is true for $\widehat{v} = {}^{(W)}\overline{b}^\alpha_{[1,k')}{}^{(+1)}$. Since $\underline{b}^\alpha_1 = (-1:2)$, we obtain that

$$v \in \Big((-1:2)\,\big((-1:3)(-1:2)^0\big)^*\,(-1:4)\,\big((-1:2)^0(-1:3)\big)^*\Big)^*\,\mathscr{E}$$

with $\mathscr{E} = \{(-1:2)^0,\ (-1:2)\,((-1:3)(-1:2)^0)^*\}$. Furthermore, the element of $\mathscr{E}$ corresponding to v cannot be $(-1:2)^0$ since this would imply that $\overline{b}^\alpha_{[2,k')} = \widehat{v}_{[2,k')}{}^{(-1)}$ ends with a factor in $(-1:2)\,(-1:3)^*\,(-1:2)$. Here, we have used that v is not the empty word because $\overline{b}^\alpha_1 \neq (1:1)$. Therefore, we have $|\widehat{v}| = |v|$. $\square$

Remark 8.13. Combining Corollaries 8.12 and 8.7 with the beginning of the proof of Theorem 5.1, we find that $[g^2, g]$ is a maximal interval of constancy for the entropy function.

9. More on the structure of Γ

Proposition 9.1. *The set $(0,1] \setminus \Gamma$ has zero Lebesgue measure.*

Proof. We have

$$\begin{aligned}(0,1]\setminus\Gamma &\subseteq \big\{\alpha\in(0,1] \mid T_\alpha^n(\alpha-1)<0 \text{ for all } n\geq 1\big\}\\ &= \big\{\alpha\in(0,1] \mid T_0^n(\alpha-1)\geq \alpha-1 \text{ for all } n\geq 1\big\}\\ &\subseteq \bigcup_{r\geq 1}\big\{\alpha\in(0,1] \mid T_0^n(\alpha-1)\geq 1/r-1 \text{ for all } n\geq 1\big\},\end{aligned}$$

where the middle equality is due to Lemma 5.6. Since T_0 is ergodic with respect to a σ-finite (infinite) invariant, which is absolutely continuous with respect to Lebesque measure on $[-1,0)$, we have that this set is the countable union of null sets. $\square$

We now prove that each synchronizing interval Γ_v has a synchronizing interval to its immediate left.

Definition 9.2. For $v \in \mathscr{F}$, let $\Theta(v) = v\,\widehat{v}^{(-1)}$.

We note that Carminati et al. in their numeric study [CMPT10] have remarked upon the following phenomenon.

Lemma 9.3. *For $v \in \mathscr{F}$, we have that $\Theta(v)$ is also in $\mathscr{F}$, and that*

$$|\Theta(v)| = |\widehat{\Theta(v)}| = |v| + |\widehat{v}|\,.$$

Moreover, the left endpoint of the interval Γ_v is the right endpoint of the interval $\Gamma_{\Theta(v)}$, i.e., $\zeta_v = \eta_{\Theta(v)}$.

Proof. It is easily verified by Proposition 7.3 that $v \in \mathscr{F}$ implies $\Theta(v) \in \mathscr{F}$. By checking the two possible cases of $c_{2\ell}$ being zero or not in the factorization $v = (-1:2)^{c_0}(-1: 3+c_1)(-1:2)^{c_2}\cdots(-1:3+c_{2\ell-1})(-1:2)^{c_{2\ell}}$, one verifies that $\widehat{\Theta(v)} = \widehat{v}\, v^{(+1)}$. The equality of the lengths of $\Theta(v)$ and $\widehat{\Theta(v)}$ with the sum of those of v and $\widehat{v}$ then follows directly. It follows from the definitions of ζ_v, η_v and $\Theta(v)$ that $\zeta_v - 1 = \eta_{\Theta(v)} - 1$. □

Definition 9.4. For $v \in \mathscr{F}$, let τ_v denote the limit point of the monotonically decreasing sequence $(\zeta_{\Theta^j(v)})_{j\ge 0}$.

We obtain the following corollary, which corrects a minor error in [NN08] (Example 3 on p. 1221).

Corollary 9.5. *If $v \in \mathscr{F}$, then, irrespective of the behavior of the entropy function $\alpha \mapsto h(T_\alpha)$ on Γ_v, this function is constant immediately to the left, that is on $[\tau_v, \zeta_v]$.*

Proof. We have that $|\Theta(v)| = \big|\widehat{\Theta(v)}\big|$, and $|\Theta^j(v)| = \big|\widehat{\Theta^j(v)}\big|$ for all $j \ge 1$, thus the result follows immediately from Corollary 8.6. □

It is mentioned in [CMPT10] that τ_v cannot be quadratic.

Theorem 9.6. *For $v \in \mathscr{F}$, the limit point τ_v is a transcendental real number.*

Proof. We argue using the transcendence results of Adamczewski and Bugeaud [AB05] given in terms of regular continued fraction expansions. Consider τ_v with $v \in \mathscr{F} \setminus \Theta(\mathscr{F})$. We have that τ_v is the limit of the sequence formed by the $\chi_{\Theta^n(v)}$. This is the sequence of zeros of the functions $x \mapsto M \cdot x$, where M is respectively:

$$R_v, L_{\Theta(v)}, R_{\Theta^2(v)}, L_{\Theta^3(v)}, \ldots = R,\, R^2,\, R^2WR^2,\, R^2WR^4WR^2,\, \ldots\,,$$

where we have set $R = R_v$ for ease.

By Lemma 3.1, there is a unique finite word w of odd length in $\mathscr{A}_+^*$ such that $R = M_w$ and thus $\chi_v = [\![w, 0]\!]$. (Since all signs are $+1$ here, we suppress them for the remainder of this proof so as to use traditional regular fraction notation, in full accordance with [AB05], writing $[w]$ for $[\![w]\!]$.) Since $\chi_{\Theta(v)}$ is the zero of $L_{\Theta(v)} = R^2$, we have $\chi_{\Theta(v)} = [w, w]$. Since $\chi_{\Theta^2(v)}$ is the zero of R^2WR^2, we have that

$$\chi_{\Theta^2(v)} = [w, w^{(-1)}, 1, w, w]\,,$$

where as usual $w^{(-1)} = w_1 \cdots w_{n-1}(w_n - 1)$, and thus in the exceptional case of $w_n = 1$ we find $[w, w^{(-1)}, 1, w, w] = [w, w_{[1,n-1)}, w_{n-1} + 1, w, w]$. We continue this analysis, and find that the regular continued fraction expansions for the various $\chi_{\Theta^{2m}(v)}$ give prefixes for the regular continued fraction expansion of τ_v . Indeed, letting V_m be the word in positive integers of our continued fraction expansion of $\chi_{\Theta^{2m}(v)}$, then (at least for $m > 1$), $V_m W_m$ where W_m denotes the prefix of length $|V_m| - 2$ of V_m. In the notation of Adamczewski and Bugeaud [AB05], we have that $V_m^{2(|V_m|-1)/|V_m|}$ is a prefix of that expansion. Now, we have (1) the regular continued fraction expansion of τ_v is clearly of bounded partial quotients; (2) the $|V_m|$ are strictly increasing; and, (3) $2(|V_m| - 1)/|V_m| \geq 8/5$. Therefore, the first theorem of [AB05] applies and we have that τ_v is transcendental. □

Remark 9.7. From the above, the largest element of $(0, 1] \setminus (\Gamma \cup Z)\}$ is τ_v, with v the empty word. Using a standard continued fraction notation, this value is

$$\begin{aligned}\tau_v &= [2, 1, 1, 2, 2, 2, 1, 1, 2, 1, 1, 2, 1, 1, 2, 2, 2, 1, 1, 2, 2, 2, 1, 1, 2, 2, 2, \dots]\\ &= 0.3867499707143007\cdots.\end{aligned}$$

Note that the partial quotients sequence of τ_v is the fixed point of the morphism on $\{1, 2\}^\omega$ defined by $2 \mapsto 211$, $1 \mapsto 2$, giving a more direct manner to prove transcendence of this particular number.

Corollary 9.8. *The point g^2 is a two sided limit of the set $\{\tau_v \,|\, v \in \mathscr{F}\}$.*

Proof. The elements of the sequence in Corollary 8.7 converging to g^2 lie in distinct Γ_v, each with $|v| = |\,\widehat{v}\,| + 1$. Since none of these words lie in $\Theta(\mathcal{F})$, the corresponding sequence of τ_v is also decreasing to g^2.

Similarly, Proposition 7.6 shows that each $(-1 : 2)(-1 : 3)^\ell$ belongs to $\mathscr{F}$. Since $\underline{b}^{g^2}_{[1,\infty)} = (-1 : 2)\,(-1 : 3)^\omega$, we find that any sequence formed by selecting a point from each Γ_v with v of the form $(-1 : 2)(-1 : 3)^\ell$ gives a sequence converging to g^2 from below. But, $v = (-1 : 2)(-1 : 3)^\ell$ gives $\widehat{v} = (-1 : 4)(-1 : 3)^\ell$ and thus $\Theta(v)$ is not of the form $(-1 : 2)(-1 : 3)^m$. Thus the set of τ_v with v of the form $(-1 : 2)(-1 : 3)^\ell$ converges from below to g^2. □

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