

# INFINITELY GENERATED VEECH GROUPS

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ABSTRACT. We give a positive response to a question of W. Veech: Infinitely generated Veech groups do exist.

## 1. INTRODUCTION

Thurston, see [Th], investigated surface diffeomorphisms by way of locally flat structures. In particular, he examined maps which were of constant derivative with respect to the atlas of such a structure. To each of these oriented, area preserving *affine diffeomorphisms*, see below, one associates the element of  $SL(2, \mathbb{R})$  which gives the differential of the diffeomorphism at the nonsingular points.

Veech [Vch] showed that the subgroup of  $SL(2, \mathbb{R})$  formed by the differentials of the affine diffeomorphisms can reveal significant aspects of the ergodic theory of the linear flow of the surface. In particular, this has important implications in the study of billiards on Euclidean tables. One now calls this group the *Veech group* of the surface. It remains an open question to classify the groups which arise as Veech groups.

We answer a question of W. Veech [Vch3, Vch5] by the following.

**Theorem 1.** *There exist translation surfaces whose Veech group is infinitely generated and of the first kind.*

A Veech group is of finite index in the stabilizer in the Teichmüller modular group of the corresponding Teichmüller disk. Thus, our result is directly related to a question of Thurston as presented by Kra, see [Kr].

Whereas Thurston [Th] and Gutkin [Gu] had given examples of arithmetic Veech groups, Veech [Vch, Vch2] was the first to find nonarithmetic lattice groups. Further examples were given by Vorobets [Vo], Ward [Wa] and Kenyon and Smillie [KS], see also Earle and Gardiner [EG].

Much of this later work was motivated by applications to billiards. In [HS] we showed in particular that the triangle of angles  $3\pi/10, 3\pi/10, 2\pi/5$ , ‘unfolds’ to give a translation surface of whose Veech group is not a lattice. In fact, it is of the type of our Theorem: infinitely generated and of the first kind, see Remark 4.

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It follows that the directions of periodic billiard trajectories on the triangle are dense in  $S^1$ .

Surfaces with lattice Veech groups are rare, see [KS] and the related [Pu] as well as [GJ]. Gutkin and the present authors [GHS] studied surfaces whose Veech groups satisfy a weaker condition. The translation surfaces of our main theorem have Veech groups which are of the type named “prelattice” groups in [GHS].

After this research was completed, Calta [Ca] and McMullen [Mc] independently announced results about genus two translation surfaces. In particular, their results show that there are infinitely many  $GL(2, \mathbb{R})$  equivalence classes of genus two translation surfaces with nonarithmetic lattice Veech groups. More recently, McMullen [Mc2] has shown that there exist genus two translation surfaces with infinitely generated Veech groups. In the final section of this article, we briefly report on these related works and their impact our setting.

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## 2. BACKGROUND

Here we briefly recall some of the basic background which we use.

**2.1. Translation Surfaces.** A *translation surface* is a compact orientable surface with an atlas such that off of finitely many points called singularities, all transition functions are translations. Each holomorphic 1-form on a Riemann surface induces a translation structure on the surface. The study of Euclidean *billiards*, the straight-line flow within subsets of the Euclidean plane, quickly leads to translation surfaces. There is a classic construction, see [KZ], which passes from a Euclidean rational polygon to an associated translation surface, determining a complex structure on the surface along with a holomorphic 1-form. This viewpoint, and the consideration of what can be called 1/2-translation surfaces which are analogously related to quadratic differentials, has lead to impressive results, in particular see [KMS], [MS], [Ma], [Vch] and [Vch4].

Away from its singularities, a translation surface has a flat structure in the sense of Thurston [Th2]. To study its geometry, one uses the *developing map*. Following Thurston’s [Th2] treatment of general  $\mathcal{G}$ -manifolds, see also [Th3], we sketch some of the basics in the setting of translation surfaces. See [GJ2] and [KS] for related discussion.

Let  $Trans$  be the group of translation isometries of  $\mathbb{R}^2$ , thus  $Trans$  is isomorphic to the group  $\mathbb{R}^2$ . A *Trans-atlas* on a topological surface  $\mathcal{S}$  is an atlas on  $\mathcal{S}$  such that all transition functions are given by restrictions of elements of  $Trans$ . As usual, two such atlases are said to be compatible if their union is also a *Trans-atlas*;

a *Trans-structure* on a surface is a maximal *Trans-atlas* for the surface. Post-composition of all local coordinate functions with a fixed element of  $\mathrm{GL}(2, \mathbb{R})$  is easily seen to give rise to a new *Trans-structure* on a surface; that is, there is an action of  $\mathrm{GL}(2, \mathbb{R})$  on the set of *Trans-structures* on a topological surface. One says that two *Trans-structures* are *affinely equivalent* if they differ by the action of an element in  $\mathrm{SL}(2, \mathbb{R})$ . Two *Trans-structures* on a topological surface are *Trans-isomorphic* if their local coordinate maps differ by post-composition with elements of *Trans*.

The only orientable compact topological surface which can be given a *Trans-structure* is the torus; however, general compact translation surfaces are such that by deleting a finite number of points they give rise to surfaces with *Trans-structures*.

To define the developing map of a simply connected surface with *Trans-structure*: one first fixes a chart of the *Trans-atlas*; the developing map restricted to the corresponding open set is defined to be the same as the local coordinate function; thereafter, analytic continuation leads to a well-defined map.

The developing map,  $\mathrm{dev} = \mathrm{dev}_{\mathcal{M}}$  of a general translation surface  $\mathcal{M}$  is defined by factoring through its universal cover. In particular, this gives that any element of the fundamental group  $a \in \pi(\mathcal{M})$  induces an element  $g(a)$  of the structure group  $\mathcal{G} = \mathrm{Trans}$  such that  $\mathrm{dev} \circ a = g(a) \circ \mathrm{dev}$ . The group of all such  $g(a)$  is the *holonomy group* of  $\mathcal{M}$ . A geodesic arc of  $\mathcal{M}$  is a curve whose image under the developing map is a line segment in  $\mathbb{R}^2$ .

Note that given a point  $p$  of  $\mathcal{M}$ , we can normalize  $\mathrm{dev}(\mathcal{M})$  such that the developed image of  $p$  lies at the origin of  $\mathbb{R}^2$ . Indeed, there is a unique translation which brings  $\mathrm{dev}(p)$  to the origin. We simply replace  $\mathcal{M}$  by the *Trans-isomorphic* surface given by post-composing all local coordinate functions with this translation.

Given a translation surface  $\mathcal{M}$ , we can puncture all of its singularities and obtain a true *Trans-surface*,  $\mathcal{M}'$ . This  $\mathcal{M}'$  is in particular a metric space, which we can complete to say  $\bar{\mathcal{M}}$ . An *admissible* translation surface  $\mathcal{M}$  is one for which each point of  $\bar{\mathcal{M}} \setminus \mathcal{M}$  has the metric of a cone point. In particular, any translation surface induced by a holomorphic one-form on a Riemann surface is admissible, see [Th3], [Bo] and [Tr]. We will only consider admissible translation surfaces in all that follows. The complete, locally compact metric space  $\bar{\mathcal{M}}$  is a length space — the distance between points is the infimum of lengths of paths between them. Therefore, the Hopf-Rinow theorem, see say [BH], shows that  $\bar{\mathcal{M}}$  is geodesically complete — any two points can be joined by a geodesic. The geodesics of  $\mathcal{M}$  restrict to be the union of geodesics on  $\mathcal{M}'$ . See Lemma 2 for the significance of this.

Now, the identity map on  $\mathcal{M}'$  extends to a homeomorphism between  $\mathcal{M}$  and  $\bar{\mathcal{M}}$ . We thus will think of  $\mathcal{M}$  as  $\bar{\mathcal{M}}$ . In particular, we will refer to geodesics and

the like of  $\mathcal{M}$ . We also speak of the images of a singularity under the development map.

A geodesic emanating from a singularity is called a *separatrix*. A geodesic, without singularities in its interior, which connects two singularities is called a *saddle connection*. The image, considered as a vector, in  $\mathbb{R}^2$  of an oriented saddle connection by the developing map is called the associated *saddle connection vector*.

By way of the translation structure, directions of linear flow are well defined on a translation surface. It is a standard result, see say [MT], that the flow in a given direction is nonminimal only if the direction is the direction of some saddle connection.

The following result is presumably well-known. Our proof is a variation of that of Vorobets' proof that the set of directions of saddle connections on a translation surface is dense in  $S^1$ , thus of part of Proposition 3.1 in [Vo].

**Lemma 1.** *Let  $p$  be a point of a translation surface with at least one singular point. Then the set of directions of geodesic segments which emanate from  $p$  and encounter some singular point of the surface is dense in  $S^1$ .*

**Proof.** Since there are only countably many saddle connection directions, if the originally chosen direction is not minimal, then we can and do choose an arbitrarily close minimal direction to replace it. Furthermore, without loss of generality, we may and do apply an element of  $SL(2, \mathbb{R})$  so as to replace our direction by the vertical.

Denote our translation surface by  $\mathcal{M}$ . Given  $p \in \mathcal{M}$ , if  $p$  lies on a vertical separatrix, we are done. Thus suppose not. Fix a singularity,  $s$  of  $\mathcal{M}$ . Choose any  $\epsilon > 0$  sufficiently small such that there is a disk of radius  $\epsilon$  on  $\mathcal{M}$  about  $s$ . Since the leaf of the flow in the vertical direction emanating from  $p$  is dense, there is a sequence of times  $t_1 < t_2 < \dots < t_n$  tending towards infinity such that for each  $n$  the distance from  $s$  to  $\phi_{\text{vert}}^{t_n}(p)$ , the image of  $p$  under the vertical flow at time  $t_n$ , is at most  $\epsilon$ . Fix one such  $n$ . The developed images of  $p$ ,  $s$  and  $\phi_{\text{vert}}^{t_n}(p)$  form a right triangle in  $\mathbb{R}^2$  with angle at  $\text{dev}(p)$  which we label as  $\theta_n$ . We have that  $\theta_n$  is bounded above by  $\tan^{-1} \epsilon/t_n$ .

If this developed triangle has no images of singularities in its interior, then it is in fact the developed image of a geodesic triangle in  $\mathcal{M}$ . There is thus a separatrix passing through  $p$  making an angle  $\theta_n$  with the vertical.

If the developed triangle does have an image of a singularity in its interior, then we can replace  $s$  by this other singularity and find that there is a separatrix passing through  $p$  making an angle less than  $\theta_n$  with the vertical.

Since the angle measurements of  $\theta_n$  converge to zero as  $n$  tends to infinity, we are done. ■

The following is well-known.

**Lemma 2.** *Let  $p$  and  $q$  be singularities of  $\mathcal{M}$ , a translation surface with at least one singular point. Then there is a sequence of saddle connections passing from  $p$  to  $q$ .*

**Proof.** If  $\mathcal{M}$  has only one singularity, then the previous result shows that there are in fact many saddle connections joining this singularity to itself.

If  $p$  and  $q$  are distinct singularities of  $\mathcal{M}$ , then this is immediate from the geodesic completeness of  $\mathcal{M}$ .

If  $p = q$ , and there is more than one singularity, then we can join  $p$  to some other singularity by a sequence of saddle connections and return to  $p$  by reversing direction. ■

*Remark 1.* There are instances in which there is no *single* saddle connection connecting a singularity to itself. This is related to the notion of *blocking points*, see [HeSn] — however, note that [Sch] and [Mo] have found corrections to [HeSn].

Following [KS], one calls the image, considered as a vector, of an element of  $H_1(\mathcal{M}, \mathbb{Z})$  under the developing map a *holonomy vector*. The *holonomy field* is the smallest field  $k$  such that after choosing  $\mathbb{R}$ -linearly independent elements  $e_1$  and  $e_2$  of the holonomy vectors, the  $k$ -span of  $e_1$  and  $e_2$  contains every holonomy vector.

**2.2. Fuchsian Groups.** We will have need of some basic facts about Fuchsian groups. For further details than we summarize here, refer to say, S. Katok's textbook [Ka].

Recall that the oriented isometry group of the Poincaré upper half-plane  $\mathcal{H}$ , with its hyperbolic metric is  $\mathrm{PSL}(2, \mathbb{R})$ , acting as fractional linear transformations. The group  $\mathrm{SL}(2, \mathbb{R})$  is naturally topologized as a subset of  $\mathbb{R}^4$ , the quotient topology is then induced onto  $\mathrm{PSL}(2, \mathbb{R})$ . A subgroup of  $\mathrm{PSL}(2, \mathbb{R})$  which is discrete with respect to this topology is called a *Fuchsian group*.

The *limit set* of a Fuchsian group  $\Gamma$  is the set of limits of  $\Gamma$ -orbits of points  $z \in \mathcal{H}$ . The limit set is a subset of the *points at infinity*, that is of  $\mathbb{R} \cup \{\infty\}$ . A Fuchsian group is said to be of the *first kind* if its limit set is the full set of points at infinity. It is well known that if the limit set of a Fuchsian group which is not of the first kind is a nowhere dense subset of  $\mathbb{R} \cup \{\infty\}$ , see say Theorem 3.4.6 of [Ka]. A Fuchsian group is *geometrically finite* if it has a convex fundamental domain for its action on  $\mathcal{H}$  which has finitely many sides. A geometrically finite Fuchsian group is finitely generated. A Fuchsian group is a *lattice* if it has a fundamental domain which has finite hyperbolic area. One also then says that the Fuchsian group is of *finite covolume*.

We will in particular use the following well-known fact.

**Lemma 3.** *Let  $\Gamma$  be a Fuchsian group of the first kind. Then  $\Gamma$  is either a lattice, or  $\Gamma$  is infinitely generated.*

**Proof.** Suppose that  $\Gamma$  is a finitely generated Fuchsian group of the first kind. We show that  $\Gamma$  is a lattice.

By Theorem 4.6.1 of [Ka], since  $\Gamma$  is finitely generated, it is also geometrically finite. By Theorem 4.5.1 of [Ka], since  $\Gamma$  is also of the first kind, it is indeed of finite covolume. ■

A *parabolic element* of  $\mathrm{PSL}(2, \mathbb{R})$  is an element which fixes exactly one point of the extended hyperbolic plane, and this fixed point is a point at infinity. Recall that subgroups of  $\mathrm{PSL}(2, \mathbb{R})$  are said to be *commensurable* if they share a common subgroup of finite index in each. They are said to be *commensurable in the wide sense* if a finite index subgroup of one conjugates within  $\mathrm{PSL}(2, \mathbb{R})$  to give a finite index subgroup of the other. A Fuchsian group with parabolic elements is called *arithmetic* if it is commensurable in the wide sense to  $\mathrm{PSL}(2, \mathbb{Z})$ .

**2.3. Veech Groups.** We will say that  $\phi$  is a diffeomorphism of a translation surface  $\mathcal{M}$  if  $\phi$  is a homeomorphism which is a diffeomorphism off of the singularities of  $\mathcal{M}$ . The differential of such a diffeomorphism is then a linear operator on the tangent space at each nonsingular point of  $\mathcal{M}$ . Using the atlas of  $\mathcal{M}$ , the differential of a diffeomorphism can be expressed at each nonsingular point as an element of  $\mathrm{GL}(2, \mathbb{R})$ .

A diffeomorphism  $\phi : \mathcal{M} \rightarrow \mathcal{N}$  induces a new atlas on the underlying topological space: the open sets are the images of open sets of  $\mathcal{M}$ ; the local coordinate functions are the post-composition of  $\phi^{-1}$  by the local coordinate functions of the atlas of  $\mathcal{M}$ . A diffeomorphism which induces a *Trans*-isomorphic atlas is called an *affine diffeomorphism*. In particular, an affine diffeomorphism must have a constant differential with respect to the atlas. Since *Trans*-isomorphism clearly preserves area and is orientation-preserving, the constant differential is an element of  $\mathrm{SL}(2, \mathbb{R})$ . In [Vch] Veech proved, in particular, that the group of these differentials is a discrete subgroup of  $\mathrm{SL}(2, \mathbb{R})$ . That is, it defines a Fuchsian group. We will call this image in  $\mathrm{PSL}(2, \mathbb{R})$  the *Veech group* of  $\mathcal{M}$  and denote it by  $V(\mathcal{M})$ .

If  $V(\mathcal{M})$  is a lattice,  $\mathcal{M}$  is called a *Veech surface*. If  $V(\mathcal{M})$  is an arithmetic Fuchsian group,  $\mathcal{M}$  is called an *arithmetic surface*.

A direction on a translation surface  $\mathcal{M}$  is called *parabolic* if there is an affine diffeomorphism which preserves the set of geodesics in this direction and whose differential is parabolic.

A direction on a translation surface  $\mathcal{M}$  is called *periodic* if the flow in this direction is periodic (off of the set of separatrices in this direction). Maximal cylinders of flow in a periodic direction are bounded by saddle connections. Veech [Vch] showed in particular that the direction of any saddle connection on a Veech

surface is a parabolic direction. Thus, on a Veech surface, the set of periodic and parabolic directions are equal.

There is a direct connection between the parabolic direction and the fixed points of these corresponding parabolic elements of the Veech group. Let  $\theta$  be the angle which the parabolic direction makes with the horizontal. Then the parabolic elements fix the (extended, that is possibly infinite) real number  $\nu = \cot \theta$ . Indeed, let  $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  be a parabolic element of the Veech group. Acting as a standard matrix on real 2-vectors, suppose that it has as nontrivial eigenvector  $(x, y)$ . One solves to find that  $\frac{a\frac{x}{y} + b}{c\frac{x}{y} + d} = \frac{x}{y}$ . But, of course  $\nu = x/y$  is the cotangent of  $\theta$ . In particular, note that the parabolic directions of a translation surface are dense in the unit circle if and only if the parabolic fixed points of the surface's Veech group are dense in  $\mathbb{R} \cup \{\infty\}$ .

**2.4. Markings and Coverings.** Let  $\mathcal{M}$  be a translation surface. Let  $\mathcal{P} = \{p_1, \dots, p_n\}$  be any set of nonsingular points of  $\mathcal{M}$ . By *marking*  $\mathcal{M}$  at  $\mathcal{P}$  we create a formally new translation surface  $(\mathcal{M}; \mathcal{P})$ . Its set of singularities is the union of the singularities of  $\mathcal{S}$  with  $\mathcal{P}$ . We define the group of affine diffeomorphisms of  $(\mathcal{M}; \mathcal{P})$  to be the subgroup of affine diffeomorphisms which preserve  $\mathcal{P}$ . The Veech group of  $(\mathcal{M}; \mathcal{P})$ ,  $V(\mathcal{S}; \mathcal{P})$ , is thus the corresponding subgroup of  $V(\mathcal{M})$ .

A *balanced translation covering* of a translation surface  $\mathcal{M}$  is a translation surface  $\mathcal{N}$  and a map  $f : \mathcal{N} \rightarrow \mathcal{M}$  such that the pull-back by  $f$  of the translation atlas of  $\mathcal{M}$  is fully compatible with the translation atlas of  $\mathcal{N}$ . More formally,  $f$  restricted to  $\mathcal{N}'$  gives a *Trans-covering* of  $\mathcal{M}'$  — where, as above  $\mathcal{X}'$  denotes the surface arising from a translation surface  $\mathcal{X}$  by puncturing it at all of its marked points — and both the image of every marked point of  $\mathcal{N}$  is a marked point of  $\mathcal{M}$  and the inverse image of every marked point of  $\mathcal{M}$  is a marked point of  $\mathcal{N}$ .

A balanced *affine covering* of a translation surface  $\mathcal{M}$  is a map  $f : \mathcal{N} \rightarrow \mathcal{M}$  of translation surfaces whose pull-back by the action of an element of  $\mathrm{SL}(2, \mathbb{R})$  on the translation atlas of  $\mathcal{N}$  gives a balanced translation covering. It is a result of [GJ2], see also [Vo], that the Veech groups of such a pair  $\mathcal{M}$  and  $\mathcal{N}$  are commensurable in the wide sense. See also [HS].

### 3. RATIONAL POINTS AND CONNECTION POINTS

As defined in [GHS], a point of a translation surface is called a *rational point* if there exist two distinct parabolic directions of the surface such that the point is periodic under both of the corresponding primitive parabolic elements of the Veech group of the surface. One can also consider the class of *periodic points*, those which are of finite orbit under the full Veech group of the given surface.

**3.1. Connection Points.** In order to find infinitely generated groups, we will need to mark a special type of point.

**Definition 1.** We say that a nonsingular point  $p$  is a *connection point* of a translation surface  $\mathcal{M}$  if every separatrix passing through  $p$  is a saddle connection.

**Proposition 1.** *Let  $p$  be a nonperiodic connection point on a Veech surface  $\mathcal{M}$ . Then the group  $V(\mathcal{M}; \{p\})$  is infinitely generated.*

**Proof.** Since  $p$  is not a periodic point of  $\mathcal{M}$ , Corollary 4 of [HS2] shows that  $V(\mathcal{M}; \{p\})$  is not a lattice. Thus, by Lemma 3, it suffices to show that  $V(\mathcal{M}; \{p\})$  is a Fuchsian group of the first kind.

By Lemma 1, the directions of segments which join  $p$  to singular points of  $\mathcal{M}$  are dense in  $S^1$ . But, each such direction is the direction of a saddle connection on the Veech surface  $\mathcal{M}$ , and is thus a parabolic direction of the Veech surface. Furthermore,  $p$  lies on a saddle connection of each of these directions. Therefore, for each direction, there is some corresponding parabolic element of the Veech group which is the differential of an affine diffeomorphism of  $\mathcal{M}$  fixing  $p$ . Each of these survives to  $V(\mathcal{M}; \{p\})$ . Hence, the parabolic directions of  $V(\mathcal{M}; \{p\})$  are dense in  $S^1$ . But, the limit set of a Fuchsian group which is not of the first kind is nowhere dense. Thus, we conclude that  $V(\mathcal{M}; \{p\})$  is in fact of the first kind, and we are done. ■

**Proposition 2.** *Each connection point on a Veech surface is a rational point.*

**Proof.** Let  $p$  be a connection point of a Veech surface  $\mathcal{M}$ . Take any two segments in transverse directions connecting  $p$  to singularities of  $\mathcal{M}$ . These segments define separatrices; by hypothesis on  $p$ , these separatrices extend to saddle connections. Since  $\mathcal{M}$  is a Veech surface, the directions of these saddle connections are parabolic.

The cylinders of flow in these parabolic directions are bounded by saddle connections. Since  $p$  is on the boundary of at least one cylinder in each of the two parabolic directions which have been defined,  $p$  is surely a rational point. ■

**Proposition 3.** *Let  $p$  be a connection point on a Veech surface  $\mathcal{M}$ . Suppose that the translation surface  $\mathcal{N}$  is a balanced affine cover of  $(\mathcal{M}; \{p\})$ . Then  $\mathcal{N}$  satisfies the following weak Veech dichotomy: For any direction of  $\mathcal{N}$ , the flow in this direction is either minimal or it is periodic.*

**Proof.** By replacing  $\mathcal{N}$  with the image of its atlas by an appropriately chosen element of  $\mathrm{SL}(2, \mathbb{R})$ , we may and do assume that the covering is a translation covering. If a direction is not minimal, then there is a saddle connection in this



direction. Since a periodic direction for  $\mathcal{M}$  is surely a periodic direction for  $\mathcal{N}$ , it suffices to show that the directions of saddle connections for  $\mathcal{N}$  are periodic directions for  $\mathcal{M}$ .

There are three types of saddle connection on  $\mathcal{N}$ : those which project to saddle connection on  $\mathcal{M}$ ; those which project to geodesic segments connecting  $p$  to a singularity; and finally those which project to a geodesic connecting  $p$  to itself.

Since  $\mathcal{M}$  is a Veech surface, our first type of direction is certainly a periodic direction there. Since  $p$  is a connection point of  $\mathcal{M}$ , the second type is also a periodic direction on this Veech surface. Finally, a geodesic of  $\mathcal{M}$  which connects  $p$  to itself is obviously a periodic orbit. But, since  $\mathcal{M}$  is a Veech surface, the direction of any periodic orbit is in fact a periodic direction. ■

**3.2. Strong Holonomy Type.** Recall that the image, considered as a vector, of an element of  $H_1(\mathcal{M}, \mathbb{Z})$  under the developing map is called a *holonomy vector*.

**Definition 2.** We say that a translation surface  $\mathcal{M}$  is of *weak holonomy type* if both: (1) every holonomy vector, and (2) every saddle connection vector of  $\mathcal{M}$ , when expressed in terms of the canonical basis of  $\mathbb{R}^2$ , has all of its components in the holonomy field of  $\mathcal{M}$ . We say  $\mathcal{M}$  is of *strong holonomy type* if  $\mathcal{M}$  is of weak holonomy type, and (3) the periodic directions are exactly the vertical and those directions whose slopes belong to the holonomy field.

Our main result of this subsection is the following. We give the proof at the end of this subsection.

**Theorem 2.** *Let  $\mathcal{M}$  be a nonarithmetic Veech surface which is of strong holonomy type. Then there are infinitely many nonperiodic connection points on  $\mathcal{M}$ . Let  $p$  be one such point. Then there are infinitely many balanced translation covers of  $(\mathcal{M}; \{p\})$ . Each of these covering translation surfaces has an infinitely generated Veech group.*

The action of  $GL(2, \mathbb{R})$  on *Trans*-structures restricts to give an action on the set of translation surfaces. It is trivial to check that the Veech groups of surfaces corresponding under this action are conjugate. Thus the following is an immediate implication of Theorem 2.

**Corollary 1.** *Suppose that  $\mathcal{M}$  is in the  $GL(2, \mathbb{R})$ -orbit of a nonarithmetic Veech surface of strong holonomy type. Then there are infinitely many points  $p$  on  $\mathcal{M}$  such that  $(\mathcal{M}; \{p\})$  admits infinitely many balanced translation covers, each having an infinitely generated Veech group.*

We record a simple remark for use in §3.3.

**Lemma 4.** *If a translation surface  $\mathcal{M}$  has exactly one singularity and every holonomy vector has all of its components in the holonomy field of  $\mathcal{M}$ , then  $\mathcal{M}$  is of weak holonomy type.*

**Proof.** Since  $\mathcal{M}$  has only one singularity, every saddle connection is a closed curve. The development of a saddle connection is thus contained in the set of holonomy vectors. By hypothesis, these have coordinates in the holonomy field, and thus  $\mathcal{M}$  satisfies the conditions of the definition to be of weak holonomy type. ■

The following indicates that the property of weak holonomy type is rather ubiquitously enjoyed.

**Lemma 5.** *Up to normalization by an element of  $GL(2, \mathbb{R})$ , any translation surface whose Veech group contains a hyperbolic element is of weak holonomy type.*

**Proof.** Recall that [KS] give the holonomy field as the smallest field  $k$  such that after choosing  $\mathbb{R}$ -linearly independent elements  $e_1$  and  $e_2$  of the holonomy vectors, the  $k$ -span of  $e_1$  and  $e_2$  contains every holonomy vector. By applying an element of  $GL(2, \mathbb{R})$  to the atlas of the translation surface, we may assume that the  $e_i$  are the standard basis vectors of  $\mathbb{R}^2$ . Then, each holonomy vector has all of its components in  $k$ .

If further the Veech group contains a hyperbolic element, then Theorem 30 of [KS] shows that the  $\mathbb{Z}$ -span of the saddle connection vectors contains the holonomy vector group with finite index. In particular, the saddle connection vectors must also have all of their components in the holonomy field. ■

The following lemma provides a means for working with the property of weak holonomy type.

**Lemma 6.** *Let  $\mathcal{M}$  be of weak holonomy type. Suppose that the developing map of  $\mathcal{M}$  is such that the image of some singular point is the origin of  $\mathbb{R}^2$ . Then the developed images of all singular points and all rational points of  $\mathcal{M}$  have coordinates lying in the holonomy field.*

**Proof.** By Lemma 2, given any singular point there is a sequence of saddle connections from our distinguished singular point to it. Thus, every singular point has a developed image whose coordinates are in the holonomy field. For  $p$  any point on a translation surface, the developed images of  $p$  differ by holonomy vectors. Thus, all developed images of singularities of  $\mathcal{M}$  have coordinates in the holonomy field.

Let  $p$  be a rational point. It thus lies in a rectangle formed by the intersection of flow cylinders for two distinct parabolic directions. Choose a developed image

of the rectangle, thus determining a development  $\tilde{p}$  of  $p$  and developed images of the two cylinders.

Each cylinder develops to a large rectangle; each of its sides parallel to the flow direction lies on a line which contains multiple developed singularities. Each of these lines is thus of equation with coefficients in the holonomy field. Therefore, each of the four vertices of the small rectangle is the intersection point of lines defined over the field. These vertices thus have coordinates in the holonomy field. Let us label in counter-clockwise order these vertices as  $v_1, \dots, v_4$ , where  $v_1$  and  $v_2$  determine a side which we call the *base*. Now,  $\tilde{p}$  lies on a line which intersects the base of the rectangle at a point  $q = v_1 + t(v_2 - v_1)$  with  $t \in \mathbb{Q} \cap [0, 1]$ . This line intersects the top of the rectangle at  $r = v_4 + t(v_3 - v_4)$  for the same value of  $t$ . Of course,  $q$  and  $r$  have coordinates in the holonomy field. Thus, the line they determine is defined over this field. Similarly, the line parallel to the base which passes through  $\tilde{p}$  also has equation in the holonomy field. The intersection of these two lines,  $\tilde{p}$ , is hence a point of coordinates in the holonomy field.

Since any other developed image of  $p$  differs from  $\tilde{p}$  by a holonomy vector, we are done. ■

Our use of the property of strong holonomy type will be as an aid in the location of connection points.

**Proposition 4.** *For  $p$  a nonsingular point on a Veech surface  $\mathcal{M}$  of strong holonomy type, the following properties are equivalent:*

- (i)  $p$  is a connection point;
- (ii)  $p$  is a rational point;
- (iii) after the development of a singular point has been fixed as origin, every developed image of  $p$  is of coordinates in the holonomy field.

**Proof.** We have already seen that (i) implies (ii) and that (ii) implies (iii). We thus prove that (iii) implies (i).

Normalize the developing map of  $\mathcal{M}$  such that the image of some singular point is the origin of  $\mathbb{R}^2$ . Suppose that a nonsingular point  $p$  now has its developed image of coordinates in the holonomy field.

Choose any separatrix passing through  $p$ . The initial segment of this separatrix develops to a line segment joining two points of coordinates in the holonomy field. Therefore, the slope of this line segment lies in the holonomy field. But since  $\mathcal{M}$  is of strong holonomy type, the direction of this segment is a periodic direction on  $\mathcal{M}$ . Therefore, any separatrix in this direction extends to a saddle connection. In particular, the separatrix with which we began extends to a saddle connection, and we have proven the result. ■

**Proposition 5.** *Let  $\mathcal{P} = \{p_1, \dots, p_n\}$  be a set of connection points on a Veech surface  $\mathcal{M}$  of strong holonomy type. Suppose that the translation surface  $\mathcal{N}$  is a balanced translation cover of  $(\mathcal{M}; \mathcal{P})$ . Then  $\mathcal{N}$  satisfies the following weak Veech dichotomy: for any direction of  $\mathcal{N}$ , the flow in this direction is either minimal or it is periodic.*

**Proof.** When  $n = 1$ , this is exactly Proposition 3. It thus suffices to prove that the segments on  $\mathcal{N}$  connecting lifts of any  $p_i$  and  $p_j$  project to segments lying in parabolic directions of  $\mathcal{M}$ . But,  $p_i$  and  $p_j$  have developed images whose coordinates (can be assumed to) belong to the holonomy field. The direction of a segment connecting them is hence of slope in the field. By the strong holonomy type of  $\mathcal{M}$ , this is a periodic direction of  $\mathcal{M}$ . Since  $\mathcal{M}$  is a Veech surface, this is indeed a parabolic direction. ■

**Lemma 7.** *If  $\mathcal{M}$  is a nonarithmetic Veech surface which is of strong holonomy type, then there exist infinitely many nonperiodic connection points on  $\mathcal{M}$ .*

**Proof.** Since  $\mathcal{M}$  is a nonarithmetic Veech group, Theorem 1 of [GHS] shows that it has at most finitely many periodic points.

Theorem 4 of [GHS] shows that the rational points of any *prelattice surface* form a dense countable set. A prelattice surface is a translation surface having at least two distinct parabolic directions; any Veech surface is certainly of this type. In particular, we conclude that  $\mathcal{M}$  has infinitely many rational points. By Proposition 4 each of these is a connection point. Thus there are infinitely many connection points which are not periodic. ■

**Proposition 6.** *If  $\mathcal{M}$  is a nonarithmetic Veech surface which is of strong holonomy type, then  $\mathcal{M}$  has infinitely many nonperiodic connection points. For each nonperiodic connection point  $p$ ,  $V(\mathcal{M}; \{p\})$  is infinitely generated.*

**Proof.** By Lemma 7 there are infinitely many nonperiodic connection points. For each such point  $p$ , Proposition 1 then implies the result. ■

We can now prove the main result of this section.

**Proof of Theorem 2.** As mentioned in [GHS], the statement that given any nonsingular point in  $\mathcal{M}$  there are infinitely many distinct balanced translation covers of  $(\mathcal{M}; \{p\})$  follows from Theorem IV.9.12 of [FK].

Now consider  $p$  a nonperiodic connection point. Fix a balanced cover of  $(\mathcal{M}; \{p\})$ . Recall, see [GJ2] and [Vo], that the Veech group of such a cover is commensurable with  $V(\mathcal{M}; \{p\})$ . Since commensurable groups have the same set of parabolic fixed points, the Veech group of the covering surface has all of  $S^1$  as its limit set. The group  $V(\mathcal{M}; \{p\})$  is not a lattice, hence no group commensurable with it can

be a lattice. Therefore, Lemma 3 shows that the covering translation surface does indeed have an infinitely generated Veech group. ■

**3.3. Examples.** It is easy to find arithmetic translation surfaces which are of strong holonomy type.

**Lemma 8.** *If  $\mathcal{M}$  is a balanced translation cover of the once-marked standard square torus, then  $\mathcal{M}$  is of strong holonomy type.*

**Proof.** Gutkin-Judge [GJ2] have shown that the holonomy field of any affine cover of the square torus is the rational field. They also show that the parabolic directions on such covers are the vertical and those of rational slope. Finally, we note that every saddle connection on a balanced translation cover of the standard square torus has rational components. ■

Strong holonomy seems a very restrictive property on the class of nonarithmetic translation surfaces. We exhibit the three nonarithmetic translation surfaces which we found to enjoy this property.

In [HS2], we defined the *cross of translation*  $\lambda$  for each real  $\lambda \geq 1$  as the symmetric rectangular cross of minor length one and major length  $\lambda$ ; see Figure 1 of [HS2]. By identifying opposite sides by translation, one obtains a surface of genus 2 with exactly one singularity. See [Ca] and [Mc] for much more on Veech groups of surfaces of this general description. Here we use the term *golden cross* to refer to the translation surface arising from the cross of translation length  $\lambda = (1 + \sqrt{5})/2$ .

**Proposition 7.** *The golden cross is of strong holonomy type.*

**Proof.** The proof of Lemma 2 of [HS2] shows that the Veech group of this cross is the so-called Hecke triangle group of index 5. Indeed, Lemma 4 of [HS2] shows that this cross is affinely equivalent to Veech's [Vch] original nonarithmetic translation surface arising from copies of the regular pentagon; see [Mc] for a pictorial proof of this. The group is in particular nonarithmetic.

The hyperbolic element  $\begin{pmatrix} 2\lambda & -1 \\ 1 & 0 \end{pmatrix}$  is easily seen to be in this Hecke group.

Thus, by Theorem 28 of [KS] the holonomy field of the surface is  $\mathbb{Q}(\sqrt{5})$ .

Referring to Figure 1 of [HS2], one sees that there are four visible opposite images of the sole singular point of the cross. In particular, the standard vectors  $e_1$  and  $e_2$  of  $\mathbb{R}^2$  are clearly saddle connection vectors. Since the cross is a translation surface with only one singularity, these are also holonomy vectors. This translation surface is thus of weak holonomy type.

Leutbecher [Le] showed that the set of parabolic fixed points (with respect to the upper half-plane model) of this group is exactly  $\mathbb{Q}(\sqrt{5}) \cup \{\infty\}$ . From

the relation between parabolic fixed points and parabolic directions mentioned in subsection 2.3, we conclude that the vertical and the directions of slope in  $\mathbb{Q}(\sqrt{5})$  form the parabolic directions for the golden cross. We have thus shown that this translation surface is indeed of strong holonomy type. ■

*Remark 2.* As mentioned in the above proof, there is an affine map from the golden cross to the double pentagon example of [Vch]. The images of the connection points of the golden cross are hence connection points on this latter surface. In particular, one checks that the centers of the pentagons are connection points.

Recall that the double pentagon surface may be seen as arising from an “unfolding” process applied to the Euclidean triangle of angles  $\pi/5, \pi/5, 3\pi/5$ . In Theorem 8.1 of [Vch], Veech considers the Euclidean triangle of angles  $\pi/n, \pi/n, (n-2)\pi/n, n > 2$ . Using that the billiard trajectories of the Euclidean triangle lift to geodesics of the translation surface, he shows that whenever a billiard trajectory begins and ends at *the same* vertex of the triangle, then every trajectory in the same direction is either closed or else begins and ends at a vertex. In a remark, Veech gives an argument for why one can remove the words “the same” when  $n$  is even. He leaves the case of odd  $n$  open.

When  $n = 5$ , we can remove the hypothesis “the same.” In brief, the corresponding translation surface has a sole singularity given by copies of the triangle meeting at vertices of angle  $3\pi/5$ ; Veech’s arrangement is such that the remaining vertices meet variously at the centers of the two pentagons. Any billiard trajectory from a vertex to a vertex lifts to a translation geodesic arc which is either: a saddle connection; an arc joining a center to itself; a separatrix meeting a center; or, an arc joining the two centers. The translation surface is a Veech surface, thus the directions of saddle connections and of single closed orbits are periodic directions. The centers are connections points, hence separatrices passing through them extend to saddle connections. Finally, any geodesic arc passing from one center to the other is affinely equivalent to a geodesic arc joining rational points on a translation surface which is of strong holonomy type — the golden cross; the original geodesic arc thus also lies in a periodic direction.

**Definition 3.** Let  $\mathcal{M}_{2n}$  be the translation surface formed from the regular  $2n$ -gon inscribed in the unit circle of the complex plane by identifying, by translation, opposite sides.

**Lemma 9.** For  $n > 3$ , the holonomy field of  $\mathcal{M}_{2n}$  is  $\mathbb{Q}(\cos \frac{\pi}{n})$ .

**Proof.** As [AH] point out, the Veech groups of the  $\mathcal{M}_{2n}$  are the same as those of the even index examples of Veech’s fundamental paper [Vch]. In particular, for

$n > 3$  fixed, the group is generated by  $P = \begin{pmatrix} 1 & 2 \cot \frac{\pi}{n} \\ 0 & 1 \end{pmatrix}$  and  $R$ , the rotation of angle  $2\pi/(2n)$ . The trace of  $P^2R$  is easily computed to be  $2(6 \cos^2 \frac{\pi}{2n} - 1)$ . This is clearly greater than 2 for  $n \in \mathbb{N}$ . Thus  $P^2R$  is a hyperbolic element. Hence, by Theorem 28 of [KS], the holonomy field of  $\mathcal{M}_{2n}$  is  $\mathbb{Q}(\cos^2 \frac{\pi}{2n})$ . But, the double angle formula for the cosine function shows that for any real  $x$ , the two fields  $\mathbb{Q}(\cos^2 x)$  and  $\mathbb{Q}(\cos 2x)$  are equal. ■

**Proposition 8.** *The translation surfaces  $\mathcal{M}_8$  and  $\mathcal{M}_{12}$  are of strong holonomy type. If  $n > 3$  is odd, then  $\mathcal{M}_{2n}$  is not of strong holonomy type.*

**Proof.** The singularities of an  $\mathcal{M}_{2n}$  arise from the vertices of the corresponding regular  $2n$ -gon. It is easily checked that if  $n$  is even, then there is only one singularity on  $\mathcal{M}_{2n}$ . In particular, by Lemma 4, to prove that either of  $\mathcal{M}_8$  and  $\mathcal{M}_{12}$  is of weak holonomy type, it suffices to show that the holonomy vectors all have their components in the holonomy field. By the definition of the holonomy field, to do this it suffices to find two  $\mathbb{R}$ -linearly independent holonomy vectors having all of their components in the holonomy field.

By Lemma 9 the holonomy fields of  $\mathcal{M}_8$  and  $\mathcal{M}_{12}$  are  $\mathbb{Q}(\sqrt{2})$  and  $\mathbb{Q}(\sqrt{3})$ , respectively. The coordinates of the vertices of the corresponding  $2n$ -gon are all in this respective field. Thus, the holonomy vector arising from the segment joining any two opposite vertices has all components in the holonomy field. Choose any two distinct pairs of opposite vertices, the corresponding holonomy vectors are clearly  $\mathbb{R}$ -linearly independent. We thus conclude that  $\mathcal{M}_8$  and  $\mathcal{M}_{12}$  are of weak holonomy type.

It is shown in [AS] that results of Leutbecher [Le2] imply that the fields  $\mathbb{Q}(\sqrt{2})$  and  $\mathbb{Q}(\sqrt{3})$  are the set of slopes of the nonvertical parabolic directions for  $\mathcal{M}_8$  and  $\mathcal{M}_{12}$ , respectively. We thus have shown that  $\mathcal{M}_8$  and  $\mathcal{M}_{12}$  are of strong holonomy type.

In [AS], it is also noted that results of Leutbecher [Le2] imply that when  $n > 3$  is odd, no rational number is the slope of a parabolic direction for  $\mathcal{M}_{2n}$ . Hence, none of these can be of strong holonomy type. ■

*Remark 3.* Wolfart [Wo] extended Leutbecher's and others' work on cusp values. In our terms, with [AS] in mind, this shows that the only remaining  $\mathcal{M}_{2n}$  which could possibly be of strong holonomy type are  $\mathcal{M}_{20}$  and  $\mathcal{M}_{24}$ .

Combining Propositions 7 and 8 and Corollary 2, give the following, which proves our Theorem 1.

**Corollary 2.** *Let  $\mathcal{M}$  be any of the golden cross,  $\mathcal{M}_8$  or  $\mathcal{M}_{12}$  and  $p$  be any nonperiodic rational point of  $\mathcal{M}$ . There are infinitely many balanced translation covers*

of  $(\mathcal{M}; \{p\})$ . Each of these covering translation surfaces is such that its Veech group is infinitely generated.

*Remark 4.* Again consider the double pentagon surface of [Vch]. As mentioned in Remark 2, the centers of the pentagons are connection points; in [HS], we show that these points are nonperiodic. Therefore, using the arguments for Proposition 5 to extend Proposition 6, we find that any balanced translation covering of the surface marked at these points is of infinitely generated Veech group. Now, in the proof of Proposition 4 of [HS] we show that the translation surface arising from the ‘unfolding’ of the Euclidean triangle of angles  $3\pi/10, 3\pi/10, 2\pi/5$  gives exactly such a balanced covering. It follows that on this ‘non-Veech’ triangle every direction of billiard trajectories is either minimal or periodic; indeed the periodic directions are dense.

*Remark 5.* By taking a double cover of either of the golden cross or  $\mathcal{M}_8$  which is ramified at a nonperiodic rational point and the singularity, one obtains a translation surface of genus 4 which has infinitely generated Veech group. This is the minimal genus which can occur in our construction.

#### 4. IMPLICATIONS OF WORKS OF CALTA AND OF McMULLEN

After the completion of this research, Calta [Ca] and McMullen [Mc], gave certain results about genus two Veech surfaces. Furthermore, McMullen [Mc2] has since given his own examples of infinitely generated Veech groups. We briefly indicate some of the implications of these works in the present setting.

By the Riemann-Roch theorem, the total multiplicity of the zeros of a nontrivial holomorphic 1-form on a compact Riemann surface of genus  $g$  is  $2g - 2$ . The set of translation surfaces of a fixed genus is stratified by sets corresponding to the partitions of  $2g - 2$  which are achievable as multiplicities of zeros. (Recall that the angle at a cone point on  $(X, \omega)$  corresponding to a zero of  $\omega$  of multiplicity  $k$  is  $2(k + 1)\pi$ .) For example, any genus two translation surface either has one sole singularity or else has two distinct singularities; the first characterizes an element of  $\mathcal{H}(2)$ , the second of  $\mathcal{H}(1, 1)$ . See [KoZo] for a detailed description of the strata in the general genus case.

With distinct methods, Calta [Ca] and McMullen [Mc] give results which may be interpreted in the following manner.

**Theorem 3.** (*Calta; McMullen*) *Any translation surface of  $\mathcal{H}(2)$  whose Veech group has more than two points in its limit set is a Veech surface. Each such surface is either an arithmetic surface, or else is of real quadratic holonomy field.*



Theorem A.1 of [Mc2] can be interpreted in the following manner. Note that Theorem 7 of [Ca] is closely related.

**Theorem 4.** (*McMullen*) *Any Veech surface having real quadratic holonomy field lies in the  $GL(2, \mathbb{R})$ -orbit of some nonarithmetic Veech surface which is of strong holonomy type.*

*Remark 6.* The above two results imply that in  $\mathcal{H}(2)$  there are many translation surfaces which admit nonperiodic connection points and are therefore covered by translation surfaces of infinitely generated Veech group.

We do not yet know of any translation surface of strong holonomy type which is not also of real quadratic holonomy field.

Our construction of translation surfaces of infinitely generated Veech group results in these translation surfaces being of genus at least four.

**Theorem 5.** (*McMullen*) *There exist genus two translation surfaces which are of infinitely generated Veech group.*

McMullen's technique for displaying his examples is, as he points out, rather general. It seems likely that in each genus there are translation surfaces of infinitely generated Veech group. The surfaces which McMullen finds are similar to those which were found in this work: there are parabolic directions which are dense in  $S^1$ . We do not yet know if these are the only possibilities for infinitely generated Veech groups.

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