METRIC AND ARITHMETIC PROPERTIES OF MEDIANT-ROSEN MAPS

COR KRAAIKAMP, HITOSHI NAKADA, AND THOMAS A. SCHMIDT

Abstract. We define maps which induce mediant convergents of Rosen continued fractions and discuss arithmetic and metric properties of mediant convergents. In particular, we show equality of the ergodic theoretic Lenstra constant with the arithmetic Legendre constant for each of these maps. This value is sufficiently small that the mediant Rosen convergents directly determine the Hurwitz constant of Diophantine approximation of the underlying Fuchsian group.

1. Introduction

Ergodic properties of a number theoretic transformation can in certain circumstances be studied by way of transformations which induce it. In the classical setting of the *simple continued fraction* (SCF) expansion S. Ito [6] studied maps corresponding to the mediant convergents for exactly this purpose. Motivated by this classical setting, we call any such inducing transformation a *mediant* map. In this paper we give mediant maps for the Rosen continued fraction maps, allowing us to extend the work of [13] and [2]. We discuss some arithmetic and metric properties of mediant convergents arising from these maps, in particular using techniques of [14] to show that the Legendre constant—determining membership in the sequence of approximations of a real number—is equal to the ergodic theoretic Lenstra constant.

One motivation for this work comes from Diophantine approximation in terms of the Rosen fractions. Diophantine approximation by simple continued fractions has of course a rich history. A geometric aspect of this is expressible in terms of the Möbius action of the modular group $\mathrm{PSL}(2, \mathbb{Z})$. In the middle of the last century, this was generalized to approximation by the orbit of infinity under a reasonably large class of Fuchsian groups. Some thirty years after Rosen [16] introduced his continued fractions to study elements of the Hecke triangle groups (a family of Fuchsian groups including the modular group), Lehner [10, 11] used these continued fractions to begin the study of the quality of approximation by the orbit of infinity under each of the Hecke groups. His goal was to determine the Hurwitz constant for this approximation, thus in a sense bounding how bad the approximation can be. These Hurwitz constants were finally determined by Haas and Series [5], using techniques of hyperbolic geometry.

One would naturally like to determine these Hurwitz values using purely continued fraction methods, and this is what we do. However, the process is not as

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straightforward as one might guess. (See the introduction to Section 5 for a more detailed discussion of the following).

The Rosen continued fraction algorithm does provide approximates to each real number that are elements of the orbit of infinity under the corresponding Hecke group; however, there exist real numbers that are yet better approximated by other elements of the orbit. Thus, what one might reasonably call the Hurwitz value of the Rosen fractions, which we easily deduce from results of [9], could a priori be greater than the Hurwitz constant of the corresponding group. The difficulty here is that the (correct, as opposed to the value given in Section 4 of [17]) Legendre value for the Rosen fractions is too large to allow direct determination of the group Hurwitz constants. In order to capture sufficiently closer approximates, we use the mediant maps. As stated above, we determine the Legendre constant for these by first showing equality with the Lenstra constant, whose value we then explicitly find. This Legendre constant is sufficiently small that we can conclude the determination of the Hurwitz constant of the corresponding Hecke group.

1.1. **Outline.** In the next section, we introduce the mediant algorithm. In Section 3, we give the underlying mediant maps. Section 4 is devoted to the construction of planar natural extensions for these, and to the study of their basic ergodic properties. Section 5 gives the continued fraction determination of the Hurwitz constants. Definitions of the Legendre and Lenstra constants for the mediant maps appear in Section 6, where their equality is proven. In Section 7, we evaluate the Lenstra constants.

2. Mediants of Rosen Fractions

Throughout this paper, $\lambda_k = 2\cos\frac{\pi}{k}$ and $\mathbb{I}_k = [-\cos\frac{\pi}{k}, \cos\frac{\pi}{k})$ for $k \geq 3$. For a fixed integer $k \geq 3$, the Rosen continued fraction map is defined by

$$T_k(x) = \begin{cases} \left|\frac{1}{x}\right| - \lambda\lfloor\left|\frac{1}{\lambda x}\right| + \frac{1}{2}\rfloor & x \neq 0; \\ 0 & x = 0 \end{cases}$$

for $x \in \mathbb{I}_k$; here and below, we omit the index "k" whenever it is clear from context. We define

$$\varepsilon_n(x) = \operatorname{sgn}(T^{n-1}x) \qquad \text{and} \qquad r_n(x) = r(T^{n-1}x)$$

with

$$\varepsilon(x) = \operatorname{sgn}(x) \qquad \text{and} \qquad r(x) = \left\lfloor \left|\frac{1}{\lambda x}\right| + \frac{1}{2} \right\rfloor .$$

Then we have the Rosen continued fraction expansion of x as follows

$$x = \frac{\varepsilon_1(x)\,|}{|\,\lambda r_1(x)} + \frac{\varepsilon_2(x)\,|}{|\,\lambda r_2(x)} + \cdots + \frac{\varepsilon_n(x)\,|}{|\,\lambda r_n(x)} + \cdots ,$$

which is denoted by $x = [\,\varepsilon_1(x) : r_1(x),\, \varepsilon_2(x) : r_2(x), \ldots,\, \varepsilon_n(x) : r_n(x), \ldots]$. Here the expansion terminates at a finite term if and only if x is a parabolic point of Hecke group of index k, denoted by G_k (see [16]). As usual we can define the convergents p_n/q_n of $x \in \mathbb{I}_k$ by

$$\begin{pmatrix} p_{-1} & p_0 \\ q_{-1} & q_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_1 \\ 1 & \lambda r_1 \end{pmatrix} \begin{pmatrix} 0 & \varepsilon_2 \\ 1 & \lambda r_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & \varepsilon_n \\ 1 & \lambda r_n \end{pmatrix}$$

for $n \geq 1$. From this definition it is easy to see that $|p_{n-1}q_n - q_{n-1}p_n| = 1$, and that we have the well-known recurrence relations

$$\begin{aligned} p_{-1} &= 1; p_0 = 0; \quad p_n = \lambda r_n p_{n-1} + \varepsilon_n p_{n-2},\ n \geq 1 \\ q_{-1} &= 0; q_0 = 1; \quad q_n = \lambda r_n q_{n-1} + \varepsilon_n q_{n-2},\ n \geq 1. \end{aligned}$$

It also follows that

$$\begin{pmatrix} p_{n-1} & q_{n-1} \\ p_n & q_n \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \varepsilon_n & \lambda r_n \end{pmatrix} \begin{pmatrix} 0 & 1 \\ \varepsilon_{n-1} & \lambda r_{n-1} \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ \varepsilon_1 & \lambda r_1 \end{pmatrix},$$

giving

$$\frac{p_n}{q_n} = \frac{\varepsilon_1 |}{| \lambda r_1} + \frac{\varepsilon_2 |}{| \lambda r_2} + \cdots + \frac{\varepsilon_n |}{| \lambda r_n} \tag{1}$$

and

$$\frac{q_{n-1}}{q_n} = \frac{1 |}{| \lambda r_n} + \frac{\varepsilon_n |}{| \lambda r_{n-1}} + \cdots + \frac{\varepsilon_2 |}{| \lambda r_1}. \tag{2}$$

Since Hecke groups are discontinuous groups, the (parabolic) value p_n/q_n uniquely determines q_n up to sign; we can and do assume q_n to be positive. We sometimes call (2) the *dual* of (1). In the case of simple continued fractions the dual continued fractions are also simple continued fractions (see [12]). However, in the case of the Rosen fractions the resulting sequence of $\varepsilon_i : r_i$ may fail to be admissible.

In the sequel, we identify any 2×2 matrix with the associated linear fractional transformation. That is, $\begin{pmatrix} a & b \\ c & d \end{pmatrix}(x)$ means $\dfrac{ax+b}{cx+d}$.

From the definition of p_n/q_n, we have

$$\frac{p_{n+1}}{q_{n+1}} = \frac{r_{n+1}\lambda p_n + \varepsilon_{n+1}p_{n-1}}{r_{n+1}\lambda q_n + \varepsilon_{n+1}q_{n-1}}.$$

If $r_{n+1} > 1$, then—following [6] in the SCF-case—we can interpolate between p_n/q_n and p_{n+1}/q_{n+1} by

$$\frac{u_{n,l}}{v_{n,l}} = \frac{l\lambda p_n + \varepsilon_{n+1}p_{n-1}}{l\lambda q_n + \varepsilon_{n+1}q_{n-1}}, \quad 1 \leq l < r_{n+1}, \tag{3}$$

and call $u_{n,l}/v_{n,vl}$ the l-th mediant convergent of x (of level n); note that such a mediant does not exist in case $r_{n+1} = 1$. In the next section, we define a map which induces mediant convergents of Rosen continued fractions.

3. Mediant Maps and Convergents

3.1. **Full Mediant Map.** For each fixed k, we define the mediant map $S = S_k$ as follows. We put $\mathbb{J} = \mathbb{J}_k = [-\lambda, \frac{1}{\lambda})$ and define S of $\mathbb{J}$ by

$$S(x) = \begin{cases} -\dfrac{1}{x} - \lambda, & x \in [-\frac{\lambda}{2}, -\frac{2}{3\lambda}) ; \\ -\dfrac{x}{\lambda x + 1}, & x \in [-\frac{2}{3\lambda}, 0) ; \\ 0, & x = 0 ; \\ \dfrac{x}{-\lambda x + 1}, & x \in (0, \frac{2}{3\lambda}] ; \\ \dfrac{1}{x} - \lambda, & x \in (\frac{2}{3\lambda}, \frac{2}{\lambda}) . \end{cases}$$

In the case of $k = 3$, T_3 is the classical nearest integer continued fraction map, and S_3 is the mediant map as defined by R. Natsui in [15]. In the sequel we always assume that $k \geq 4$. To describe the relationship between the mediant map and the Rosen map, we consider matrix actions. The following two lemmas are trivially verified.

Lemma 1. *The Rosen map T_k can be expressed in the following manner.*

$$T_k(x) = \begin{cases} \begin{pmatrix} \lambda & 1 \\ -1 & 0 \end{pmatrix} (x), & x \in [-\frac{\lambda}{2}, -\frac{2}{3\lambda}); \\ \begin{pmatrix} t\lambda & 1 \\ -1 & 0 \end{pmatrix} (x), & x \in [-\frac{2}{(2t-1)\lambda}, -\frac{2}{(2t+1)\lambda}),\ t \in \mathbb{N}_{\geq 2}; \\ \begin{pmatrix} -t\lambda & 1 \\ 1 & 0 \end{pmatrix} (x), & x \in (\frac{2}{(2t+1)\lambda}, \frac{2}{(2t-1)\lambda}],\ t \in \mathbb{N}_{\geq 2}; \\ \begin{pmatrix} -\lambda & 1 \\ 1 & 0 \end{pmatrix} (x), & x \in (\frac{2}{3\lambda}, \frac{\lambda}{2}). \end{cases}$$

Definition 1. We define the following matrices.

$$U_- = \begin{pmatrix} 0 & -1 \\ 1 & \lambda \end{pmatrix} ; \quad U_+ = \begin{pmatrix} 0 & 1 \\ 1 & \lambda \end{pmatrix} ; \quad V_- = \begin{pmatrix} -1 & 0 \\ \lambda & 1 \end{pmatrix} ; \quad V_+ = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} .$$

The inverses of these are of course:

$$U_-^{-1} = \begin{pmatrix} \lambda & 1 \\ -1 & 0 \end{pmatrix} ; \quad U_+^{-1} = \begin{pmatrix} -\lambda & 1 \\ 1 & 0 \end{pmatrix} ; \quad V_-^{-1} = \begin{pmatrix} -1 & 0 \\ \lambda & 1 \end{pmatrix},$$

and

$$V_+^{-1} = \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix} .$$

Lemma 2. *The mediant map S can be expressed in the following terms.*

$$S(x) = \begin{cases} U_-^{-1}(x), & x \in [-\frac{\lambda}{2}, -\frac{2}{3\lambda})\ ; \\ V_-^{-1}(x), & x \in [-\frac{2}{3\lambda}, 0)\ ; \\ V_+^{-1}(x), & x \in (0, \frac{2}{3\lambda}]\ ; \\ U_+^{-1}(x), & x \in (\frac{2}{3\lambda}, \frac{2}{\lambda})\ . \end{cases}$$

The next lemma is verified by direct computation.

Lemma 3. *The following equalities hold:*

$$\begin{pmatrix} t\lambda & 1 \\ -1 & 0 \end{pmatrix} = U_+^{-1} \cdot V_+^{-(t-2)} \cdot V_-^{-1} \ \ and \ \ \begin{pmatrix} -t\lambda & 1 \\ 1 & 0 \end{pmatrix} = U_+^{-1} \cdot V_+^{-(t-1)}\ .$$

Direct calculation also shows the following.

Lemma 4. *Viewed as a linear fractional transformation, the matrix V_-^{-1} is a bijective map from $\left(-\frac{2}{(2l-1)\lambda}, -\frac{2}{(2l+1)\lambda}\right)$ to $\left(\frac{2}{(2l-1)\lambda}, \frac{2}{(2l-3)\lambda}\right)$, with*

$$-\frac{2}{(2l+1)\lambda} \mapsto \frac{2}{(2l-1)\lambda} \qquad and \qquad -\frac{2}{(2l-1)\lambda} \mapsto \frac{2}{(2l-3)\lambda}.$$

Furthermore, the linear fractional transformation V_+^{-1} maps $\left(\frac{2}{(2l+1)\lambda}, \frac{2}{(2l-1)\lambda}\right)$ bijectively to $\left(\frac{2}{(2l-1)\lambda}, \frac{2}{(2l-3)\lambda}\right)$, with

$$\frac{2}{(2l+1)\lambda} \mapsto \frac{2}{(2l-1)\lambda} \qquad and \qquad \frac{2}{(2l-1)\lambda} \mapsto \frac{2}{(2l-3)\lambda}\ .$$

Lemma 5. *For each $x \in \mathbb{I}_k$, let $\ell(x)$ be defined as follows.*

$$\ell(x) := \min\left\{\ell \geq 0 :\ S^{\ell}(x) \in [-\frac{\lambda}{2}, -\frac{2}{3\lambda}) \cup (\frac{2}{3\lambda}, \frac{2}{\lambda})\right\}\ .$$

Then for each $x \in \mathbb{I}_k$, one has the following equality:

$$S^{\ell(x)+1}(x) = T(x).$$

Proof. This follows from our definitions and an application of Lemma 3. □

3.2. **Mediant Convergents.** For $x \in \mathbb{I}_k$ and $i \in \mathbb{N}$, we let

$$M_i = \begin{cases} U_- & \text{if} \quad S^{i-1}(x) \in [-\frac{\lambda}{2}, -\frac{2}{3\lambda}); \\ U_+ & \text{if} \quad S^{i-1}(x) \in (\frac{2}{3\lambda}, \frac{2}{\lambda}); \\ V_- & \text{if} \quad S^{i-1}(x) \in [-\frac{2}{3\lambda}, 0); \\ V_+ & \text{if} \quad S^{i-1}(x) \in (0, \frac{2}{3\lambda}]; \\ \mathrm{Id} & \text{if} \quad S^{i-1}(x) = 0. \end{cases}$$

for $i \geq 1$, where as usual Id denotes the identity.

Then we have a sequence of matrices from x which is denoted by

$$x \sim M_1 M_2 \cdots M_j \cdots \tag{4}$$

Of course, the mediant map acts as a shift on each such sequence.

Definition 2. Fix $x \in \mathbb{I}_k$ and consider the above sequence of M_i. Let $k_1, k_2, \ldots$ be the increasing sequence of indices for which $M_{k_i} \in \{ U_- , U_+ \}$.

The following lemma records the fact that the sequence of Rosen convergents p_n/q_n of $x \in \mathbb{I}_k$ is a subsequence of the sequence $u_{n,l}/v_{n,vl}$, $n \geq 1$, of mediant convergents of x.

Lemma 6. *For each $x \in \mathbb{I}_k$, consider the corresponding sequence of Equation* (4). *Then, for each k_m as above, one has the following equality.*

$$M_1 M_2 \cdots M_{k_m} = \begin{pmatrix} p_{m-1} & p_m \\ q_{m-1} & q_m \end{pmatrix} .$$

Furthermore, $k_{m+1} = k_m + r_{m+1}$ where $(\varepsilon_m : r_m)$ is the m-th coefficient of the Rosen continued fraction expansion of x.

We have the following result.

Proposition 1. *With notation as above, we have*

$$M_1 \cdots M_{k_m} \cdots M_{k_m+l} = \begin{pmatrix} u_{m,l} & p_m \\ v_{m,l} & q_m \end{pmatrix}$$

for $1 \leq l < r_{m+1}$.

By this proposition, we see that the sequence $(M_1 \cdots M_i(\infty) : i \geq 1)$ is

$$\frac{u_{0,1}}{v_{0,1}}, \frac{u_{0,2}}{v_{0,2}}, \ldots, \frac{u_{0,r_1-1}}{v_{0,r_1-1}}, \frac{p_0}{q_0}, \frac{u_{1,1}}{v_{1,1}}, \frac{u_{1,2}}{v_{1,2}}, \ldots, \frac{u_{1,r_2-1}}{v_{1,r_2-1}}, \frac{p_1}{q_1}, \cdots$$

$$\cdots, \frac{u_{n,1}}{v_{n,1}}, \frac{u_{n,2}}{v_{n,2}}, \cdots, \frac{u_{n,r_{n+1}-1}}{v_{1,r_{n+1}-1}}, \frac{p_n}{q_n}, \cdots$$

It is easy to see that

$$x = \begin{pmatrix} u_{m,l} & p_m \\ v_{m,l} & q_m \end{pmatrix} (S^n(x)) ,$$

for $n = k_m + l$. We put $x_n = S^n(x)$. It follows that

$$\left| x - \frac{u_{m,l}}{v_{m,l}} \right| = \left| \frac{u_{m,l}x_n + p_m}{v_{m,l}x_n + q_m} - \frac{u_{m,l}}{v_{m,l}} \right| = \frac{1}{v_{m,l}^2 (x_n - (-\frac{q_m}{v_{m,l}}))}, \tag{5}$$

where $-\frac{q_m}{v_{m,l}} = (M_1 \cdots M_n)^{-1}(\infty)$. We recall that $(M_1 \cdots M_n)^{-1}$ is the linear fractional transformation which defines S^n. It also follows that

$$\left| x - \frac{p_{m-1}}{q_{m-1}} \right| = \frac{1}{q_{m-1}^2 (x_{k_m} - (-\frac{q_m}{q_{m-1}}))} \tag{6}$$

where $-q_m/q_{m-1} = (M_1 \cdots M_{k_m})^{-1}(\infty)$. Consequently, the distribution of

$$\big((M_1 \cdots M_n)^{-1}(x) - (M_1 \cdots M_n)^{-1}(\infty) \, : \, n \geq 1 \big)$$

determines the distribution of the error (after normalization by the square of the denominator) of the principal and the mediant convergents.

If $k = 3$, since $\lambda_3 = 1$, it is easy to see that

$$\begin{pmatrix} u_{m,1} \\ v_{m,1} \end{pmatrix} = \begin{pmatrix} u_{m-1,r_m-1} \\ v_{m-1,r_m-1} \end{pmatrix}$$

when $\varepsilon_{m+1} = -1$. This equality never holds for $k \geq 4$. Indeed, since $\lambda_k > 1$ for $k \geq 4$, we have—due to (3),

$$v_{m,1} = \lambda q_m - q_{m-1} > q_m - \lambda q_{m-1} = v_{m-1,r_m-1}.$$

This implies that all values in

$$\left\{ \frac{u_{m,l}}{v_{m,l}}, \frac{p_m}{q_m} \right\}_{m \geq 0}$$

are different from each other.

4. Natural extensions

Since the work of [12], planar natural extensions of continued fractions maps have provided a significant tool in the study of ergodic properties of number theoretic transformations.

In this section we construct planar natural extensions of the Rosen mediant maps. Here "natural extension" means (i) one-to-one onto (a.e.) and (ii) the projection map to the first coordinate coincides with S. Because of the different behavior of the orbit of $\pm\frac{\lambda}{2}$ by S, we discuss the even index case and the odd index case separately.

In our construction, we rely on [2]. However, we proceed slightly differently. There the invariant measure for the natural extension of the Rosen maps are $\frac{\mathrm{d}x\,\mathrm{d}y}{(1+xy)^2}$, up to normalizing constants. Indeed, this is the invariant measure for the action of the determinant ± 1

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ sending } (x, y) \text{ to } \left(\frac{ax+b}{cx+d}, \frac{dy-c}{-by+a} \right).$$

Here, we use the more natural action directly related to hyperbolic geometry:

$$(x, y) \mapsto \left(\frac{ax+b}{cx+d}, \frac{ay+b}{cy+d} \right)$$

with invariant measure $\frac{\mathrm{d}x\,\mathrm{d}y}{(x-y)^2}$. These maps are conjugate, thus there is no loss in proceeding in our manner. As in the regular mediant case (cf. [6]), the mediant

Rosen maps have σ-finite, infinite absolutely continuous invariant measures. Thus, we cannot directly apply the classical Birkhoff's individual ergodic theorem.

4.1. **Even index case: $k = 2\ell$.** Because we apply (5) and (6) in the subsequent section, our construction of the region of the natural extension differs from [2]. However, it is easy to recover one region from the other by use of the conjugating map $(x, y) \mapsto (x, -1/y)$. First we recall some notations from [2].

$$\phi_0 = -\frac{\lambda}{2}, \qquad \phi_j = T^j\left(-\frac{\lambda}{2}\right),\ 0 \le j \le \ell - 1, \qquad \text{and} \quad \phi_{\ell-1} = 0,$$

and

$$L_1 = \frac{1}{\lambda + 1}, \qquad L_j = \frac{1}{\lambda - L_{j-1}},\ 2 \le j \le \ell - 1, \qquad \text{and} \quad 1 = \frac{1}{\lambda - L_{\ell-1}}.$$

With the various ϕ_i and L_i the natural extension Ω of the ergodic system underlying the Rosen fraction was defined in [2]. As stated before, we consider here an isomorphic copy Ω_0 of this system, which will be a subset of the region of natural extension Ω^* of the ergodic system underlying the mediant map S. Setting for $1 \le j \le \ell - 1$,

$$\left\{\begin{array}{lcl} J_j & = & [\phi_{j-1},\, \phi_j) \\ J_\ell & = & [0,\, \frac{\lambda}{2}) \\ J_{\ell+1} & = & [\frac{\lambda}{2},\, \frac{2}{\lambda}) \end{array}\right. \qquad \text{and} \qquad \left\{\begin{array}{lcl} \bar{K}_j & = & [-\infty,\, -\frac{1}{L_j}\,] \\ \bar{K}_\ell & = & [-\infty,\, 0\,] \\ \bar{K}_{\ell+1} & = & [-1,\, 0], \end{array}\right.$$

we define the region of the natural extension by

$$\Omega^* = \bigcup_{j=1}^{l+1} \left(J_j \times \bar{K}_j\right),$$

see Figure 1. The map $\hat{S} : \Omega^* \to \Omega^*$ is given by

$$\hat{S}(x, y) = \left(M_1^{-1}(x),\, M_1^{-1}(y)\right),$$

for $(x, y) \in \Omega^*$. We will call this map the natural extension of S; indeed we show below that it has the requisite properties. From the definition of $M_1 = M_1(x)$, it follows that $M_1^{-1}(x) = S(x)$.

Theorem 1. *The map $\hat{S}$ is surjective from Ω^* onto itself and is injective off of the boundaries of $J_j \times \bar{K}_j$, $1 \le j \le l + 1$. Moreover, $\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^2}$ is an invariant measure for $\hat{S}$.*

Remark. Note that

$$\iint_{\Omega^*} \frac{\mathrm{d}x\,\mathrm{d}y}{|x - y|^2} = \infty.$$

Proof. We must simply check that, up to measure zero niceties, the map $\hat{S}(x, y)$ does indeed act bijectively on Ω^*. This is a matter of elementary calculations, which we now outline.

- $x \in [-\frac{\lambda}{2}, -\frac{2}{3\lambda})$; On this interval, $S(x) = -\frac{1}{x} - \lambda$. Recall that the ϕ_i are in fact the orbit of $\phi_0 = -\frac{\lambda}{2}$ under iteration of this map; in particular, $\phi_{\ell-1} = 0$. Then $\hat{S}$ sends $\cup_{i=1}^{\ell-1} [\phi_{i-1},\, \phi_i) \times [-\infty,\, -\frac{1}{L_i}]$ to $\cup_{i=1}^{\ell-1} [\phi_i,\, \phi_{i+1}) \times [-\lambda_q,\, -1/L_{i+1}\,]$. Also $[\phi_{\ell-1},\, -\frac{2}{3\lambda}) \times [-\infty, -\frac{1}{L_{\ell-1}}]$ is now sent to $[0,\, \frac{\lambda}{2}) \times [-\lambda, -1]$.

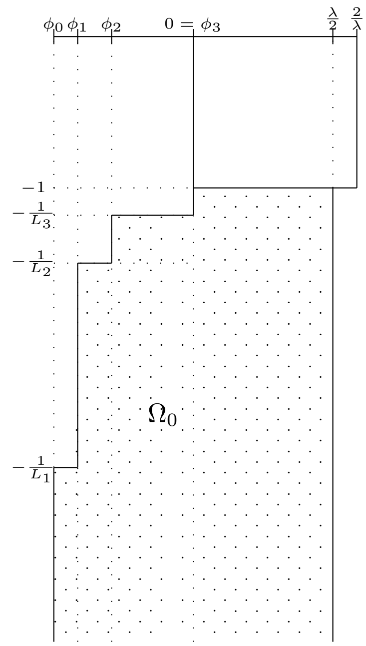

FIGURE 1. The region Ω^* contains Ω_0, here $k = 8$.

- $x \in [-\frac{2}{3\lambda},\, 0)$; On this interval, $\hat{S}(x, y) = \left(\frac{-x}{\lambda x+1},\, \frac{-y}{\lambda y+1}\right)$ for $(x, y) \in \Omega^*$. Since $L_{\ell-1} = \lambda - 1$, we easily find that $\hat{S}$ sends $[-\frac{2}{3\lambda},\, 0) \times [-\infty,\, -\frac{1}{L_{l-1}}]$ to $[0,\, \frac{2}{\lambda}) \times [-1,\, -\frac{1}{\lambda}]$.
- $x \in [0,\, \frac{2}{3\lambda})$; On this interval, $\hat{S}(x, y) = \left(\frac{x}{1-\lambda x},\, \frac{y}{1-\lambda y}\right)$ for $(x, y) \in \Omega^*$. One immediately finds that $[0,\, \frac{2}{3\lambda}) \times [-\infty, 0]$ is sent to $[0,\, \frac{2}{\lambda}) \times [-\frac{1}{\lambda},\, 0]$.
- $x \in [\frac{2}{3\lambda},\, \frac{\lambda}{2})$; On this interval, $\hat{S}(x, y) = (\frac{1}{x} - \lambda,\, \frac{1}{y} - \lambda)$ for $(x, y) \in \Omega^*$. One finds that $[\frac{2}{3\lambda},\, \frac{\lambda}{2}) \times [-\infty,\, 0)$ is sent to $[\phi_1,\, \frac{\lambda}{2}) \times [-\infty,\, -\lambda]$.
- $x \in [\frac{\lambda}{2},\, \frac{2}{\lambda})$; On this interval, also $\hat{S}(x, y) = (\frac{1}{x} - \lambda,\, \frac{1}{y} - \lambda)$ for $(x, y) \in \Omega^*$. Hence $[\frac{2}{3\lambda},\, \frac{\lambda}{2}) \times [-1,\, 0]$ is sent to $[-\frac{\lambda}{2},\, \phi_1) \times [-\infty,\, -\lambda - 1]$.

Consequently, we see that $\hat{S}$ is bijective except for failing to be injective on the (measure zero) boundaries. The invariance of the measure follows from

$$h(\hat{S}(x, y)) \cdot \left|\det D\hat{S}(x, y)\right| \cdot h^{-1}(x, y) \;=\; 1 \quad \text{for} \quad h(x, y) = \frac{1}{(x - y)^2}$$

Indeed, this invariance holds for any linear fractional transformation $(x, y) \mapsto (Ax,\, Ay)$, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{R})$:

$$\frac{1}{\left(\frac{ax+b}{cx+d} - \frac{ay+b}{cy+d}\right)^2} \cdot \frac{1}{(cx + d)^2(cy + d)^2} \cdot (x \,-\, y)^2 \;=\; 1.$$

□

4.2. **Odd index case:** $k = 2\ell + 3$**.** We recycle notation, now using ϕ_j and L_j as follows (all necessary calculations are in [2]):

$$\phi_0 = -\frac{\lambda}{2}, \qquad \text{and} \quad \phi_j = T^j\left(-\frac{\lambda}{2}\right),\ 0 \le j \le 2\ell+1.$$

We recall that

$$\begin{cases} -\frac{\lambda}{2} \le \phi_j < -\frac{2}{3\lambda} \quad \text{for} \quad j \in \{0,1,\dots,\ell-1\} \cup \{\ell+1,\dots,2\ell\} \\ -\frac{2}{3\lambda} < \phi_\ell < -\frac{2}{5\lambda} \\ \phi_{2\ell+1} = 0. \end{cases}$$

Also we put, with R the positive root of $R^2 + (2-\lambda)R - 1 = 0$,

$$\begin{cases} L_{2\ell} &= \lambda - 1/R \\ L_{2\ell+1} &= \lambda - R \\ L_1 &= \frac{1}{2\lambda - L_{2\ell}} \\ L_2 &= \frac{1}{2\lambda - L_{2\ell+1}} \\ L_j &= \frac{1}{\lambda - L_{j-2}}, \quad 2 < j \le 2\ell+2\,, \end{cases}$$

which are well-defined (see Subsection 3.2 of [2]). Then we define

$$\Omega^* = \bigcup_{j=1}^{2\ell+4} J_j \times \bar{K}_j \ :$$

where

$$\begin{cases} J_{2j} &= [\phi_{\ell+j},\, \phi_j), \quad 1 \le j \le \ell \\ J_{2j-1} &= [\phi_{j-1},\, \phi_{j+l}), \quad 1 \le j \le \ell+1 \\ J_{2\ell+2} &= [0,\, \frac{\lambda}{2}), \\ J_{2\ell+3} &= [\frac{\lambda}{2}, 1) \\ J_{2\ell+4} &= [1,\, \frac{2}{\lambda}), \end{cases}$$

$1 = -\frac{\phi_\ell}{\lambda\phi_\ell + 1}$, and

$$\begin{cases} \bar{K}_j &= [-\infty,\, -\frac{1}{L_j}], \quad 1 \le j \le 2\ell+1 \\ \bar{K}_{2\ell+2} &= [-\infty,\, 0,], \\ \bar{K}_{2\ell+3} &= [-\frac{1}{\lambda - L_{2\ell+1}},\, 0] \,=\, [-\frac{1}{R},\, 0] \\ \bar{K}_{2\ell+4} &= [-\frac{1}{\lambda - L_{2\ell}},\, 0] \,=\, [-R,\, 0]; \end{cases}$$

see Figure 2.

Theorem 2. *The map* $\hat{S}(x,y) = \left(M_1^{-1}(x),\, M_1^{-1}(y)\right)$ *of* Ω^* *is bijective off of the boundaries of* $J_j \times \bar{K}_j$, $1 \le j \le 2l+2$*. Moreover,* $\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^2}$ *is an invariant measure for* $\hat{S}$*.*

Proof. The invariance of the measure has been given in the proof of Theorem 1. The first part of the assertion also follows similarly:

- $x \in [-\frac{\lambda}{2},\, -\frac{2}{3\lambda})$; Here, $\hat{S}(x,y) = \left(-\frac{1}{x} - \lambda,\, -\frac{1}{y} - \lambda\right)$. Thus, the corresponding image of Ω^* is fibred below $[\phi_1,\, \frac{\lambda}{2})$, with y values in $[-\lambda,\, -\frac{1}{L_j}]$ for $S(x)$ negative, with the appropriate value of of L_j, and $y \in [-\lambda,\, -\frac{1}{R}]$ for $S(x)$ non-negative.

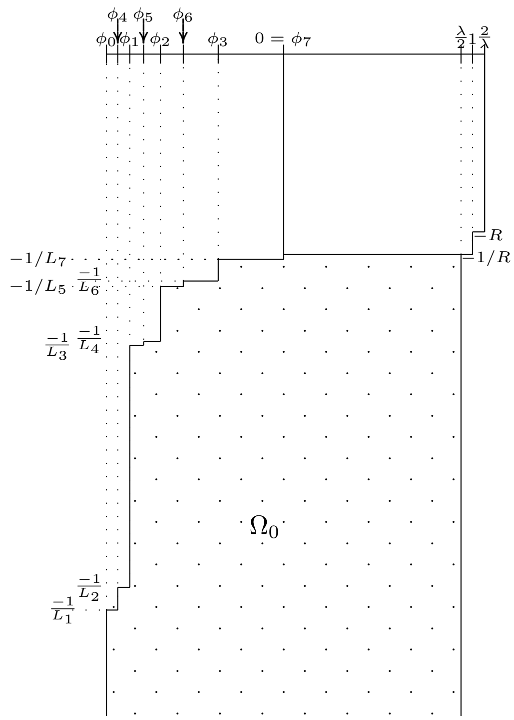

FIGURE 2. The region Ω^* contains Ω_0, here $k = 9$.

- $x \in [-\frac{2}{3\lambda},\, 0)$; Here $\hat{S}(x,y) = \left(\frac{-x}{\lambda x+1},\, \frac{-y}{\lambda y+1}\right)$ for $(x,y) \in \Omega^*$. Thus $[-\frac{2}{3\lambda},\, 0) \times [-\infty,\, -\frac{1}{L_{2\ell+1}}]$ is sent to $[0,\, \frac{2}{\lambda}) \times [-\frac{1}{\lambda - L_{2\ell+1}},\, -\frac{1}{\lambda}]$.
- $x \in [0,\, \frac{2}{3\lambda})$; For these values of x, $\hat{S}(x,y) = \left(\frac{x}{1-\lambda x},\, \frac{y}{1-\lambda y}\right)$ for $(x,y) \in \Omega^*$. Thus $[0,\, \frac{2}{3\lambda}) \times [-\infty, 0]$ is sent to $[0,\, \frac{2}{\lambda}) \times [-\frac{1}{\lambda},\, 0]$.
- $x \in [\frac{2}{3\lambda},\, \frac{\lambda}{2})$; Here $\hat{S}(x,y) = (\frac{1}{x} - \lambda,\, \frac{1}{y} - \lambda)$ for $(x,y) \in \Omega^*$. Thus $[\frac{2}{3\lambda},\, \frac{\lambda}{2}) \times [-\infty,\, 0)$ is sent to $[\phi_1,\, \frac{\lambda}{2}) \times [-\infty,\, -\lambda]$.
- $x \in [\frac{\lambda}{2},\, \alpha)$; Here again $\hat{S}(x,y) = (\frac{1}{x} - \lambda,\, \frac{1}{y} - \lambda)$ for $(x,y) \in \Omega^*$. Then $[\frac{2}{3\lambda},\, \alpha) \times [-\frac{1}{R},\, 0]$ is sent to $[\phi_{\ell+1},\, \phi_1) \times [-\infty,\, -R-\lambda] = [\phi_{\ell+1},\, \phi_1) \times [-\infty,\, -\frac{1}{L_2}]$.
- $x \in [\alpha,\, \frac{2}{\lambda})$; Once again, $\hat{S}(x,y) = (\frac{1}{x} - \lambda,\, \frac{1}{y} - \lambda)$ for $(x,y) \in \Omega^*$ and $[\alpha,\, \frac{2}{\lambda}) \times [-\frac{1}{\lambda - L_{2\ell}},\, 0]$ is sent to $[-\frac{\lambda}{2},\, \phi_{\ell+1}) \times [-\infty,\, -\lambda - \frac{1}{L_1}]$.

Combining the above, we get the first part of the assertion of the theorem. □

4.3. **Ergodicity.** We denote by $\hat{\mu}$ the measure defined by $\frac{1}{(x-y)^2}$ as its density function and by μ its marginal distribution on the first coordinate.

Theorem 3. *The dynamical system $(\Omega^*,\, \hat{S},\, \hat{\mu})$ is ergodic, and its entropy $h(\hat{S},\, \hat{\mu})$ is equal to $\frac{(k-2)\pi^2}{2k}$.*

Proof. An easy calculation shows that

$$\hat{S}^{k_m}(x,y) \;=\; \big((M_1 \cdots M_{k_m})^{-1}(x),\, (M_1 \cdots M_{k_m})^{-1}(y)\big)$$

and

$$(M_1 \cdots M_{k_m})^{-1}(x) = T^m(x).$$

Now we define

$$\Omega_0 = \{(x, y) : (x, -1/y) \in \Omega\} \tag{7}$$

where Ω is the set defined in Theorem 3.1 and Theorem 3.2 of [2], i.e., Ω_0 is an isomorphic copy of the region of the natural extension of the Rosen map T. For reasons which will become clear in a moment, we use Ω_0 in this paper. Then $\hat{S}^{k_m}$ is the induced transformation $\hat{S}_{\Omega_0}$ of $\hat{S}$ to Ω_0, which is conjugate to $\mathcal{T}$ by the isomorphism $(x, y) \to (x, -1/y)$. Since $\mathcal{T}$ is ergodic (see also [2]), so is $\hat{S}_{\Omega_0}$. This also implies the ergodicity of $\hat{S}$.

The entropy $h(\hat{S}, \hat{\mu})$ of $(\hat{S}, \hat{\mu})$ is given by the entropy of its induced transformation on the region Ω_0, as

$$h(\hat{S}, \hat{\mu}) = h(\hat{S}_{\Omega_0}, \hat{\mu}_{\Omega_0}) \cdot \hat{\mu}(\Omega_0),$$

where $\hat{\mu}_{\Omega_0}$ is the restricted normalized measure of $\hat{\mu}$ to Ω_0; see [8]. Since $(\hat{S}_{\Omega_0}, \hat{\mu}_{\Omega_0})$ is a natural extension of the Rosen map, and its entropy is

$$C \cdot \frac{(k-2)\pi^2}{2k},$$

where C is the normalizing constant of the invariant measure, i.e.,

$$C^{-1} = \iint_{\Omega_0} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^2},$$

(see [14]), the result follows. □

The following result is an immediate consequence of Theorem 3.

Corollary 1. *The dynamical system (S, μ) is ergodic, and its entropy $h(S, \mu)$ is equal to $\frac{(k-2)\pi^2}{2k}$.*

Remark. The density function f of the measure μ is given by

$$f(x) \; = \; \int_{\{y\,:\,(x,y)\in\Omega^*\}} \frac{\mathrm{d}y}{(x-y)^2}$$

which diverges at $x = 0$.

Corollary 2. *For a.e. $x \in \mathbb{I}_k$, $\left\{ v_{m,l} \left| x - \frac{u_{m,l}}{v_{m,l}} \right| \,:\, 1 \le l \le r_m - 1, m \ge 1 \right\}$ is unbounded.*

Proof. From the ergodicity of $\hat{S}$, for a.e. $(x, y) \in \Omega^*$, its forward $\hat{S}$-orbit is dense in Ω^*. It follows that distance between the second coordinates of $\hat{S}^n(x, y)$ and $\hat{S}^n(x, -\infty)$ tends to 0. Thus we see that the forward $\hat{S}$-orbit of $(x, -\infty)$ is also dense in Ω^*. By Fubini's theorem, this holds for a.e. $x \in \mathbb{I}_k$. Because

$$\hat{S}^n(x, -\infty) \; = \; \left(S^n(x), -\frac{q_m}{v_{m,s}} \right)$$

for $n \;=\; k_m + s$ and by (4), we have the assertion. □

5. A Hurwitz-type result

In this section we use the natural extensions to find the analog for the Rosen fraction of a classical result of Hurwitz on the quality of Diophantine approximations by continued fraction approximates.

For the regular continued fraction expansion we have the following classical *Borel result* (cf. [3]): if x has SCF-expansion $x = [0; a_1, a_2, \dots]$, and convergents P_n/Q_n, $n \geq 0$, and if the approximation coefficients $\vartheta_n = \vartheta_n(x)$, $n \geq 0$, are defined by

$$\vartheta_n = \vartheta_n(x) = Q_n^2 \left| x - \frac{P_n}{Q_n} \right|, \qquad n \geq 0,$$

then for every $n \geq 1$ and every irrational x we have

$$\min(\vartheta_{n-1}, \vartheta_n, \vartheta_{n+1}) < 1/\sqrt{5},$$

and the constant $1/\sqrt{5}$ is 'best possible,' in the sense that it cannot be replaced by a smaller constant. Borel's result, together with the yet older *Legendre result*, which states that if $p, q \in \mathbb{Z}$, $q > 0$, $\gcd(p, q) = 1$, then

$$\left| x - \frac{p}{q} \right| < \frac{1}{2q^2} \quad \Rightarrow \quad \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p_n \\ q_n \end{pmatrix}, \text{ for some } n \geq 0,$$

implies the also classical *Hurwitz result*: for every irrational x there are infinitely many rationals p/q, such that

$$\left| x - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2}.$$

In our setting, we put

$$\Theta\left(x, \frac{a}{c}\right) = c^2 \left| x - \frac{a}{c} \right|$$

for $\begin{pmatrix} a & \cdot \\ c & \cdot \end{pmatrix} \in G_k$. Hereafter, when we write a/c we assume that there exists a matrix $\begin{pmatrix} a & \cdot \\ c & \cdot \end{pmatrix} \in G_k$. The set of these real numbers a/c is called the G_k-rationals, denoted $G_k(\infty)$. In case a/c is equal to the nth mediant convergent of x — with indexing as for Equation (5) —, we write $\Theta_n(x)$ instead of $\Theta(x, a/c)$. These numbers $\Theta_n(x)$, $n \geq 1$, are the *approximation coefficients* of x, and express the 'quality of approximation' of x by these convergents.

In this section, we use continued fraction methods to prove the analogue of Hurwitz' result for λ_k-irrationals. This was first proved obtained by Haas and Series using hyperbolic geometry see [5], after continued fractions based work of Lehner [10, 11].

Theorem 4. (Haas and Series) *For $x \in \mathbb{R} \setminus G_k(\infty)$, let*

$$\mu_k(x) = \inf\{h\,;\ \left| x - \frac{a}{c} \right| < \frac{h}{c^2} \textit{ has infinitely many solutions } \frac{a}{c} \in G_k(\infty)\},$$

and set $C(k) = \sup\{\mu_k(x)\,;\ x \in \mathbb{R} \setminus G_k(\infty)\}$. Then

$$C(k) = \begin{cases} 1/2 & \textit{if } k \textit{ is even} \\ \dfrac{1}{2\sqrt{(1 - \lambda_k/2)^2 + 1}} & \textit{if } k \textit{ is odd.} \end{cases}$$

One could imagine that a continued fractions proof proceeds by way of analogs of Borel and Legendre results. And, [9] gives a Borel-type result: for $x \in \mathbb{R} \setminus G_k(\infty)$ and $(p_n/q_n)_{n\geq 1}$ the λ_k-Rosen convergents of x, one has that

$$\theta_n(x) := q_n^2 \left| x - \frac{p_n}{q_n} \right| < C(k) \quad \text{infinitely often,}$$

and the constant $C(k)$ is optimal. Furthermore, due to results of Nakada ([13]), a Legendre-result exists for the Rosen fractions. Unfortunately, this analog of the Legendre constant is strictly *less* than $C(k)$, so the Borel-type result from [9] does not immediately imply the Hurwitz analog. That is, one cannot rule out the existence of some constant D, with $D < C(k)$, such that for some G_k-irrational x there exist infinitely many G_k-rationals a/c, which are *not* Rosen convergents of x, but do satisfy

$$\left| x - \frac{a}{c} \right| < \frac{D}{c^2}.$$

To address this difficulty, one turns to our Section 7: the Legendre constant ℓ_k for the Rosen mediant transformation *is* larger that $C(k)$. Therefore, any G_k-rational a/c satisfying

$$\left| x - \frac{a}{c} \right| < \frac{C(k)}{c^2},$$

which is not a Rosen convergent *must* be a mediant convergent of x. Finally, in order to show that the Borel-type result from [9] implies Theorem 4, it suffices to exhibit a G_k-irrational number x (a *witness*), for which the statement

$$\Theta_n(x) < C, \quad \text{for at most finitely many } n$$

holds for any $C < C(k)$. For each $k > 3$, we call our witness τ_0; this value is suggested by the geometry of our planar natural extension. The approach in both the even and odd index cases is quite similar, the odd case is –as usual– a little bit more complicated. We give full detail of the even case, and outline the odd case.

5.1. **Even index case:** $k = 2\ell$**.** One of the main ingredients of the aforementioned Borel-type result from [9] is the fact that if we set

$$\tau_0 := [\,\overline{(-1:1)^{\ell-2},\,(-1:2)}\,] = 1 - \lambda = -L_{\ell-1}, \quad \eta_0 := L_1 = \frac{1}{\lambda+1}$$

(here the bar indicates periodicity), the sequence

$$(\tau_i, \eta_i) := \mathcal{T}^i(\tau_0, \eta_0), \quad \text{for } i \geq 0,$$

is purely periodic, with period-length $\ell - 1$. Here $\mathcal{T} : \Omega \to \Omega$ is the natural extension map from [2], given by

$$\mathcal{T}(x,y) = \left(T(x), \frac{1}{\lambda \lfloor |\frac{1}{\lambda x}| + \frac{1}{2} \rfloor + \operatorname{sign}(x) y} \right).$$

Note that $\tau_{\ell-2} = \frac{-1}{\lambda+1} = -L_1$, $\eta_{\ell-2} = \lambda - 1 = L_{\ell-1}$, and that $\mathcal{T}(\tau_{\ell-2}, \eta_{\ell-2}) = (\tau_0, \eta_0)$. Furthermore, in [2] it was shown that if

$$(t_n, v_n) = \mathcal{T}^n(x, 0), \quad \text{for } x \in [-\lambda/2, \lambda/2), \text{ and } n \geq 0,$$

one has that

$$\theta_{n-1}(x) = \frac{v_n}{1 + t_n v_n}, \qquad \text{and} \quad \theta_n(x) = \frac{|t_n|}{1 + t_n v_n}, \qquad n \geq 1.$$

Due to this

$$\theta(\tau_i,\eta_i) := \frac{\eta_i}{1+\tau_i\eta_i} \geq C(k) = \frac{1}{2}$$

for $i \geq 0$ (with equality if $i \equiv 0 \pmod{\ell-1}$ or $i \equiv \ell-2 \pmod{\ell-1}$), and since $\theta_{i-1+n(\ell-1)}(\tau_0) \uparrow \theta(\tau_i,\eta_i)$, as $n \to \infty$, it follows that for any $C < C(k) = 1/2$,

$$\theta_n(\tau_0) < C, \quad \text{for at most finitely many } n \geq 0.$$

Recall that Ω_0 is an isomorphic copy of Ω; for $i = 0, 1, \dots, \ell-2$, the points $(\tau_i,\eta_i) \in \Omega$ correspond to the points $(\tau_i, K_{i+1}) \in \Omega_0$. Since the isomorphic copy of the system $(\Omega_0, \mathcal{T})$ is induced from $(\Omega^*, \hat{S})$, the $\hat{S}$-orbit of (τ_0, K_1) has cardinality at least $\ell-1$. In fact, this $\hat{S}$-orbit is also purely periodic, but is of cardinality ℓ: since $\tau_i < -2/3\lambda$, for $i = 0, 1, \dots, \ell-3$, and $\tau_{\ell-2} > -2/3\lambda$, we have that

$$\hat{S}(\tau_{\ell-2}, K_{\ell-1}) = \hat{S}\left(\frac{-1}{\lambda+1}, \frac{1}{\lambda-1}\right) = (1,-1),$$

and

$$\hat{S}(1,-1) = \left(\frac{1}{1} - \lambda, \frac{1}{-1} - \lambda\right) = (1-\lambda, -1-\lambda) = (\tau_0, K_1)\ .$$

Setting

$$(T_n, V_n) = \hat{S}^n(\tau_0, -\infty), \qquad n \geq 0,$$

it follows from (6), and the fact that (an isomorphic copy of) $\mathcal{T}$ is an induced transformation of $\hat{S}$, that $(\theta_n(\tau_0))_{n\geq 0}$ is a subsequence of $(\Theta_n(\tau_0))_{n\geq 0}$. In fact the only points in the latter sequence which are not in the former are among the numbers $\Theta(1,y)$, with $-1 < y < 0$. For these numbers we have that $\Theta(1,y) = \frac{1}{1-y} > \Theta(1,-1) = \frac{1}{2}$. Consequently, we find for any $C < C(k) = 1/2$,

$$\Theta_n(\tau_0) < C, \quad \text{for at most finitely many } n \geq 0.$$

5.2. **Odd index case:** $k = 2\ell + 3$**.** Analogous to the even index case, we set τ_0 equal to the "left-top height" of Ω, and η_0 equal to the lowest "height" of Ω, i.e.,

$$\tau_0 := -L_{2\ell+1} = R - \lambda, \quad \eta_0 := L_1 = \frac{1}{\lambda + \frac{1}{R}},$$

and we set

$$(\tau_i, \eta_i) := \mathcal{T}^i(\tau_0, \eta_0), \quad \text{for } i \geq 0.$$

Again the sequence $(\tau_i,\eta_i)_{i\geq 0}$ is purely periodic, with period-length 2ℓ; see [9]. Contrary to the even case, this sequence is more "complicated," with a kind of "double loop." On Ω_0 the sequence corresponding with $(\tau_i,\eta_i)_{i\geq 0}$ is the (purely periodic) sequence $(\tau_i, K_{i+1})_{i\geq 0}$; see Figure 3.

As in the even case, $\bar{S}$ "picks up" a few extra points in the orbit of (τ_0, K_1), since both $\tau_\ell = -L_1$ and $\tau_{2\ell} = -L_2$ are larger than $-\frac{2}{3\lambda}$. Similar to the even case, we find for any $C < C(k) = 1/(2\sqrt{(1-\lambda_k/2)^2+1})$,

$$\Theta_n(\tau_0) < C, \quad \text{for at most finitely many } n \geq 0.$$

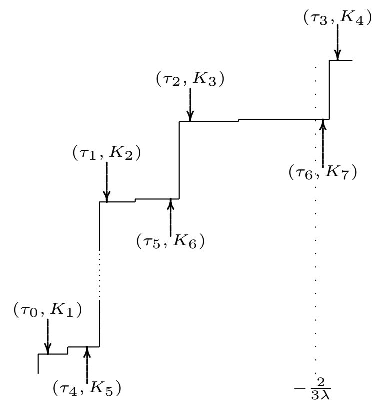

FIGURE 3. The sequence $(\tau_i, K_{i+1})_{i\geq 0}$ in Ω_0 for $k = 9$.

6. Equality of Legendre and Lenstra constants

We define each of the Legendre and Lenstra constants for the Rosen mediant maps. The first of these is the exact analog of the value 1/2 in the aforementioned classical Legendre result. The second is the analog of the value 1/2 in a conjecture of H. W. Lenstra, Jr., for the setting of the simple continued fraction map, confirmed by Bosma et al [1], on the value of the endpoint of the linearity in the average value of small approximation coefficients (for almost every x). Haas [4] has recently shown that Lenstra constants are related to universal behavior of geodesic excursions into cusps of hyperbolic surfaces. Nakada [14] has proved that whenever a continued fraction map has a Legendre constant, then it also has a Lenstra constant, which is at least as large as the Legendre constant. For the Rosen fraction maps and the Rosen mediant maps, we show equality of these constants.

6.1. **Definitions.** Fix an index k, and suppose that there exists $\ell_k > 0$ such that (i) for any G_k-irrational x and any finite G_k-rational a/c,

$$\left| x - \frac{a}{c} \right| < \frac{\ell_k}{c^2}$$

implies a/c is either a Rosen convergent p_n/q_n for some $n \geq 0$, or a mediant Rosen convergent $u_{n,l}/v_{n,l}$ of x; and, (ii) for any $C > \ell_k$, there exist x and a/c such that

$$\left| x - \frac{a}{c} \right| < \frac{C}{c^2}$$

and a/c is neither a Rosen convergent nor a mediant Rosen convergent. Then we call $\ell_k > 0$ the *Legendre constant* for mediant Rosen convergents (of index k). The Legendre constant certainly exists for any index $k \geq 4$ because the Legendre constant for the Rosen continued fractions exists (see [17]), and the mediant Legendre constant is certainly larger than or equal to it.

We again fix an index k, and now suppose that there exists $\mathcal{L}_k > 0$ such that both: for any $0 < t_1, t_2 < \mathcal{L}_k$,

$$\lim_{N\to\infty} \frac{\sharp\{1 \leq n \leq N \,:\, \Theta(M_1 M_2 \cdots M_n(\infty), x) \,<\, t_1\}}{\sharp\{1 \leq n \leq N \,:\, \Theta(M_1 M_2 \cdots M_n(\infty), x) \,<\, t_2\}} = \frac{t_1}{t_2} \tag{8}$$

holds for a.e. $x \in \mathbb{J}$; and, for any $0 < t_2 < \mathcal{L}_k < t_1$,

$$\lim_{N\to\infty} \frac{\sharp\{1 \le n \le N : \Theta(M_1 M_2 \cdots M_n(\infty), x) < t_1\}}{\sharp\{1 \le n \le N : \Theta(M_1 M_2 \cdots M_n(\infty), x) < t_2\}} < \frac{t_1}{t_2} \tag{9}$$

for a.a. $x \in \mathbb{J}$. We call $\mathcal{L}_k$ the *Lenstra constant* for the mediant Rosen convergents (of index k).

6.2. **Legendre $\le$ Lenstra.** The following is a direct consequence of the corollary of Section 2 in [14].

Theorem 5. *Fix any $t_0 > 0$, then for any $t > 0$ we have*

$$\lim_{Q\to\infty} \frac{\#\{\frac{a}{c} \in G_k(\infty) : \Theta(\frac{a}{c}, x) < t, \quad 0 < c \le Q\}}{\#\{\frac{a}{c} \in G_k(\infty) : \Theta(\frac{a}{c}, x) < t_0, \quad 0 < c \le Q\}} = \frac{t}{t_0} \tag{10}$$

for a.e. x. Here we note that $G_k(\infty)$ is the set of parabolic points of the Hecke group G_k.

From this, we have the following result.

Proposition 2. *The Legendre constant is less than or equal to the Lenstra constant, i.e., $\ell_k \le \mathcal{L}_k$.*

Proof. Let

$$C(n, x, t) = \#\{j : 1 \le j \le n,\ \Theta(M_1 \cdots M_j(\infty), x) < t\}. \tag{11}$$

If t is smaller than ℓ_k, then we have

$$\lim_{N\to\infty} \frac{\#\{\frac{a}{c} \in G_k(\infty) : \Theta(\frac{a}{c}, x) < t, \quad 0 < c \le q_N\}}{\#\{\frac{a}{c} \in G_k(\infty) : \Theta(\frac{a}{c}, x) < t_0, \quad 0 < c \le q_N\}} = \lim_{n\to\infty} \frac{C(n, x, t)}{C(n, x, t_0)},$$

for almost every $x \in I$. But this implies that for each such t and for each of these x, the limit as N tends to infinity of the average of the counting function, $C(n, x, t)/N$, is a linear function in t. That is, $\mathcal{L}_k \ge \ell_k$. □

6.3. **Harder Inequality.** The idea of the following proof is to begin with an x_0 which is fairly well approximated by some G_k-rational not arising as a mediant map convergent. We then identify, in terms of the Rosen fraction expansion of this x_0, a whole cylinder set of x all of which are fairly well approximated by such G_k-rationals. There is then a deficit in the numerator of the fraction of the fundamental Equation (10). This then gives the desired inequality. We fix $t_0 > 0$ sufficiently small so that it is smaller than the Legendre constant associated to Rosen continued fractions.

Proposition 3. *Let $C(n, x, t)$ be defined as in* (11)*. For $t > \ell_k$ and a.e. $x \in \mathbb{I}_k$, one has*

$$\lim_{n\to\infty} \frac{C(n, x, t)}{C(n, x, t_0)} < \frac{t}{t_0}.$$

In particular, one has $\mathcal{L}_k \le \ell_k$.

Proof. Fix t such that $t > \ell_q$. Then there exist $x_0 \in I$ and $\frac{p}{q} \in G_k(\infty)$ such that

$$\Theta\left(x_0, \frac{p}{q}\right) < t\,, \quad \text{with } \frac{p}{q} \neq \frac{a_m}{b_m} \quad \text{for any } m \geq 1\,.$$

Consider the Rosen fraction expansions:

$$x_0 = [\,\varepsilon_1 : c_1,\, \varepsilon_2 : c_2,\, \cdots,\, \varepsilon_n : c_n,\, \cdots\,]\,,$$

and

$$\frac{p}{q} = [\,\varepsilon'_1 : d_1,\, \varepsilon'_2 : d_2,\, \cdots,\, \varepsilon'_l : d_l]\,.$$

Since $\frac{p}{q}$ is not a mediant convergent, at least one of the following does not hold:

$$\varepsilon_j = \varepsilon'_j \text{ for } 1 \leq j \leq l\,;\; c_j = d_j \text{ for } 1 \leq j \leq l-1\,;\; \text{and, } 1 \leq d_l \leq c_l - 1\,.$$

We can choose a large integer L such that

$$\left| y - \frac{p}{q} \right| < \frac{t}{q^2} \quad (\,\text{equivalently, such that} \Theta\left(y, \frac{p}{q}\right) < t\,)$$

holds whenever

$$y = [\,\varepsilon_1 : c_1,\, \varepsilon_2 : c_2,\, \cdots,\, \varepsilon_L : c_L,\, \varepsilon''_{L+1} : c''_{L+1},\, \varepsilon''_{L+2} : c''_{L+2},\, \cdots]\,.$$

We consider all such y. Now suppose that

$$z = \frac{\eta_1|}{|a_1} + \frac{\eta_2|}{|a_2} + \cdots + \frac{\eta_N|}{|a_N + y}$$

and

$$\frac{P}{Q} = \frac{\eta_1|}{|a_1} + \frac{\eta_2|}{|a_2} + \cdots + \frac{\eta_N|}{|a_N} + \frac{p|}{|q}\,.$$

For any such pair $(z, \frac{P}{Q})$ we have

$$\left| z - \frac{P}{Q} \right| < \frac{t}{Q^2}$$

if a_N is large enough, where the choice of a_N depends only on x_0, p/q, and L. One checks that P/Q is not a mediant convergent of z (and thus in particular not a Rosen convergent). Fix some such a_N and denote it by a and η_N by η.

Let $\mathcal{C}$ be the cylinder set of all x such that the initial segment of the Rosen expansion of x matches these:

$$\mathcal{C} := \{\, x \in I \;:\; x = [\,\eta : a,\, \varepsilon_1 : c_1,\, \varepsilon_2 : c_2,\, \ldots\, \varepsilon_L : c_L,\, \ldots\,]\,\}\,.$$

By the above discussion, whenever $T^n(x) \in \mathcal{C}$ there exists a G_k-rational $\frac{P}{Q}$ such that

$$\left| x - \frac{P}{Q} \right| < \frac{t}{Q^2} \quad \text{and} \quad q_{n-c} \leq Q < q_{n+c}\,,$$

where c is a constant independent of n. Since the Rosen map is ergodic with respect to the invariant probability measure given in [2], the Ergodic Theorem applies and shows that for a.e. $x \in \mathbb{I}_k$, we have

$$\lim_{N\to\infty} \frac{\#\{n \leq N : T^n(x) \in \mathcal{C}\}}{N} = \delta\,,$$

where δ is the measure of $\mathcal{C}$ with respect to the invariant ergodic measure. In particular, this limit is positive.

Now let $\Xi_N(x)$ be the number of $\frac{P}{Q} \in G_k(\infty)$ such that

$$\left| x - \frac{P}{Q} \right| < \frac{t}{Q^2}, \quad Q < q_N,$$

and P/Q is neither convergent nor mediant convergent of x. From the above, we conclude that

$$\liminf_{N\to\infty} \frac{\Xi_N}{C_0(N,x,t)} > 0$$

for a.e. $x \in \mathbb{I}_k$, where $C_0(N,x,t) = \#\{1 \le n \le N : \Theta\left(M_1 \cdots M_{k_n}(\infty), x\right) < t\}$.

Now,

$$\begin{aligned}
& \limsup_{n\to\infty} \frac{C(n,x,t)}{C(n,x,t_0)} \\
\le\ & \limsup_{N\to\infty} \frac{\#\{\frac{a}{c} \in G_k(\infty) \ : \Theta(x, \frac{a}{c}) < t,\ 0 < c \le q_n\} \ - \ \Xi_N(x)}{\#\{\frac{a}{c} \in G_k(\infty) : \ \Theta(x, \frac{a}{c}) < t_0,\, 0 < c \le q_N\}} \\
=\ & \frac{t}{t_0} - \liminf_{N\to\infty} \frac{\Xi_N(x)}{\#\{\frac{a}{c} \in G_k(\infty) : \ \Theta(x, \frac{a}{c}) < t_0,\, 0 < c \le q_N\}} \\
\le\ & \frac{t}{t_0} - \liminf_{N\to\infty} \frac{C_0(N,x,t)}{C_0(N,x,t_0)} \\
\le\ & \frac{t}{t_0} - \delta \quad < \quad \frac{t}{t_0}.
\end{aligned}$$

Consequently, the Lenstra constant of the mediant map cannot be larger than its Legendre constant. □

We have thus demonstrated the equality of the Legendre and Lenstra constants for the mediant convergents.

7. Evaluating the Lenstra constant

In this section, we determine the exact value of the Lenstra constant, and hence of the Legendre also, for the mediant and the principal Rosen convergents. Note that for the principal Rosen convergents, the value of the Lenstra constant was stated—without proof—in Corollary 4.1 of [2].

7.1. **Reduction to Geometry of Natural Extension.**

First we consider the natural extension $\hat{T}$ of the Rosen continued fraction map T, defined as follows: the region of the natural extension Ω_0 is given by (7), and for $(x,y) \in \Omega_0$, we define

$$\hat{T}(x,y) \ = \left(\begin{pmatrix} -r(x)\lambda & \operatorname{sgn}(x) \\ 1 & 0 \end{pmatrix} (x), \begin{pmatrix} -r(x)\lambda & \operatorname{sgn}(x) \\ 1 & 0 \end{pmatrix} (y) \right).$$

This is bijective on Ω_0 a.e., and the absolutely continuous invariant probability measure is given by

$$C \cdot \frac{\mathrm{d}x\, \mathrm{d}y}{|x-y|^2}$$

where C is the normalizing constant (see [2] for the exact value of C in both even and odd cases).

For $(x,y) \in \Omega_0$, we set $(x_n, y_n) = \hat{T}^n(x,y)$. In all that follows, we can extend to the setting of $y = -\infty$. In this case, (6) implies that $\theta_{n-1}(x) = 1/(x_n - y_n)$. (The similarity of the denominator of this last with the denominator in the expression for our invariant measure facilitates the following ergodic theoretic approach.)

As we observed in Section 4.3, the measure is ergodic. By the individual ergodic theorem and the standard approximation method (see say Chapter 4 in [7]), we have

$$\lim_{N\to\infty} \frac{1}{N}\sharp\{n : 1 \le n \le N,\, x_n - y_n > t\} = C \iint_{\{(x,y)\in\Omega_0 : x-y>t\}} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^2} \tag{12}$$

for any $t > 0$ (a.e. $(x, y) \in \Omega_0$). Elementary calculus applies to show that that the right hand side is equal to $C\lambda \cdot \frac{1}{t}$ if t is sufficiently large. By a simple calculation, we see that $|y_n - y'_n| \to 0$ as $n \to \infty$ whenever (x, y), $(x, y') \in \Omega_0$. This implies that if (12) holds for (x, y), then it holds for (x, y') too. Thus we get (12) for a.e. $x \in \mathbb{I}_k$ and any y such that $(x, y) \in \Omega_0$; from this, we also have that the property holds also for these values of x and with $y = -\infty$. Therefore, we have

$$\lim_{N\to\infty} \frac{1}{N}\sharp\{n : 1 \le n \le N,\, \theta_n(x) < c\} = C \cdot c \tag{13}$$

if c is sufficiently small (a.e. x). Thus to determine the Lenstra constant, we only need to find the infimum of those $t > 0$, for which

$$\iint_{\{(x,y)\in\Omega_0 : x-y>t\}} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^2} = \frac{\lambda}{t} \tag{14}$$

holds. It is easily seen, compare with Figures 1 and 2, that this can be determined by the infimum t_0 of those $t > 0$, for which the points on the line segment

$$y = x + t, \quad x \in \mathbb{I}_k, \tag{15}$$

are all in Ω_0.

Now for the mediant Rosen convergents, analogous arguments apply. We use the ratio ergodic theorem, instead of the individual ergodic theorem, and we obtain the same conclusion, i.e., finding the infimum t_1 of t such that

$$\iint_{\{(x,y)\in\Omega^* : x-y>t\}} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^2} = \frac{\lambda}{t} \tag{16}$$

holds. We consider the even and the odd indices cases separately.

7.2. **Even index case:** $k = 2\ell$**.** First we show that the Lenstra constant for the Rosen convergent is $\frac{\lambda}{\lambda+2}$, confirming Corollary 4.1 of [2].

To find t_0 with property (15), it is enough to check the lines of slope 1 passing through the interior corners of Ω_0. The associated equations are

$$\begin{cases} y = x - (\frac{1}{L_j} + \phi_j) & 1 \le j \le \ell - 1 \\ y = x - (1 - \frac{\lambda}{2}). \end{cases}$$

From these, we see that $t_0 = \max\{\frac{1}{L_j} + \phi_j\ (1 \le j \le \ell - 1),\, 1 - \frac{\lambda}{2}\}$. Since

$$L_j = \frac{1}{\lambda - L_{j-1}} \qquad \text{and} \qquad \phi_j = -\frac{1}{\phi_{j-1}} - \lambda,$$

it follows that

$$\frac{1}{L_j} + \phi_j = \frac{\frac{1}{L_{j-1}} + \phi_{j-1}}{\left|\frac{1}{L_{j-1}}\phi_{j-1}\right|}$$

for $2 \le j \le \ell - 1$. Because $(1/L_j)_{j=0}^{\ell-1}$ and $(|\phi_j|)_{j=0}^{\ell-1}$ are monotonically decreasing sequences, the maximum is either $\frac{1}{L_1} + \phi_1$, $\frac{1}{L_{\ell-1}} + \phi_{\ell-1}$, or $1 - \lambda/2$. Now recall that $\phi_{\ell-1} = 0$ and $L_{\ell-1} = \lambda - 1$ and $\lambda \geq \sqrt{2}$. These yield the estimate $\frac{\lambda+2}{\lambda} = \frac{1}{L_1} + \phi_1 \geq \frac{1}{L_{\ell-1}} + \phi_{\ell-1}$. Note that it is easy to show that $\frac{\lambda+2}{\lambda} = \frac{1}{L_1} + \phi_1 > 1 - \frac{\lambda}{2}$. Consequently, we have that $t_0 = \frac{1}{L_1} + \phi_1 = \frac{\lambda+2}{\lambda}$. And, the result holds.

The result for the mediant Rosen convergents is the following.

Proposition 4. *The Lenstra constant for the mediant Rosen convergent is $\lambda - 1$ when the index is even and not equal to 4. If $k = 4$, then the Lenstra constant is equal to $\sqrt{2}/2$.*

Proof. It is obvious that the measure $\frac{dx\,dy}{|x-y|^2}$ is invariant under the translation $(x, y) \mapsto (x + z, y + z)$ for any real number z. We translate the set $J_{l+1} \times \bar{K}_{l+1}$ by $-\lambda$. Then the image is $[\phi_0, \phi_1) \times [-\frac{1}{L_1}, -\frac{1}{L_1} + 1) = [\phi_0, \phi_1) \times [-\lambda - 1, -\lambda)$ and we see that $-\lambda < -\frac{1}{L_2} = -\lambda + \frac{1}{\lambda+R}$. This shows that for the mediant case, we can get t_1 by $\max\{-\lambda + \phi_1, \frac{1}{L_j} + \phi_j, (2 \leq j \leq \ell - 1), \frac{\lambda}{2}\}$. Similarly to the above, the maximum is given by either $\lambda + \phi_1$, $\frac{1}{L_2} + \phi_2$, $\frac{1}{L_{\ell-1}} + \phi_{\ell-1} = \frac{1}{\lambda-1}$, or $\frac{\lambda}{2}$. Thus we get $t_1 = \frac{1}{\lambda-1}$ when $\ell \geq 3$; see Figure 4. If $l = 2$, a simple calculation shows that $t_1 = \sqrt{2} + 1$. □

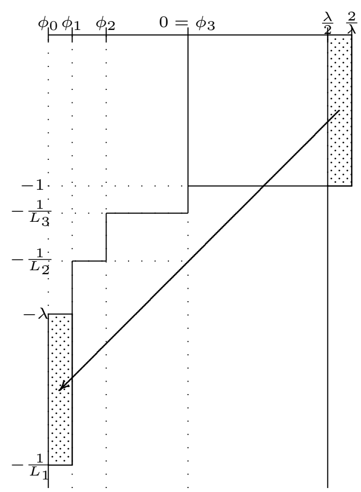

FIGURE 4. The translation of the set $J_{l+1} \times \bar{K}_{l+1}$ by $-\lambda$. Here $k = 8$.

7.3. **Odd index case:** $k = 2\ell + 3$. Here also we first confirm Corollary 4.1 of [2]: the Lenstra constant of the Rosen convergents equals $\frac{R}{R+1}$.

The idea of the calculation is the same as the even case. Considering the slope 1 lines through the corners of Ω_0, we find that

$$t_0 = \max\left\{\frac{1}{L_{2j}} + \phi_j \,(1 \leq j \leq \ell),\ \frac{1}{L_{2j-1}} + \phi_{\ell+j},\ (1 \leq j \leq \ell + 1),\ \frac{1}{R} + \frac{\lambda}{2}\right\}.$$

Since

$$\phi_{j+1} = -\frac{1}{\phi_j} - \lambda \quad \text{for} \quad 0 \leq j < \ell,\ \ell + 1 \leq j < 2\ell + 1$$

and

$$L_{j+2} = \frac{1}{\lambda_j - L_j} \quad \text{for} \quad 1 \le j \le 2\ell - 1,$$

we have

$$\phi_{j+1} + \frac{1}{L_{2(j+1)}} = \frac{\phi_j + \frac{1}{L_{2j}}}{-\phi_j \cdot \frac{1}{L_{2j}}} \qquad \text{and} \qquad \phi_{\ell+j+1} + \frac{1}{L_{2j+)}} = \frac{\phi_{\ell+j} + \frac{1}{L_{2j-1}}}{-\phi_{l+j} \cdot \frac{1}{L_{2j-1}}}$$

for $1 \le j \le \ell - 1$. Moreover, $\phi_{\ell+1} (= 1 - \lambda) = -\frac{1}{\phi_l} - 2\lambda$ and $L_1 = \frac{1}{2\lambda - L_\ell}$ implies

$$\phi_\ell + \frac{1}{L_{2\ell}} = \frac{\phi_\ell + \frac{1}{L_{2\ell}}}{-\phi_\ell \cdot \frac{1}{L_2}}.$$

Again, $(|\phi_j| : 1 \le j \le \ell + 1)$, $(|\phi_{\ell+j}| : 1 \le j \le \ell + 1)$, and $\left(\frac{1}{L_j} : 1 \le j \le 2\ell - 1\right)$ are decreasing sequences. So the above maximum is equal to

$$\begin{aligned} & \max \left\{ \phi_1 + \frac{1}{L_2}, \phi_{\ell+1} + \frac{1}{L_1}, \phi_{2\ell+1} + \frac{1}{L_{2\ell+1}}, \frac{1}{R} + \frac{\lambda}{2} \right\} \\ = \; & \max \left\{ \frac{2}{\lambda} + R, \frac{R+1}{R}, \frac{1}{\lambda - R}, \frac{1}{R} + \frac{\lambda}{2} \right\}. \end{aligned}$$

Due to the facts that $R^2 + (2 - \lambda)R - 1 = 0$ and $\lambda/2 < R < 1$, we see that the maximum is equal to $\frac{R+1}{R}$, and the result follows.

For the mediant Rosen convergents, we have the following.

Proposition 5. *In case of odd index k, the Lenstra constant for the mediant Rosen convergents is $\lambda - R$.*

Proof. We translate $(J_{2\ell+3} \times \bar{K}_{2\ell+3}) \cup (J_{2\ell+4} \times \bar{K}_{2\ell+4})$ by $-\lambda$; see Figure 5. Then its image is

$$[-\frac{\lambda}{2}, \phi_{\ell+1}) \times [-\frac{1}{R} - \lambda, -\lambda) \cup [\phi_{\ell+1}, \frac{2}{\lambda} - \lambda) \times [-R - \lambda, -\lambda)$$

This just fits on

$$J_1 \times \hat{K}_1 \cup J_2 \times \hat{K}_2 = [-\frac{\lambda}{2}, \phi_{\ell+1}) \times [-\infty, -\frac{1}{R} - \lambda) \cup [\phi_{\ell+1}, \frac{2}{\lambda} - \lambda) \times [-\infty, -R - \lambda),$$

(note that $\phi_{\ell+1} = 1 - \lambda$). Now we find that the maximum t_0 in the above is cancelled by this justification and have the new value $\phi_1 + \lambda = \frac{2}{\lambda}$ because $-\lambda < -\frac{1}{L_3} = -\lambda + \frac{1}{\lambda + \frac{1}{R}}$. Then we have

$$t_1 = \max \left\{ \frac{2}{\lambda}, \frac{1}{L_{2j}} + \phi_j \; (2 \le j \le \ell), \frac{1}{L_{2j-1}} + \phi_{\ell+j} \; (2 \le j \le \ell + 1), \frac{\lambda}{2} \right\}.$$

This is the same as

$$\max \left\{ \frac{2}{\lambda}, \frac{1}{L_4} + \phi_2, \frac{1}{L_{2\ell}} + \phi_\ell, \frac{1}{L_3} + \phi_{\ell+2}, \frac{1}{L_{2\ell+1}} + \phi_{2\ell+1}, \frac{\lambda}{2} \right\}.$$

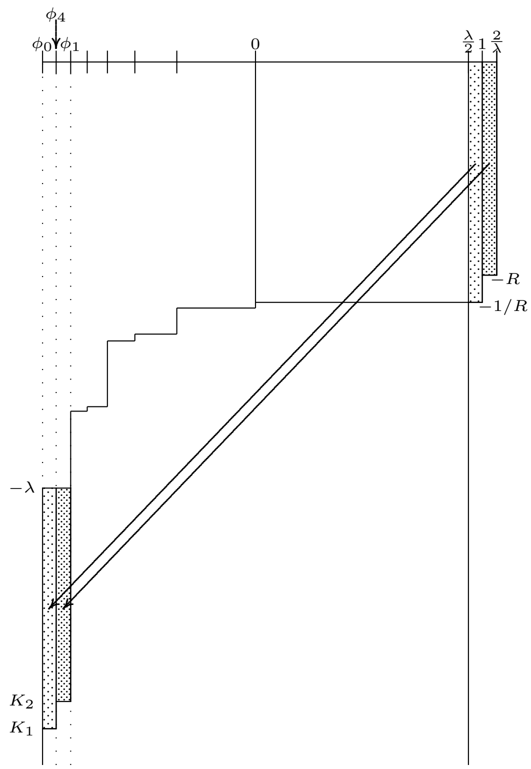

FIGURE 5. The translation of the set $(J_{2l+3} \times \bar{K}_{2l+3}) \cup (J_{2l+4} \times \bar{K}_{2l+4})$ by $-\lambda$. Here $k = 9$.

One has the following relations,

$$\begin{aligned}
\tfrac{1}{L_4} + \phi_2 &= \tfrac{1}{\lambda - \frac{2}{\lambda}} - \tfrac{1}{\lambda + R} \\
\tfrac{1}{L_{2\ell}} + \phi_\ell &= \tfrac{R}{\lambda R - 1} - \tfrac{1}{\lambda + 1} \\
\tfrac{1}{L_3} + \phi_{\ell+2} &= \tfrac{1}{\lambda - 1} - \tfrac{R}{\lambda R + 1} \\
\tfrac{1}{L_{2\ell+1}} + \phi_{2\ell+1} &= \tfrac{1}{\lambda - R}
\end{aligned}$$

Here $\frac{1}{L_4} + \phi_2$ and $\frac{1}{L_{2\ell}} + \phi_\ell$ do not appear in the above when $k = 5$. After some calculation, we see the maximum is $\frac{1}{\lambda - R}$. In order to see that this is indeed the case, note that we obviously have that $\lambda/2 < 2/\lambda$, and that

$$\frac{2}{\lambda} < \frac{1}{\lambda - R}$$

follows from $\lambda/2 < R$. Since $R^2 + (2 - \lambda)R - 1 = 0$ and $\lambda \geq \frac{1+\sqrt{5}}{2}$, we have

$$\frac{1}{\lambda - 1} - \frac{R}{\lambda R + 1} \leq \frac{1}{\lambda - R}.$$

We see

$$\frac{1}{\lambda - \frac{2}{\lambda}} - \frac{1}{\lambda + R} < \frac{1}{\lambda - R},$$

when $k > 5$. Here we used the fact that $\lambda^2 > 4 - R^2$ for $k > 5$, which has to be checked somehow. Because λ and R is increasing as k increases, it is sufficient to prove it for $k = 7$.

Finally we show that

$$\frac{R}{\lambda R - 1} - \frac{1}{\lambda + 1} < \frac{1}{\lambda - \frac{2}{\lambda}} - \frac{1}{\lambda + R}.$$

for $k \geq 7$. This inequality is equivalent to

$$\frac{1}{\lambda - \frac{1}{R}} + \frac{1}{\lambda + R} < \frac{\lambda}{\lambda^2 - 2} + \frac{1}{\lambda + 1}$$

Since $R - 1/R = \lambda - 2$, this is equivalent to

$$\frac{3\lambda - 1}{\lambda^2 + (\lambda - 2)\lambda - 1} < \frac{2\lambda^2 + \lambda - 2}{\lambda^3 + \lambda^2 - 2\lambda - 2}.$$

Note that both denominators are positive (for $\lambda > \sqrt{3}$). The last inequality follows from

$$\lambda^4 - 3\lambda^3 + 5\lambda - 2 > 0 \qquad \text{for } \lambda > \sqrt{3}.$$

□

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Technische Universiteit Delft and Thomas Stieltjes Institute of Mathematics, EWI, Mekelweg 4, 2628 CD Delft, the Netherlands
E-mail address: c.kraaikamp@tudelft.nl

Department of Mathematics, Keio University, Yokohama, Japan
E-mail address: nakada@math.keio.ac.jp

Oregon State University, Corvallis, OR 97331, USA
E-mail address: toms@math.orst.edu