Tong's Spectrum for Rosen Continued Fractions

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Résumé. Dans les années 90, J.C. Tong a donné une borne supérieure optimale pour le minimum de k coefficients d'approximation consécutifs dans le cas des fractions continues à l'entier le plus proche. Nous généralisons ce type de résultat aux fractions continues de Rosen. Celles-ci constituent une famille infinie d'algorithmes de développement en fractions continues, où les quotients partiels sont certains entiers algébriques réels. Pour chacun de ces algorithmes nous déterminons la borne supérieure optimale de la valeur minimale des coefficients d'approximation pris en nombres consécutifs appropriés. Nous donnons aussi des résultats métriques pour des plages de "mauvaises" approximations successives de grande longueur.

Abstract. In the 1990s, J.C. Tong gave a sharp upper bound on the minimum of k consecutive approximation constants for the nearest integer continued fractions. We generalize this to the case of approximation by Rosen continued fraction expansions. The Rosen fractions are an infinite set of continued fraction algorithms, each giving expansions of real numbers in terms of certain algebraic integers. For each, we give a best possible upper bound for the minimum in appropriate consecutive blocks of approximation coefficients. We also obtain metrical results for large blocks of "bad" approximations.

1. Introduction

It is well-known that every $x \in [0,1) \setminus \mathbb{Q}$ has a unique (regular) continued fraction expansion of the form

$$(1.1) \qquad x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots + \cfrac{1}{a_n + \ldots}}} = [\,a_1,\, a_2, \ldots, a_n, \ldots\,].$$

Here the partial quotients a_n are positive integers for $n \geq 1$. Finite truncation in (1.1) yields the convergents p_n/q_n of x, i.e., for $n \geq 1$

$$\frac{p_n}{q_n} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots + \cfrac{1}{a_n}}} = [\,a_1,\, a_2, \ldots, a_n\,],$$

and throughout it is assumed that p_n/q_n is in its lowest terms. Note that (1.1) is a shorthand for $\lim_{n\to\infty} p_n/q_n = x$.

Underlying the regular continued fraction (RCF) expansion (1.1) is the map $T : [0,1) \to [0,1)$, defined by

$$T(x) = \frac{1}{x} \bmod 1 = \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor, \; x \neq 0; \quad T(0) = 0.$$

Here $\left\lfloor \frac{1}{x} \right\rfloor$ denotes the integer part of $\frac{1}{x}$. The RCF-convergents of $x \in [0,1) \setminus \mathbb{Q}$ have strong approximation properties. We mention here that

$$\left| x - \frac{p_n}{q_n} \right| < \frac{1}{q_n^2}, \quad \text{for } n \geq 0,$$

which implies, together with the well-known recurrence relations for the p_n and q_n, that the rate of convergence of p_n/q_n to x is exponential (see e.g. [DK]). One thus defines the approximation coefficients $\theta_n(x)$ of x by $\theta_n = \theta_n(x) = q_n^2 |x - p_n/q_n|$, $n \geq 0$. We usually suppress the dependence on x in our notation.

For the RCF-expansion we have the following classical theorems by Borel (1905) and Hurwitz (1891) about the quality of the approximations.

Theorem 1. (Borel) *For every irrational number x, and every $n \geq 1$*

$$\min\{\theta_{n-1}, \theta_n, \theta_{n+1}\} < \frac{1}{\sqrt{5}}.$$

The constant $1/\sqrt{5}$ is best possible.

Borel's result, together with a yet earlier result by Legendre [L], which states that if $p, q \in \mathbb{Z}$, $q > 0$, and $\gcd(p, q) = 1$, then

$$\left| x - \frac{p}{q} \right| < \frac{1}{2q^2} \quad \text{implies that} \quad \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p_n \\ q_n \end{pmatrix}, \quad \text{for some } n \geq 0,$$

implies the following result by Hurwitz.

Theorem 2. (Hurwitz) *For every irrational number x there exist infinitely many pairs of integers p and q, such that*

$$\left| x - \frac{p}{q} \right| < \frac{1}{\sqrt{5}} \frac{1}{q^2}.$$

The constant $1/\sqrt{5}$ is best possible.

By removing all irrational numbers which are equivalent to the 'golden mean' $g = \frac{1}{2}(\sqrt{5} - 1)$ (i.e., those irrationals whose RCF-expansion consists of 1s from some moment on), we have that

$$\left| x - \frac{p}{q} \right| < \frac{1}{\sqrt{8}} \frac{1}{q^2},$$

for infinitely many pairs of integers p and q. These constants $1/\sqrt{5}$ and $1/\sqrt{8}$ are the first two points in the so-called *Markoff spectrum*; see [CF] or [B] for further information on this spectrum, and the related Lagrange spectrum.

Note that the theorem of Borel does not suffice to prove Hurwitz's theorem; one needs Legendre's result to rule out the existence of rationals p/q which are not RCF-convergents, but still satisfy $|x - p/q| < 1/(\sqrt{5} q^2)$.

In [T1, T2], Tong generalized Borel's results to the nearest integer continued fraction expansion (NICF). These are continued fractions of the form

$$x = \cfrac{\varepsilon_1}{b_1 + \cfrac{\varepsilon_2}{b_2 + \ldots + \cfrac{\varepsilon_n}{b_n + \ldots}}},$$

generated by the operator $T_{\frac{1}{2}} : \left[-\frac{1}{2}, \frac{1}{2} \right) \to \left[-\frac{1}{2}, \frac{1}{2} \right)$, defined by

$$(1.2) \qquad T_{\frac{1}{2}}(x) = \frac{\varepsilon}{x} - \left\lfloor \frac{\varepsilon}{x} + \frac{1}{2} \right\rfloor, \quad x \neq 0; \quad T(0) = 0,$$

where ε denotes the sign of x. Since the NICF-expansion of any number x can be obtained from the RCF-expansion via a process called singularization (see [DK] or [IK] for details), the sequence of NICF-convergents $(r_k/s_k)_{k \geq 0}$ is a subsequence $(p_n/q_n)_{n \geq 0}$ of the sequence of RCF-convergents of x. Due to this, the approximation by NICF-convergents is faster;

see e.g. [A], or [IK]. In [BJW] it was shown that the approximation by NICF-convergents is also closer; for almost all x one has that

$$\lim_{k\to\infty} \frac{1}{k} \sum_{i=0}^{k-1} \vartheta_k = \frac{\sqrt{5}-2}{2\log G} = 0.24528\ldots$$

whereas

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{k-1} \theta_i = \frac{1}{4\log 2} = 0.36067\ldots,$$

where $\vartheta_k = \vartheta_k(x) = s_k^2 |x - r_k/s_k|$ is the kth NICF-approximation coefficient of x, and $G = g + 1$.

In contrast to this, it was shown in [JK] that for almost every x there are infinitely many arbitrary large blocks of NICF-approximation coefficients $\vartheta_{n-1}, \ldots, \vartheta_{n+k}$, which are all larger than $1/\sqrt{5}$. In spite of this, it is also shown in [JK] that for *all* irrational numbers x there exist infinitely many k for which $\vartheta_k < 1/\sqrt{5}$.

In [T1] and [T2], Tong sharpened the results from [JK], by showing that for the NICF there exists a 'pre-spectrum,' i.e., there exists a sequence of constants $(c_k)_{k\geq 1}$, monotonically decreasing to $1/\sqrt{5}$, such that for all irrational numbers x the minimum of any block of $k+2$ consecutive NICF-approximation coefficients is smaller than c_k.

Theorem 3. (Tong) *For every irrational number x and all positive integers n and k one has*

$$\min\{\vartheta_{n-1}, \vartheta_n, \ldots, \vartheta_{n+k}\} < \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{5}} \left(\frac{3-\sqrt{5}}{2}\right)^{2k+3}.$$

The constant $c_k = \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{5}} \left(\frac{3-\sqrt{5}}{2}\right)^{2k+3}$ is best possible.

In [HK], Hartono and Kraaikamp showed how Tong's result follows from a geometrical approach based on the natural extension of the NICF. We further this approach to find Tong's spectrum for an infinite family of continued fractions generalizing the NICF; these *Rosen fractions* are briefly described in the next section.

Although the appropriate terms are only defined in the following section, the reader may wish to compare Tong's Theorem with the following, whose proof appears in Section 3. (The constants τ_k are given in the statement of Theorem 10.)

Theorem 4. *Fix an even $q = 2p$, with $p \geq 2$. For every G_q-irrational number x and all positive n and k, one has*

$$\min\{\Theta_{n-1}, \Theta_n, \ldots, \Theta_{n+k(p-1)}\} < \frac{-\tau_{k-1}}{1 + (\lambda - 1)\tau_{k-1}}.$$

The constant $c_{k-1} = \frac{-\tau_{k-1}}{1+(\lambda-1)\tau_{k-1}}$ is best possible.

We prove an analogous result for all odd indices of these G_q in Section 4. In both cases, we prove a Borel-type result. Furthermore, our approach allows us to give various metric results.

2. Rosen continued fractions

In 1954, David Rosen (see [R]) introduced a family of continued fractions now bearing his name. The Rosen fractions form an infinite family of continued fractions generalizing the NICF. Although Rosen introduced his continued fractions to study certain Fuchsian groups, we are only concerned with their Diophantine approximation properties.

Define $\lambda_q = 2\cos\frac{\pi}{q}$ for each $q \in \{3, 4, \ldots\}$. For q fixed, to simplify notation we usually write λ for λ_q. For each q the *Rosen* or λ*-expansion* (λCF) of x is found by using the map $f_q : \left[-\frac{\lambda}{2}, \frac{\lambda}{2}\right) \to \left[-\frac{\lambda}{2}, \frac{\lambda}{2}\right)$, defined by

(2.1) $$f_q(x) = \frac{\varepsilon}{x} - \lambda r(x), \quad x \neq 0; \quad f_q(0) = 0,$$

where $r(x) = \left\lfloor \frac{\varepsilon}{\lambda x} + \frac{1}{2} \right\rfloor$ and $\frac{\varepsilon}{x} = 1/|x|$. We usually write r instead of $r(x)$. Since $\lambda_3 = 1$, we see that for $q = 3$ the map f_q is the NICF-operator $T_{\frac{1}{2}}$ from (1.2). For $x \in [-\lambda/2, \lambda/2)$, the map f_q yields a continued fraction of the form

$$x = \cfrac{\varepsilon_1}{r_1\lambda + \cfrac{\varepsilon_2}{r_2\lambda + \ldots + \cfrac{\varepsilon_n}{r_n\lambda + \ldots}}} =: [\varepsilon_1 : r_1, \varepsilon_2 : r_2, \ldots, \varepsilon_n : r_n, \ldots],$$

where $\varepsilon_i \in \{\pm 1\}$ and $r_i \in \mathbb{N}$. As usual, finite truncations yield the convergents R_n/S_n, for $n \geq 0$, i.e., $R_n/S_n = [\varepsilon_1 : r_1, \varepsilon_2 : r_2, \ldots, \varepsilon_n : r_n]$. The (Rosen) approximation coefficients of x are defined by

$$\Theta_n = \Theta_n(x) = S_n^2 \left| x - \frac{R_n}{S_n} \right|, \quad \text{for } n \geq 0.$$

For $x \in [-\lambda/2, \lambda/2)$, we define the *future* (t_n) and the *past* (v_n) of x at time n by

$$t_n = [\varepsilon_{n+1} : r_{n+1}, \varepsilon_{n+2} : r_{n+2}, \ldots], \quad v_n = [1 : r_n, \varepsilon_n : r_{n+1}, \ldots, \varepsilon_2 : r_1].$$

The map f_q acts as a one-sided shift on the Rosen expansion of x: $f_q^n(x) = t_n$. We define the natural extension operator to keep track of both t_n and v_n.

Definition. For a fixed q the natural extension map $\mathcal{T}$ is given by

$$\mathcal{T}(x, y) = \left(f_q(x), \frac{1}{r\lambda + \varepsilon y} \right).$$

In [BKS] it was shown that for every $q \geq 3$ there exists a region $\Omega_q \subset \mathbb{R}^2$, for which $\mathcal{T} : \Omega_q \to \Omega_q$ is bijective almost everywhere (with respect to an invariant measure, see Equation (3.1), that is absolutely continuous with respect to Lebesgue measure). In Section 3 (for q even) and Section 4 (for q odd) we recall the exact form of Ω_q. See also [N1], where Ω_q was obtained for q even.

For $x = [\varepsilon_1 : r_1, \varepsilon_2 : r_2, \ldots]$ one has $\mathcal{T}^n(x, 0) = (t_n, v_n)$. The approximation coefficients of x can be given in terms of t_n and v_n (see also [DK]) as

$$(2.2) \qquad \Theta_{n-1} = \frac{v_n}{1 + t_n v_n}, \quad \Theta_n = \frac{\varepsilon_{n+1} t_n}{1 + t_n v_n}.$$

For simplicity, we say that a real number r/s is a G_q-*rational* if it has finite Rosen expansion, all other real numbers are called G_q-*irrationals*. In [HS], Haas and Series derived a Hurwitz-type result using non-trivial hyperbolic geometric techniques. They showed that for every G_q-irrational x there exist infinitely many G_q-rationals r/s, such that $\Theta(x, r/s) \leq \mathcal{H}_q$, where $\mathcal{H}_q$ is given by

$$\mathcal{H}_q = \begin{cases} \dfrac{1}{2} & \text{if } q \text{ is even,} \\[2ex] \dfrac{1}{2\sqrt{(1 - \frac{\lambda}{2})^2 + 1}} & \text{if } q \text{ is odd.} \end{cases}$$

In this paper we derive a Borel-type result, by showing that for every G_q-irrational x there are infinitely many $n \geq 1$ such that $\Theta_n \leq \mathcal{H}_q$. The even and odd case differ and we treat them separately. In both cases we focus on regions where $\min\{\Theta_{n-1}, \Theta_n, \ldots\} < \mathcal{H}_q$.

In fact, the Borel-type result we derive does not immediately imply the Hurwitz-type result of Haas and Series. Nakada [N2] showed that the Legendre constant L_q is smaller than $\mathcal{H}_q$ (recall that for the RCF this Legendre constant is $1/2$, thus is larger than the Hurwitz constant $1/\sqrt{5}$). Still, the Haas and Series results can be proved using continued fraction properties by means of a map which yields the Rosen-convergents and the so-called first medians; see [KNS].

3. Tong's spectrum for even indices $q = 2p$

In this section q is even, we fix $q = 2p$. The region of the natural extension Ω_q is the smallest region where $\mathcal{T}$ is bijective. Usually we write Ω instead of Ω_q. We have the following result from [BKS].

Theorem 5. ([BKS]) *The domain Ω upon which $\mathcal{T}$ is bijective is given by*

$$\Omega = \bigcup_{j=1}^{p} J_j \times K_j.$$

Here J_j is defined as follows: Let $\phi_j = T^j \left(-\frac{\lambda}{2} \right)$, then $J_j = [\phi_{j-1}, \phi_j)$ for $j \in \{1, 2, \ldots, p-1\}$ and $J_p = \left[0, \frac{\lambda}{2} \right)$. Further, $K_j = [0, L_j]$ for $j \in \{1, 2, \ldots, p-1\}$ and $K_p = [0, R]$, where L_j and R are derived from the relations

$$\begin{cases} (\mathcal{R}_0): & R = \lambda - L_{p-1}, \\ (\mathcal{R}_1): & L_1 = 1/(\lambda + R), \\ (\mathcal{R}_j): & L_j = 1/(\lambda - L_{j-1}) \qquad \text{for } j \in \{2, \cdots, p-1\}, \\ (\mathcal{R}_p): & R = 1/(\lambda - L_{p-1}). \end{cases}$$

The map f_q sends each interval J_i to J_{i+1} for $i = 1, \ldots, p-1$. Further, we denote $\Omega_+ = \{(t, v) \in \Omega \mid t > 0\}$.
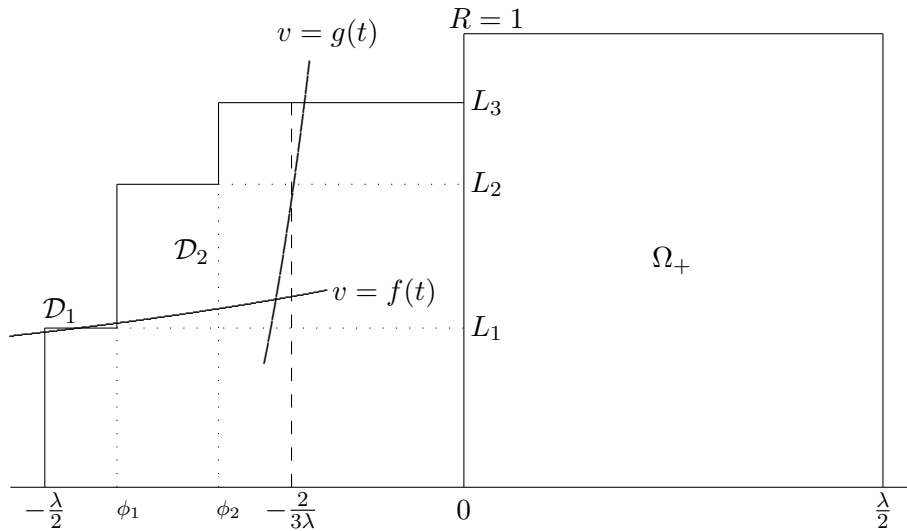

FIGURE 1. The region of the natural extension Ω_8, with $\mathcal{D}$ of Lemma 7.

In [BKS] it is shown that $\mathcal{T}$ preserves a probability measure, ν, that is absolutely continuous with respect to Lebesgue measure. Its density is

$$(3.1) \qquad g_q(t,v) = \begin{cases} \dfrac{C_q}{(1+tv)^2}, & \text{for } (t,v) \in \Omega_q, \\ 0, & \text{otherwise,} \end{cases}$$

where $C_q = \dfrac{1}{\log[(1+\cos\frac{\pi}{q})/\sin\frac{\pi}{q}]}$ is a normalizing constant. It is also shown in [BKS], that the dynamical system $(\Omega, \nu, \mathcal{T})$ is weak Bernoulli (and therefore ergodic).

The following proposition on the distribution of the Θ_n, also in [BKS], is a consequence of the Ergodic Theorem and the strong approximation properties of the Rosen fractions; see [DK], or [IK], Chapter 4.

Proposition 6. *Let $q \geq 3$ be even. For almost all G_q-irrational numbers x the two-dimensional sequence*

$$\mathcal{T}^n(x,0) = (t_n, v_n), n \geq 1$$

is distributed over Ω_q according to the density function $g_q(t,v)$ given in Equation (3.1).

3.1. Consecutive pairs of large approximation constants: The region $\mathcal{D}$. To find Tong's spectrum we start by looking at two consecutive large approximation coefficients Θ_{n-1} and Θ_n. In view of (2.2) we define $\mathcal{D} \subset \Omega$ by

$$(3.2) \qquad \mathcal{D} = \left\{ (t,v) \in \Omega \,\middle|\, \min\left\{ \frac{v}{1+tv}, \frac{|t|}{1+tv} \right\} > \frac{1}{2} \right\}.$$

So $(t_n, v_n) \in \mathcal{D}$ if and only if $\min\{\Theta_{n-1}, \Theta_n\} > \frac{1}{2}$. We have the following result describing $\mathcal{D}$.

Lemma 7. *Define functions f and g by*

$$(3.3) \qquad f(x) = \frac{1}{2-x} \quad and \quad g(x) = \frac{2|x|-1}{x}.$$

For all even q, $\mathcal{D}$ consists of two connected components $\mathcal{D}_1$ and $\mathcal{D}_2$. The subregion $\mathcal{D}_1$ is bounded by the lines $t = -\frac{\lambda}{2}, v = L_1$ and the graph of f; $\mathcal{D}_2$ is bounded by the graph of g from the right, by the graph of f from below and by the boundary of Ω; see Figure 1.

Region		r_{n+1}	ε_{n+1}	ε_{n+2}	Θ_{n+1}
$\mathcal{A}:$	$\dfrac{-2}{3\lambda} \le t_n \le \dfrac{-1}{\lambda+1}$	2	-1	-1	$\dfrac{(1+2t_n\lambda)(v_n-2\lambda)}{1+t_nv_n}$
$\mathcal{B}:$	$\dfrac{-1}{\lambda} \le t_n < \dfrac{-2}{3\lambda}$	1	-1	1	$\dfrac{(1+t_n\lambda)(\lambda-v_n)}{1+t_nv_n}$
$\mathcal{C} \cup \mathcal{D}_1:$	$\dfrac{-\lambda}{2} \le t_n < \dfrac{-1}{\lambda}$	1	-1	-1	$\dfrac{(1+t_n\lambda)(v_n-\lambda)}{1+t_nv_n}$

TABLE 1. Subregions of $\mathcal{D}$ giving constant coefficients.

Proof. For the approximation coefficients one has

$$\frac{v}{1+tv} \le \frac{1}{2} \Leftrightarrow \qquad\qquad v \le f(t),$$

$$\frac{|t|}{1+tv} \le \frac{1}{2} \Leftrightarrow \begin{cases} v \le g(t) & \text{if } t < 0 \\[2mm] v \ge g(t) & \text{if } t \ge 0. \end{cases}$$

Since for $t > 0$ the graphs of f and g meet at $t = 1$, and $1 > \frac{\lambda}{2}$, it follows that points for which $\min\left\{\frac{v}{1+tv}, \frac{|t|}{1+tv}\right\} < \frac{1}{2}$ must satisfy $t < 0$ and $v > g(t)$. It is easy to check that for all even q the only intersection points of the graphs of the functions f and g in the region of the natural extension are given by $\left(-\frac{\lambda}{2}, \frac{2}{\lambda+4}\right)$, $(-L_1, L_{p-1})$ and $(-L_{p-1}, L_1)$. The fact that $\mathcal{D}$ consists of the two pieces follows from $\phi_0 \le -L_{p-1} = 1 - \lambda \le \phi_1$. $\qquad\square$

Having control on the approximation coefficients Θ_{n-1} and Θ_n, we turn our attention to Θ_{n+1} on $\mathcal{D}$. It follows from (2.2) and the definition of $\mathcal{T}$ that

(3.4) $$\Theta_{n+1} = \frac{\varepsilon_{n+2}(1 - \varepsilon_{n+1}r_{n+1}t_n\lambda)(\lambda r_{n+1} + \varepsilon_{n+1}v_n)}{1+t_nv_n}.$$

In order to express Θ_{n+1} locally as a function of only t_n and v_n, we divide $\mathcal{D}$ into regions where $r_{n+1}, \varepsilon_{n+1}$ and ε_{n+2} are constant. This gives three regions; see Table 1 for the definition of $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, the new subregions involved in this. See also Figure 2.
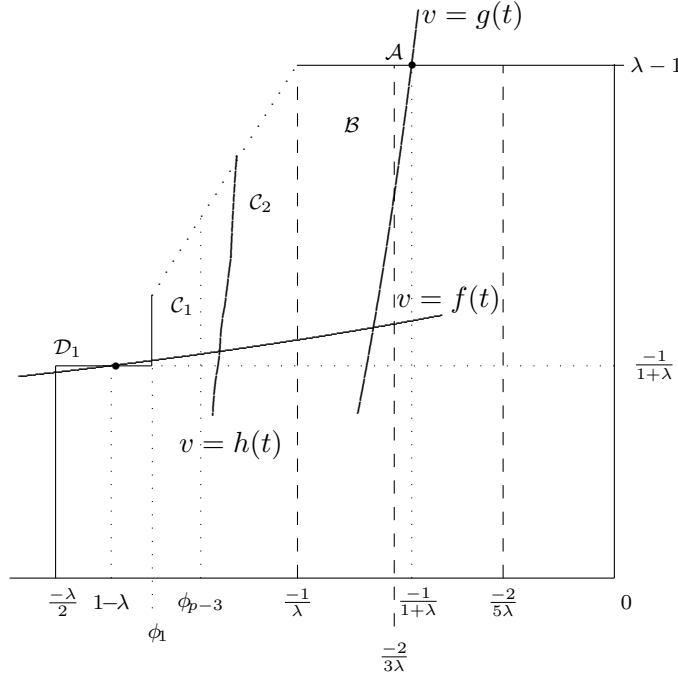
FIGURE 2.   The regions in $\mathcal{D}$.

Solving for $\Theta_{n+1} = 1/2$, leads to

$$h(t) = \frac{2\lambda^2 t + 2\lambda + 1}{2\lambda t - t + 2} \, ,$$

whose graph divides $\mathcal{C}$ into two parts. Let $\mathcal{C}_1$ denote the left-hand side of this graph, there and on $\mathcal{D}_1$ one has $\Theta_{n+1} > \frac{1}{2}$; on the remainder, $\mathcal{C}_2$, one has $\Theta_{n+1} < \frac{1}{2}$. Note that $\mathcal{T}$ takes the graph of h to the graph of g.

On its right-hand side region $\mathcal{B}$ is bounded by the graph of g. In view of Equation (3.4) and Table 1, we consider the graph of

$$\ell(t) = \frac{2\lambda^2 t + 2\lambda - 1}{2\lambda t + t + 2}, \quad \text{for } t \neq \frac{-2}{2\lambda + 1}.$$

An easy calculation shows that the graphs of ℓ and g intersect only at the point $(-1/(\lambda+1), \lambda-1)$, and that for $t > -2/(2\lambda+1)$ the graph of g lies above that of ℓ. Furthermore, $\ell'(t) > 0$ for $t \neq \frac{-2}{2\lambda+1}$, and $\ell\left(\frac{-1}{\lambda}\right) = \lambda$; we conclude that $\Theta_{n+1} < 1/2$ on region $\mathcal{B}$.

Lemma 8. *With notation as above, the subset of $\mathcal{D}$ on which $\Theta_{n+1} > 1/2$ is exactly the union of regions $\mathcal{D}_1$, $\mathcal{C}_1$ and region $\mathcal{A}$. On region $\mathcal{A}$ one has $\Theta_{n+1} > \Theta_{n-1} > \Theta_n$.*

Proof. The remarks directly above show that we need now only consider region $\mathcal{A}$.

It immediately follows from (2.2) and the fact that $v_n > -t_n$ on $\mathcal{A}$, that $\Theta_{n-1} > \Theta_n$. To show that $\Theta_{n+1} > \Theta_{n-1}$ we need to show

$$(1 + 2t_n\lambda)(v_n - 2\lambda) > v_n.$$

or equivalently

$$-4\lambda^2 t_n + 2\lambda t_n v_n - 2\lambda > 0.$$

We use $v_n > -t_n$ again, so it is enough to show

$$-4\lambda^2 t_n - 2\lambda t_n^2 - 2\lambda \geq 0.$$

The last statement is true if $t_n \in [-\lambda - \sqrt{\lambda^2 - 1}, -\lambda + \sqrt{\lambda^2 - 1}]$, which does indeed hold on region $\mathcal{A}$.

Since $\min\{\Theta_{n-1}, \Theta_n\} > \frac{1}{2}$ on $\mathcal{D}$, the result follows.   □

Now, by definition, $f_q(t) = -1/t - \lambda$ for $t \in [-\lambda/2, -2/3\lambda)$. It follows that the $\mathcal{T}$ orbit of any point of the $\mathcal{C}_i$ either eventually leaves $\mathcal{D}$, or eventually enters $\mathcal{A}$. Thus, we naturally focus on the interval $t \in J_{p-1} = [\phi_{p-2}, 0)$. For almost every G_q-irrational $x \in \left[-\frac{\lambda}{2}, \frac{\lambda}{2}\right)$ there is an n such that $\mathcal{T}^n(x,0) = (t_n, v_n)$ and $t_n \in J_{p-2}$. Divide the interval J_{p-2} into three parts. If $t_n \in \left[\phi_{p-2}, \frac{-2}{3\lambda}\right)$, then $\min\{\Theta_{n-1}, \Theta_n, \Theta_{n+1}\} < \frac{1}{2}$. If $t_n \in \left[\frac{-1}{\lambda+1}, 0\right)$, then $(t_n, v_n) \notin \mathcal{D}$ so $\min\{\Theta_{n-1}, \Theta_n\} < \frac{1}{2}$. However, if $t_n \in \left[\frac{-2}{3\lambda}, \frac{-1}{\lambda+1}\right)$, then it may be that $(t_n, v_n) \in \mathcal{A}$ and thus $\min\{\Theta_{n-1}, \Theta_n, \Theta_{n+1}\} > \frac{1}{2}$.

Lemma 9. *The transformation $\mathcal{T}$ maps $\mathcal{A}$ bijectively onto region $\mathcal{D}_1$.*

Proof. The vertices of $\mathcal{A}$ are mapped onto the vertices of region $\mathcal{D}_1$ by $\mathcal{T}$:

$$\left(\frac{-2}{3\lambda}, \lambda - 1\right) \;\mapsto\; \left(\frac{-\lambda}{2}, \frac{1}{\lambda+1}\right),$$

$$\left(\frac{-1}{\lambda+1}, \lambda - 1\right) \;\mapsto\; \left(1 - \lambda, \frac{1}{\lambda+1}\right),$$

$$\left(\frac{-2}{3\lambda}, \frac{3\lambda - 4}{2}\right) \;\mapsto\; \left(\frac{-\lambda}{2}, \frac{2}{\lambda+4}\right).$$

It is easily checked that $\mathcal{T}$ takes the graph of g to the graph of f. Since $\mathcal{T}$ is continuous and bijective and sends straight lines to straight lines, this completes the proof.   □

Under $\mathcal{T}$, region $\mathcal{D}_1$ is mapped onto a region with upper vertices (ϕ_1, L_2) and $(-L_{p-2}, L_2)$. For $i = 2, \ldots, p-1$ we find a region with upper vertices

(ϕ_{i-1}, L_i) and $(-L_{p-i}, L_i)$ after applying $\mathcal{T}^i$ to $\mathcal{A}$. The lower part of this region is bounded by the ith transformation of the curve $g(t)$ under $\mathcal{T}$. After $p-1$ applications of $\mathcal{T}$ this results in a region with upper vertices $(\phi_{p-2}, L_{p-1}), (-L_1, L_{p-1})$. Clearly, this region intersects with $\mathcal{A}$. We call $p-1$ consecutive applications of $\mathcal{T}$ a *round*.

That part of $\mathcal{T}^{p-1}(\mathcal{A})$ lying to the left of $t = \frac{-2}{3\lambda}$ is in region $\mathcal{B}$; the images of these points under a subsequent application of $\mathcal{T}$ have positive t-coordinate. We call *flushing* an application of $\mathcal{T}$ to such points — the points are flushed from $\mathcal{D}$. The remainder of $\mathcal{T}^{p-1}(\mathcal{A})$ is a subset of $\mathcal{A}$.

Theorem 10. *Any point (t, v) of $\mathcal{A}$ is flushed after exactly k rounds if and only if*

$$\tau_{k-1} \leq t < \tau_k,$$

where $\tau_0 = \frac{-2}{3\lambda}$ and

$$\tau_k = \left[\left(-1 : 2, (-1 : 1)^{p-2} \right)^k, \left(\frac{-2}{3\lambda} : \right) \right].$$

For any x with $\tau_{k-1} \leq t_n < \tau_k$,

$$\min\{\Theta_{n-1}, \Theta_n, \ldots, \Theta_{n+k(p-1)-1}, \Theta_{n+k(p-1)}\} > \frac{1}{2},$$

while

$$\min\{\Theta_{n-1}, \Theta_n, \ldots, \Theta_{n+k(p-1)}, \Theta_{n+k(p-1)+1}\} < \frac{1}{2}.$$

Proof. A point $(t, v) \in \mathcal{A}$ gets flushed after exactly k rounds if k is minimal such that $\mathcal{T}^{k(p-1)}(t, v)$ has its first coordinate smaller than $\frac{-2}{3\lambda}$. We look at pre-images of $t = \frac{-2}{3\lambda}$ under f_q:

$$
\begin{aligned}
f_q^{-1}\left(\frac{-2}{3\lambda}\right) &= \left[-1; 1, \left(\frac{-2}{3\lambda} :\right)\right], \\
f_q^{-(p-1)}\left(\frac{-2}{3\lambda}\right) &= \left[-1; 2, (-1:1)^{p-2}, \left(\frac{-2}{3\lambda} :\right)\right], \\
&\vdots \\
f_q^{-k(p-1)}\left(\frac{-2}{3\lambda}\right) &= \left[(-1; 2, (-1:1)^{p-2})^k, \left(\frac{-2}{3\lambda} :\right)\right] = \tau_k.
\end{aligned}
$$

The result thus clearly follows.   □

3.2. Metrical results. We define $\mathcal{A}_k = \{(t,v) \in \mathcal{A} \mid t \geq \tau_k\}$ for $k \geq 0$. From Theorem 10 and the ergodicity of $\mathcal{T}$ (cf. Proposition 6) we have the following metrical theorem on the distribution of large blocks of "big" approximation coefficients.

Theorem 11. *For almost all x (with respect to Lebesgue measure) and $k \geq 1$, the limit*

$$\lim_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leq j \leq N \mid \min\{\Theta_{j-1}, \Theta_j, \ldots, \Theta_{j+k(p-1)}\} > \frac{1}{2} \right.$$

$$\left. and \ \Theta_{j+k(p-1)+1} < \frac{1}{2} \right\}$$

exists and equals $\nu(\mathcal{A}_{k-1} \setminus \mathcal{A}_k) = \nu(\mathcal{A}_{k-1}) - \nu(\mathcal{A}_k)$.

In order to apply Theorem 11 we compute $\nu(\mathcal{A})$ and $\nu(\mathcal{A}_k)$:

$$\nu(\mathcal{A}) = C_q \int_{t=\frac{-2}{3\lambda}}^{-L_1} \int_{v=g(t)}^{L_{p-1}} \frac{1}{(1+tv)^2} \, dv \, dt$$

$$= -C_q \int_{\frac{-2}{3\lambda}}^{-L_1} \left(\frac{1}{2t^2} + \frac{1}{t} - \frac{L_{p-1}}{1 + L_{p-1}t} \right) dt$$

$$(3.5) \qquad = C_q \left(\frac{-1}{2L_1} + \log\left| L_{p-1} + \frac{-1}{L_1} \right| + \frac{3\lambda}{4} - \log\left| L_{p-1} - \frac{3\lambda}{2} \right| \right).$$

Using the normalizing constant C_q from Definition 3.1, $L_1 = \dfrac{1}{\lambda+1}$ and $L_{p-1} = \lambda - 1$, it follows from (3.5) that

$$\nu(\mathcal{A}) = \frac{1}{\log[(1+\lambda/2)/\sin\pi/q]} \left(\frac{\lambda-2}{4} + \log\frac{4}{\lambda+2} \right).$$

Similarly,

$$\nu(\mathcal{A}_k) = \qquad\qquad C_q \int_{t=\tau_k}^{-L_1} \int_{v=g(t)}^{L_{p-1}} \frac{1}{(1+tv)^2} \, dv \, dt$$

$$= C_q \left[\frac{(-\lambda-1)\tau_k - 1}{2\tau_k} + \log\left| \frac{2\tau_k}{(\lambda-1)\tau_k + 1} \right| \right].$$

Example. If $q = 8$ we have $\nu(\mathcal{A}) = 4.6 \cdot 10^{-4}, \nu(\mathcal{A}_1) = 7.6 \cdot 10^{-7}$, and $\nu(\mathcal{A}_2) = 6.7 \cdot 10^{-10}$. So Theorem 11 yields that for almost every x about $4.6 \cdot 10^{-2}\,\%$ of the blocks of consecutive approximation coefficients of length 6 have the property that

$$\min\{\Theta_{j-1}, \Theta_j, \ldots, \Theta_{j+3}\} > \frac{1}{2} \quad \text{and} \ \Theta_{j+4} < \frac{1}{2},$$

while about $7.6 \cdot 10^{-5}\,\%$ of the blocks of length 9 have the property that

$$\min\{\Theta_{j-1}, \Theta_j, \ldots, \Theta_{j+6}\} > \frac{1}{2} \quad \text{and} \quad \Theta_{j+7} < \frac{1}{2}.$$

3.3. Tong's spectrum for even q. We are now ready to determine the Tong spectrum for these Rosen continued fractions. Recall that Theorem 4 states that the minimum of $1 + k(p-1)$ consecutive values of Θ_j, beginning with Θ_{n-1} is less than $c_{k-1} := \frac{-\tau_{k-1}}{1+(\lambda-1)\tau_{k-1}}$, where the τ_j are given in Theorem 10.

Proof of Theorem 4. We start with $k = 1$. Assume that $(t_n, v_n) \in \mathcal{D}$, otherwise we are done. For a certain $0 \le i \le p - 2$ we have (t_{n+i}, v_{n+i}) is either in $\mathcal{A}$ or flushed to Ω_+. In the latter case $\Theta_{n+i} < \frac{1}{2}$ and we are done. So assume that $(t, v) = (t_{n+i}, v_{n+i}) \in \mathcal{A}$. It follows from Lemma 8 that $\min\{\Theta_{n-1}, \Theta_n, \Theta_{n+1}\} = \Theta_n$ on $\mathcal{A}$. We give an upper bound for this minimum, by determining the maximum value of Θ_n.

On $\mathcal{A}$ one has $\Theta_n = \frac{-t_n}{1+t_n v_n}$, so we must find the point $(t, v) \in \mathcal{A}$ where $m(t, v) := \frac{-t}{1+tv}$ attains its maximum. We have

$$\frac{\partial m(t, v)}{\partial t} = \frac{-1}{(1 + tv)^2} < 0 \quad \text{and} \quad \frac{\partial m(t, v)}{\partial v} = \frac{t^2}{(1 + tv)^2} > 0.$$

Thus Θ_n attains its maximum at the upper left corner $\left(\frac{-2}{3\lambda}, \lambda - 1\right)$ of $\mathcal{A}$. This maximum equals $\frac{2}{\lambda+2} = c_0$.

It is left to show that c_0 is the best possible constant. We need to check that for the first $p - 2$ points in either direction on the orbit of $\left(\frac{-2}{3\lambda}, \lambda - 1\right)$ one has $\min\{\Theta_{n-1}, \Theta_n\} \ge c_0$. These points on the orbit of $\left(\frac{-2}{3\lambda}, \lambda - 1\right)$ are

$$\left(\left[(-1 : 1)^{p-2}, \left(\frac{-2}{3\lambda} :\right)\right], L_1\right), \left(\left[(-1 : 1)^{p-1}, \left(\frac{-2}{3\lambda} :\right)\right], L_2\right), \ldots,$$

$$\left(\left[(-1 : 1), \left(\frac{-2}{3\lambda} :\right)\right], L_{p-2}\right), \left(\frac{-2}{3\lambda}, \lambda - 1\right), \left(\frac{-\lambda}{2}, L_1\right),$$

$$(\phi_1, L_2), (\phi_2, L_3), \ldots, (\phi_{p-3}, L_{p-2}).$$

We consider the curves $\Theta_{n-1} = c_0$ and $\Theta_n = c_0$, thus

$$f_1(x) = \frac{2}{\lambda + 2 - 2x} \quad \text{and} \quad g_1(x) = \frac{-(\lambda + 2)x - 2}{2x}.$$

The graph of $f_1(x)$ intersects with the x-axis at the point $\left(0, \frac{2}{\lambda+2}\right)$. For the heights L_i we easily find that $L_{p-2} > \ldots > L_3 > L_2 > \frac{2}{\lambda+2}$. Thus, each of the first $p-2$ points in either direction of the orbit has either y-coordinate greater than L_2 or is one of the points $\left(-\frac{\lambda}{2}, L_1\right)$ or $\left(\left[(-1 : 1)^{p-2}, \left(\frac{-2}{3\lambda} :\right)\right], L_1\right)$.

At the point $\left(-\frac{\lambda}{2}, L_1\right)$ the value of Θ_{n-1} is exactly $c_1 = \frac{2}{\lambda+2}$. Further, we know that $\frac{\partial \Theta_{n-1}}{\partial t_n} > 0$ on region $\mathcal{C}$, so

$$\Theta_{n-1}\left(\left[(-1:1)^{p-2}, \left(\frac{-2}{3\lambda}:\right)\right], L_1\right) > \Theta_{n-1}\left(-\frac{\lambda}{2}, L_1\right).$$

Together this means that for all these points one finds $\Theta_{n-1} \geq c_0$. The graph of $g_1(x)$ intersects with the x-axis in the point $\left(\frac{-2}{\lambda+2}, 0\right)$. We have

$$-\frac{\lambda}{2} < \phi_1 < \ldots < \phi_{p-1} < \left[-1:1, \left(\frac{-2}{3\lambda}:\right)\right] < \frac{-2}{\lambda+2},$$

so for the relevant points on the orbit of $\left(\frac{-2}{3\lambda}, \lambda-1\right)$ we have $\Theta_n > c_0$.

This proves the case $k = 1$. For general k, assume that the starting point $(t_n, v_n) \in \mathcal{A}$ did not get flushed during the first $k-1$ rounds, otherwise we are done. Consider again a point $(t, v) = (t_{n+i}, v_{n+i})$ in $\mathcal{A}$ with $0 \leq i \leq p-2$. From Theorem 10 this means that $t \geq \tau_{k-1}$. We find that the maximum of Θ_n occurs at the upper-left corner of the region where (t, v) is located. This maximum is given by

$$\Theta_n(\tau_{k-1}, \lambda-1) = \frac{-\tau_{k-1}}{1 + (\lambda-1)\tau_{k-1}} = c_{k-1}.$$

The proof that this is the best possible is similar to the case $k = 1$. $\qquad \square$

Note that

$$\lim_{k \to \infty} \tau_k = \overline{[(-1:2, (-1:1)^{p-2})]} = \frac{-1}{\lambda+1},$$

yielding

(3.6)
$$\lim_{k \to \infty} \frac{-\tau_k}{1 + (\lambda-1)\tau_k} = \frac{1}{2}.$$

Lemma 12. *Let $\mathcal{F}$ denote the fixed point set in $\mathcal{D}$ of $\mathcal{T}^{(p-1)}$. Then*

(i) $\mathcal{F} = \{\mathcal{T}^i(-1/(\lambda+1), \lambda-1) \mid i = 0, 1, \ldots, p-1\}$;

(ii) *For every x and every $n \geq 0$, $(t_n, v_n) \notin \mathcal{F}$;*

(iii) *For every G_q-irrational number x there are infinitely many n for which $(t_n, v_n) \notin \mathcal{D}$;*

(iv) *For each $i = 0, 1, \ldots, p-1$, let $x_i = f_q^i(-1/(\lambda+1))$. Then for all $n \geq 0$, $\mathcal{T}^n(x_i, 0) \notin \mathcal{D}$. However, $\mathcal{T}^{k(p-1)}(x_i, 0)$ converges from below along the vertical line $x = x_i$ to $\mathcal{T}^i(-1/(\lambda+1), \lambda-1)$.*

Proof. (i) Since $\mathcal{T}^{(p-1)}$ fixes $(-1/(\lambda+1), \lambda-1)$, $\mathcal{F}$ certainly contains the $\mathcal{T}$-orbit of $(-1/(\lambda+1), \lambda-1)$. On the other hand, any point of $\mathcal{D}$ not in this orbit is eventually flushed from $\mathcal{D}$.

(ii) The second coordinate of any element of $\mathcal{F}$ is G_q-irrational, but for all x, the points (t_n, v_n) have G_q-rational second coordinate.

(iii) From the previous item, for any G_q-irrational x the $\mathcal{T}$-orbit of $(x, 0)$ avoids $\mathcal{F}$, hence each time this orbit encounters $\mathcal{D}$, it is flushed out within a finite number of iterations. We conclude indeed that for every G_q-irrational x there are infinitely many n for which $(t_n, v_n) \notin \mathcal{D}$.

(iv) The final item is easily checked.   □

Combining Theorem 4, Equation (3.6) and the previous lemma, we have the following result.

Theorem 13. *For every G_q-irrational x there are infinitely many $n \in \mathbb{N}$ for which*

$$\Theta_n \leq \mathcal{H}_q = \frac{1}{2}.$$

The constant $1/2$ is best possible.

4. Tong's spectrum for odd indices $q = 2h + 3$

The classical case of $q = 3$ is Tong's result itself; a geometric argument is given in [HK]. In our treatment, the case of $q = 5$ is also exceptional and we do not give full details for it. See in particular the remarks after Lemmata 15 and 16.

The results for the odd case are derived similarly to those of the even case. However, this case has more complicated dynamics, complicating the arguments. We begin with the definition of the natural extension.

Again let $\phi_j = f_q^j(-\lambda/2)$. Set $h = \frac{q-3}{2}$ and define J_j, $j \in \{1, \cdots, 2h+2\}$, by

$$J_{2k} = [\phi_{h+k}, \phi_k), \text{ for } k \in \{1, \cdots, h\},$$
$$J_{2k+1} = [\phi_k, \phi_{h+k+1}), \text{ for } k \in \{0, 1, \cdots, h\},$$

and $J_{2h+2} = [0, \frac{\lambda}{2})$. Let $K_j = [0, L_j]$ for $j \in \{1, \cdots, 2h+1\}$ and let $K_{2h+2} = [0, R]$, where R is the solution of

$$R^2 + (2 - \lambda)R - 1 = 0.$$
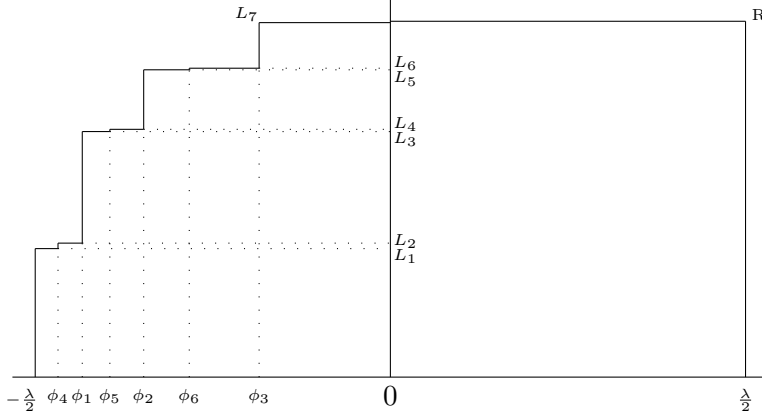
FIGURE 3. The region of the natural extension Ω_9.

Theorem 14. ([BKS]) *Let $q = 2h+3$, with $h \geq 1$ and $\Omega = \bigcup_{j=1}^{2h+2} J_j \times K_j$. With R defined as above, we have*

$$
\begin{cases}
(\mathcal{R}_0): & R = \lambda - L_{2h+1}, \\
(\mathcal{R}_1): & L_1 = 1/(2\lambda - L_{2h}), \\
(\mathcal{R}_2): & L_2 = 1/(2\lambda - L_{2h+1}), \\
(\mathcal{R}_j): & L_j = 1/(\lambda - L_{j-2}) \quad \text{for } 2 < j < 2h+2, \\
(\mathcal{R}_{2h+2}): & R = 1/(\lambda - L_{2h}),
\end{cases}
$$

while $\mathcal{T} : \Omega \to \Omega$ is bijective off a set of Lebesgue measure zero.

In [BKS] it is shown that $\mathcal{T}$ preserves the probability measure ν, with density $\dfrac{C_q}{(1 + xy)^2}$, where $C_q = \dfrac{1}{\log(1 + R)}$ is a normalizing constant. It is also shown in [BKS] that the process $(\Omega, \nu, \mathcal{T})$ is weak Bernoulli and therefore ergodic. Proposition 6 also holds in the odd case.

For odd indices q we define again the region $\mathcal{D}$ by

$$
\mathcal{D} = \left\{ (t, v) \in \Omega \mid \min \left\{ \frac{v}{1 + tv}, \frac{|t|}{1 + tv} \right\} > \mathcal{H}_q \right\}.
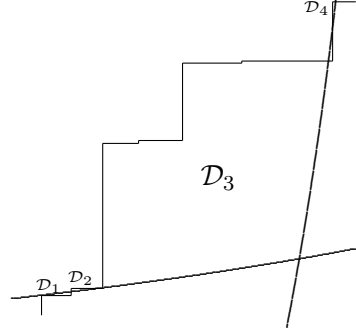$$

Clearly $\min\{\Theta_{n-1}, \Theta_n\} > \mathcal{H}_q$ if and only if $(t_n, v_n) \in \mathcal{D}$.

Lemma 15. *Define functions f and g by*

(4.1) $$ f(x) = \frac{\mathcal{H}_q}{1 - \mathcal{H}_q x} \quad \text{and } g(x) = \frac{|x| - \mathcal{H}_q}{\mathcal{H}_q x}. $$

For all odd $q \geq 7$, the region $\mathcal{D}$ consists of four disjoint regions:

$\mathcal{D}_1$ *bounded by the lines $x = \frac{-\lambda}{2}$ and $y = L_1$ and the graph of f;*

FIGURE 4.   The region $\mathcal{D}$ for $q = 9$.

Region		r_{n+1}	ε_{n+1}	ε_{n+2}	Θ_{n+1}
$\mathcal{A} \cup \mathcal{D}_4$:	$\dfrac{-2}{3\lambda} \le t_n \le \dfrac{-1}{\lambda+\frac{1}{R}}$	2	-1	-1	$\dfrac{(1+2t_n\lambda)(v_n-2\lambda)}{1+t_nv_n}$
$\mathcal{B}$:	$\dfrac{-1}{\lambda} \le t_n < \dfrac{-2}{3\lambda}$	1	-1	1	$\dfrac{(1+t_n\lambda)(\lambda-v_n)}{1+t_nv_n}$
$\mathcal{C} \cup \mathcal{D}_1 \cup \mathcal{D}_2$:	$\dfrac{-\lambda}{2} \le t_n < \dfrac{-1}{\lambda}$	1	-1	-1	$\dfrac{(1+t_n\lambda)(v_n-\lambda)}{1+t_nv_n}$

TABLE 2.   Regions of constant coefficients, the odd case.

$\mathcal{D}_2$ *bounded by the line* $x = \phi_{h+1}$ *and* $y = L_2$ *and the graph of* f;

$\mathcal{D}_3$ *(the largest region) bounded from below by the graph of* f, *from the right by the graph of* g *and by the boundary of* Ω; *and,*

$\mathcal{D}_4$ *bounded by the lines* $x = \phi_h$ *and* $y = L_{2h+1}$ *and the graph of* g.

Remark. In the exceptional case where $q = 5$, one finds that $\mathcal{D}$ consists of a single connected component.

We begin our study of Θ_{n+1} by decomposing $\mathcal{D}$ into regions where $r_{n+1}, \varepsilon_{n+1}$ and ε_{n+2} are constant; see Table 2 for the definition of the new subregions involved in this. See also Figure 5. Again, due to the dynamics of the situation, we will need only focus on the $\mathcal{T}$-orbit of $\mathcal{A}$.

On $\mathcal{A} \cup \mathcal{D}_4$, one easily finds that $\Theta_{n+1} > \Theta_{n-1} > \Theta_n$ holds. Again flushing occurs to the left of the line $t = \frac{-2}{3\lambda}$; note that $\phi_{2h} \le \frac{-2}{3\lambda} < \phi_h$. Thus we study the interval $[\phi_{2h}, \phi_h)$ (instead of the interval $[\phi_h, 0)$, as in the even
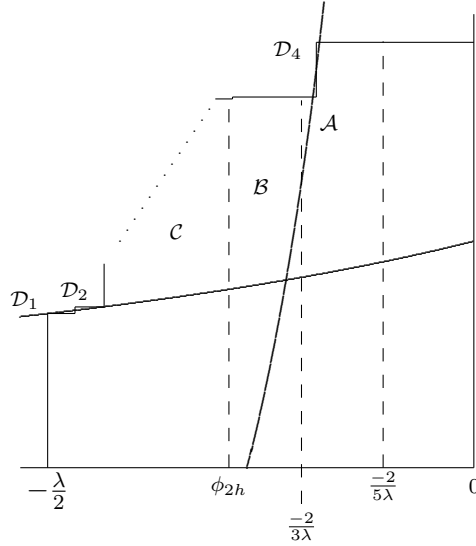
FIGURE 5. The regions for Θ_{n+1} on $\mathcal{D}$.

case). Note that the only one of our regions in the strip defined by the interval $\phi_{2h} \leq t \leq \phi_h$ is indeed $\mathcal{A}$.

One easily shows the following result.

Lemma 16. *The transformation $\mathcal{T}$ maps $\mathcal{A}$ bijectively to region $\mathcal{D}_1$.*

Remark. In the exceptional case where $q = 5$, one can easily see that $\mathcal{A}$ can be defined as in general, and has a similar image. The rest of our arguments can be appropriately adjusted so that the results announced below go through for $q = 5$.

The transformations of $\mathcal{A}$ follow a more complex orbit than in the even case: here, the regions make a "double loop," related to the orbit of the ϕ_j. Region $\mathcal{D}_1$ gets transformed into a region with upper right vertex $(-L_{2h-1}, L_3)$, this region gets transformed in a region with upper right vertex $(-L_{2h-3}, L_5)$ and so on until we reach a region with upper right corner $(-L_1, L_{2h+1})$, which lies in region $\mathcal{D}_4$. This region gets transformed into a region with upper right vertex $(-L_{2h}, L_2)$. Thereafter we find a region with upper right corner $(-L_{2h-2}, L_4)$ and so on, until finally the image intersects with $\mathcal{A}$ after $2h + 1$ steps. Here, we call a *round* these $2h + 1$ transformations of $\mathcal{T}$.

Theorem 17. *Let the constants τ_k be defined by*

$$\tau_k = \left[\left(-1 : 2, (-1 : 1)^h, -1 : 2, (-1 : 1)^{h-1} \right)^k, \left(\frac{-2}{3\lambda} : \right) \right].$$

Then for any $(t, v) \in \mathcal{A}$ such that $\tau_{k-1} \leq t < \tau_k$, the point (t_n, v_n) is flushed after k rounds. In particular, for any $x \in \mathbb{R}$ with $\mathcal{T}^n(x, 0) = (t_n, v_n)$ satisfying $\tau_{k-1} \leq t_n < \tau_k$ one has

$$\min\{\Theta_{n-1}, \Theta_n, \ldots, \Theta_{n+k(2h+1)}, \Theta_{n+k(2h+1)}\} > \mathcal{H}_q,$$

while

$$\min\{\Theta_{n-1}, \Theta_n, \ldots, \Theta_{n+k(2h+1)}, \Theta_{n+k(2h+1)+1}\} < \mathcal{H}_q.$$

4.1. Metric results.

As in the even case, we define $\mathcal{A}_k = \{(t, v) \in \mathcal{A} \,|\, t > \tau_k\}$ for $k \geq 0$.

Theorem 18. *For almost all x (with respect to the Lebesgue measure) and $k \geq 1$ the limit*

$$\lim_{N \to \infty} \frac{1}{N} \# \left\{ 1 \leq j \leq N \,|\, \min\{\Theta_{j-1}, \Theta_j, \ldots, \Theta_{j+k(2h+1)}\} > \frac{1}{2} \right.$$

$$\left. and\ \Theta_{j+k(2h+1)+1} < \frac{1}{2} \right\}$$

exists and equals $\nu(\mathcal{A}_{k-1} \setminus \mathcal{A}_k) = \nu(\mathcal{A}_{k-1}) - \nu(\mathcal{A}_k)$.

We compute again $\nu(\mathcal{A})$ and $\nu(\mathcal{A}_k)$. A calculation similar to that in the even case yields

$$\nu(\mathcal{A}) = D\left(\log\left(R + \frac{1}{R}\right) - \log\left(\frac{\lambda R + 2}{2R}\right) + \frac{\lambda C - 2CR}{2}\right),$$

$$\nu(\mathcal{A}_k) = D\left(\log\left(R + \frac{1}{R}\right) - \log\left|\frac{1 + \tau_k(\lambda - 1/R)}{\tau_k}\right| - C(\lambda + R) - \frac{C}{\tau_k}\right).$$

Example. If $q = 9$ we have $\nu(\mathcal{A}) = 6.2 \cdot 10^{-7}$, $\nu(\mathcal{A}_1) = 6.5 \cdot 10^{-13}$, and $\nu(\mathcal{A}_2) = 6.8 \cdot 10^{-19}$.

So we find that for almost every x about $6.5 \cdot 10^{-11}\%$ of the blocks of consecutive approximation coefficients of length 10 have the property that

$$\min\{\Theta_{j-1}, \Theta_j, \ldots, \Theta_{j+7}\} > \frac{1}{2} \quad \text{and}\ \Theta_{j+8} < \frac{1}{2},$$

while about $6.8 \cdot 10^{-17}\%$ of the blocks of length 17 have the property that

$$\min\{\Theta_{j-1}, \Theta_j, \ldots, \Theta_{j+14}\} > \frac{1}{2} \quad \text{and}\ \Theta_{j+15} < \frac{1}{2}.$$

4.2. Tong's spectrum for odd q**.** We have the following result, which is proved similarly to Theorem 4.

Theorem 19. *For every G_q-irrational number x and all positive n and k, one has*

$$\min\{\Theta_{n-1}, \Theta_n, \ldots, \Theta_{n+k(2h+1)}\} < \frac{-\tau_{k-1}}{1 + (\lambda - R)\tau_{k-1}}.$$

The constant $c_{k-1} = \frac{-\tau_{k-1}}{1+(\lambda-R)\tau_{k-1}}$ is best possible.

Note that $\lim_{k\to\infty} c_k = \mathcal{H}_q$. Due to this, and reasoning as in the proof of Lemma 12 (here one considers the fixed-points in $\mathcal{D}$ of $\mathcal{T}^{2h+1}$ instead of those of $\mathcal{T}^{p-1}$), we get the following result.

Theorem 20. *For every G_q-irrational x there are infinitely many $n \in \mathbb{N}$, such that*

$$\Theta_n(x) \le \mathcal{H}_q.$$

The constant $\mathcal{H}_q$ is best possible.

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