AFFINE DIFFEOMORPHISMS OF TRANSLATION SURFACES: PERIODIC POINTS, FUCHSIAN GROUPS, AND ARITHMETICITY

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Abstract. Nous étudions des surfaces de translation ayant un grand groupe de difféomorphisms affines — les surfaces "préréseaux". Parmi celles-ci se trouvent les surfaces de translation réseaux étudiées par W. Veech. Nous montrons qu'il existe des surfaces de translation préréseaux qui ne sont pas réseaux. Nous donnons une nouvelle caractérisation des surfaces arithmétiques : ce sont les surfaces préréseaux qui ont un nombre infini de points périodiques sous l'action du groupe des difféomorphisms affines. Nous exhibons des exemples de surfaces de translation dont les points périodiques et points de Weierstrass coïncident.

We study translation surfaces with rich groups of affine diffeomorphisms — "prelattice" surfaces. These include the lattice translation surfaces studied by W. Veech. We show that there exist prelattice but nonlattice translation surfaces. We characterize arithmetic surfaces among prelattice surfaces by the infinite cardinality of their set of points periodic under affine diffeomorphisms. We give examples of translation surfaces whose periodic points and Weierstrass points coincide.

1. Introduction

A *translation surface* is a flat surface with conical singularities (see say [Th88]), whose transition functions are (restrictions of) translations. Translation surfaces arise in several contexts: mathematical billiards, Riemann surfaces and their moduli, classification of surface diffeomorphisms and measured foliations. In this paper, we focus on the geometry and arithmetic of translation surfaces.

In [Th88] W. Thurston studied flat Riemannian metrics with conical singularities. In a particular setting, these give rise to translation surfaces. Let S be a Riemann surface and ϕ a holomorphic 1-form on S. Integrating ϕ, we obtain a translation atlas off of the zeros of ϕ. A zero of ϕ of multiplicity $m-1$ yields a cone point with angle $2m\pi$. See [MT01, Wrd98] for details.

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The affine diffeomorphisms of a translation surface form a group, $\mathrm{Aff}(\mathcal{S})$. Assigning to $g \in \mathrm{Aff}(\mathcal{S})$ its (constant) differential, we obtain the *differential homomorphism* $D : \mathrm{Aff}(\mathcal{S}) \to \mathrm{SL}(2, \mathbb{R})$. Let $\Gamma(\mathcal{S}) \subset \mathrm{SL}(2, \mathbb{R})$ be its range. W. Veech showed that $\Gamma(\mathcal{S})$ is a Fuchsian group, and related it to the geometry and dynamics of the geodesic flow of $\mathcal{S}$ [Vch89]. It is customary to call $\Gamma(\mathcal{S})$ the *Veech group* of $\mathcal{S}$. [1]

We say that $\mathcal{S}$ has the *lattice property*, or simply that $\mathcal{S}$ is a *lattice surface*, if $\Gamma(\mathcal{S})$ is a lattice, i.e., $\mathrm{SL}(2, \mathbb{R})/\Gamma$ has finite volume. (It is also common to call $\mathcal{S}$ a *Veech surface*.) The lattice $\Gamma(\mathcal{S})$ is necessarily nonuniform [Vch89]. For instance, the standard flat torus is a lattice surface — its Veech group is $\mathrm{SL}(2, \mathbb{Z})$.

A nonuniform lattice is *arithmetic* if it admits a finite index subgroup which is conjugate to a subgroup of $\mathrm{SL}(2, \mathbb{Z})$. A lattice translation surface is *arithmetic* if its Veech group is arithmetic. Arithmetic translation surfaces were investigated already in [Gut84]. In [Vch89, Vch92] Veech gave the first examples of nonarithmetic lattice surfaces. He also showed that the geodesics on lattice surfaces satisfy the *Veech dichotomy*: Every geodesic is either closed or uniformly distributed.

There are two major branches to the study of translation surfaces. One is the study of the general or, at least, the generic translation surface. See, for instance, [EM01, KMS86] and the survey [MT01]. The other is the study of special translation surfaces, e.g., lattice surfaces. This branch naturally subdivides: The purely geometric one [Vo96, Gut00] and the algebro-geometric one [Wrd98, GJ96, KS00, GJ00, HS00, HS01]. The present work is of the latter type.

Every element of $\mathrm{SL}(2, \mathbb{R}) \setminus \{\pm 1\}$ is either *parabolic, elliptic*, or *hyperbolic*. By convention, we consider the elements ± 1 elliptic. We say that $\phi \in \mathrm{Aff}(\mathcal{S})$ is a *parabolic, elliptic*, or a *hyperbolic* diffeomorphism, if $D\phi \in \mathrm{SL}(2, \mathbb{R})$ is parabolic, elliptic, or hyperbolic respectively.

The generic translation surface has no affine symmetries, while we study the surfaces with infinitely many of them. We emphasize the diffeomorphisms "generated" by *parabolic directions*. A direction is parabolic for $\mathcal{S}$ if

(i) Every geodesic in this direction is either periodic or a saddle connection;
(ii) The moduli of the cylinders in $\mathcal{S}$, formed by the geodesics in this direction are commensurate.

To each parabolic direction, θ, one can associate a parabolic diffeomorphism, $\phi_\theta \in \mathrm{Aff}(\mathcal{S})$, [Vch89]. The restriction of ϕ_θ to a cylinder in the direction θ is a power of the Dehn twist of that cylinder. This then allows us to identify ϕ_θ

[1]The flat structures considered by Veech and many others are induced by quadratic differentials. Translation surfaces correspond to the quadratic differentials which are squares of linear ones. From the geometric viewpoint, these more general flat structures are the *half-translation surfaces* [GJ00]. All of our results extend mutatis mutandis to the half-translation surfaces. This follows from the standard 2-sheeted covering of a half-translation surface by a translation surface [HM79].

with its differential, a parabolic element of $\Gamma(\mathcal{S})$. Furthermore, for any parabolic $g \in \mathrm{Aff}(\mathcal{S})$, there exist $m, n \in \mathbb{N}$, and a parabolic direction θ such that $g^m = \phi_\theta^n$; see Proposition 2.4 of [Vch89].

If α, β are arbitrary directions, then either $\alpha = \pm\beta$ or α and β are transversal.

Definition 1. A discrete group $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ is a *prelattice* if it contains noncommuting parabolic elements. A translation surface $\mathcal{S}$ is a *prelattice surface* if the group $\Gamma(\mathcal{S})$ is a prelattice. Equivalently, $\mathcal{S}$ is a prelattice surface if it has a pair of *transversal parabolic directions.*

In § 9 of [Vch89] Veech briefly considered translation surfaces satisfying Definition 1. We show that such surfaces need not be lattice surfaces — see Corollary 7.

Let $\mathcal{S}$ be a translation surface and let $G \subset \mathrm{Aff}(\mathcal{S})$ be a subgroup of infinite order. A point of $\mathcal{S}$ is *G-periodic* if its G-orbit is finite. When $G = \mathrm{Aff}(\mathcal{S})$, we simply speak of the *periodic points* of $\mathcal{S}$. It follows from Theorem 5.5 of [GJ00] that the periodic points of an arithmetic translation surface form a countable, dense set.

Theorem 1. *Let $\mathcal{S}$ be a prelattice translation surface. Then the following dichotomy holds:*

(i) *The surface $\mathcal{S}$ is an arithmetic lattice surface; the periodic points form a dense, countable subset;*
(ii) *The surface $\mathcal{S}$ is not an arithmetic lattice surface; the set of periodic points is finite.*

Theorem 1 will follow from Theorem 7 in § 3, which gives an upper bound on the number of periodic points that a nonarithmetic prelattice translation surface can have.

Recall that an action of a group on a compact set is *minimal* if there are no nontrivial closed invariant subsets. We show that the action of the affine group on a prelattice surface is "nearly" minimal.

Theorem 2. *Let $\mathcal{S}$ be a prelattice translation surface. Then the only closed infinite subset of $\mathcal{S}$ invariant under Aff($\mathcal{S}$) is $\mathcal{S}$ itself.*

Definition 2. Let $\mathcal{S}$ be a translation surface, and let $C(\mathcal{S})$ be its set of cone points. Let $s_1, \ldots, s_p \in \mathcal{S} \setminus C(\mathcal{S})$. Marking $s_1, \ldots, s_p$ we create a formally new translation surface $(\mathcal{S}; s_1, \ldots, s_p)$. It is the surface $\mathcal{S}$ *punctured* (or *marked*) at the points $s_1, \ldots, s_p$. Its set of cone points is $C(\mathcal{S}) \cup \{s_1, \ldots, s_p\}$. Its group of affine diffeomorphisms consists of the elements of $\mathrm{Aff}(\mathcal{S})$ that preserve $C(\mathcal{S}) \cup \{s_1, \ldots, s_p\}$.

Although puncturing a translation surface does not change the geodesics, it may drastically change its Veech group and the counting functions [Gut00, HS00, HS01]. Marked translation surfaces naturally arise in the context of affine coverings. See § 2.2 and § 5.2.

Let $\mathcal{S}$ be a prelattice surface. We say that $s \in \mathcal{S}$ is a *rational point* if there exist two transversal parabolic directions for $\mathcal{S}$ such that for each direction, s is periodic with respect to the Dehn twist of the cylinder in which it lies. See § 5 for a formal definition.

Theorem 3. *Let $\mathcal{S}$ be a prelattice translation surface. Let $\mathcal{S}_{\mathbb{Q}} \subset \mathcal{S}$ be the set of rational points and let $P(\mathcal{S})$ be the set of periodic points.*

(a) *The set $\mathcal{S}_{\mathbb{Q}}$ is dense, countable, and $P(\mathcal{S}) \subset \mathcal{S}_{\mathbb{Q}}$.*
(b) *The surface $\mathcal{S}$ is arithmetic if and only if $P(\mathcal{S}) = \mathcal{S}_{\mathbb{Q}}$.*
(c) *Let $s \in \mathcal{S}$. Then $(\mathcal{S}; s)$ is a prelattice surface if and only if $s \in \mathcal{S}_{\mathbb{Q}}$.*

Claim (b) is a new characterization of arithmeticity for translation surfaces. Theorem 3 yields a classification of points in lattice surfaces.

Corollary 1. *Let $\mathcal{S}$ be a lattice translation surface, and let $s \in \mathcal{S}$. Then the following trichotomy is satisfied.*

(i) *We have $s \in P(\mathcal{S})$ if and only if $(\mathcal{S}; s)$ is a lattice surface.*
(ii) *We have $s \in \mathcal{S}_{\mathbb{Q}} \setminus P(\mathcal{S})$ if and only if $(\mathcal{S}; s)$ is a prelattice, but not a lattice surface.*
(iii) *We have $s \in \mathcal{S} \setminus \mathcal{S}_{\mathbb{Q}}$ if and only if $(\mathcal{S}; s)$ is not a prelattice surface.*

Our further results concern *balanced coverings* [Gut00] of translation surfaces. See Definition 4 in § 2.2. Veech groups behave naturally under balanced coverings: The lattice property is preserved. A *translation covering* is an affine covering $p : \mathcal{R} \to \mathcal{S}$, whose differential satisfies $Dp = 1$. The group $\mathrm{GL}(2, \mathbb{R})$ acts on translation surfaces, by composition with coordinate functions. Let $\mathcal{S} \to g \cdot \mathcal{S}$ denote the action. Translation surfaces $\mathcal{S}, \mathcal{S}'$ are *equivalent* (resp. *equivalent in the extended sense*) if $\mathcal{S}' = g \cdot \mathcal{S}$ with $g \in \mathrm{SL}(2, \mathbb{R})$ (resp. $g \in \mathrm{GL}(2, \mathbb{R})$). This equivalence allows us to replace a (balanced) affine covering by a (balanced) translation covering.

Recall that Fuchsian groups Γ, Γ' are *commensurable* if $\Gamma \cap \Gamma'$ is of finite index in both Γ, Γ'. The groups are called *commensurable in the wide sense* if there is some $g \in \mathrm{SL}(2, \mathbb{R})$ such that Γ and $g\Gamma' g^{-1}$ are commensurable.

Definition 3. Let $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ be a Fuchsian group. We say that Γ is *realizable* (resp. *almost realizable*) as a Veech group if there exists a translation surface $\mathcal{S}$ such that $\Gamma = \Gamma(\mathcal{S})$ (resp. Γ is commensurable with $\Gamma(\mathcal{S})$).

Theorem 4. *Let $\mathcal{S}$ be a lattice translation surface, and let Γ be its Veech group. Then $\mathcal{S}$ is nonarithmetic if and only if there exists a prelattice subgroup $\Gamma' \subset \Gamma$ of infinite index which is almost realizable as a Veech group.*

Perhaps the most striking implication of Theorem 4 is the following.

Corollary 2. *Let $\Gamma \subset SL(2, \mathbb{Z})$ be a prelattice. Then Γ is almost realizable as a Veech group if and only if it is a lattice.*

Recall that a Riemann surface, S, is *hyperelliptic* if it admits a holomorphic (*hyperelliptic*) involution, σ, such that the quotient S/σ is the Riemann sphere. The fixed points of σ are called the *Weierstrass* points of S. Since a translation surface defines a Riemann surface, we can speak of *hyperelliptic translation surfaces* and their Weierstrass points.

Theorem 5. *There are hyperelliptic translation surfaces whose sets of Weierstrass points and of periodic points coincide.*

This last cannot be true in general: If $\mathcal{S}$ is arithmetic, then the set of its periodic points is infinite, while the set of Weierstrass points is always finite.

Organization of Paper In Section 2 we discuss background material. In § 3 we prove key quantitative results. In § 4 we prove our main quantitative result, Theorem 7, and then Theorems 1 and 2. In § 5 we study rational points of prelattice surfaces, proving in particular Theorems 3 and 4. In § 6 we study hyperelliptic surfaces, and give explicit examples proving Theorem 5.

Thanks We thank the referee for a careful reading.

2. Background and Preliminaries

2.1. **Parabolic Diffeomorphisms of a Translation Surface.** We recall the main concepts, referring the reader to the survey [MT01] for elaboration. We consider only closed, connected translation surfaces. A translation surface, $\mathcal{S}$, has a finite set, $C(\mathcal{S})$, of cone points. The points in $\mathcal{S} \setminus C(\mathcal{S})$ are called regular. A nonzero tangent vector at a regular point of $\mathcal{S}$ has a *direction*. For any $\theta \in [0, 2\pi)$, the unit tangent vectors in direction θ form a vector field, V_θ, with singularities at the cone points. Integral curves of V_θ are the *geodesics on $\mathcal{S}$ in direction θ*. We parametrize them by arclength. If $\gamma(t)$ is a geodesic such that $\gamma(t+\ell) = \gamma(t)$, $\gamma(t+\ell/n) \neq \gamma(t)$ for $n > 1$, then γ is a (prime) *periodic geodesic* of length ℓ. If $\gamma(t), 0 \leq t \leq \ell$, is a geodesic, whose endpoints belong to $C(\mathcal{S})$, and whose interior points are regular, then γ is a *saddle connection of length ℓ*. We designate by *closed geodesics* both periodic geodesics and the saddle connections.

The only closed translation surfaces without cone points are the flat tori. To unify our treatment, we always mark a point of a flat torus and call this the *origin*. Any regular closed geodesic determines a maximal flat cylinder, $\mathcal{C} \subset \mathcal{S}$. The flat cylinder, $\mathcal{C}(\ell, w)$, of *length* ℓ and *width* w, is obtained by identifying the two vertical sides of the rectangle $\mathcal{R}(\ell, w) = \{(x, y), 0 \leq x \leq \ell, 0 \leq y \leq w\}$. Although various cylinders are affinely equivalent, the modulus $\mu = \ell/w = \mu(\mathcal{C})$ gives a conformal invariant. The interior of any (maximal) cylinder $\mathcal{C} \in \mathcal{S}$ is isometric to $\mathrm{Int}(\mathcal{C}(\ell, w))$, where $\ell = \ell(\mathcal{C})$ and $w = w(\mathcal{C})$ are respectively the length and the width of $\mathcal{C}$. If $\mathcal{C} \subset \mathcal{S}$ is a cylinder of length ℓ, width w, and direction θ, then

L_y, $0 < y < w$, are the periodic geodesics in $\mathcal{S}$ of length ℓ and direction θ. The curves L_0 and L_w are the unions of saddle connections in the same direction.

The basic affine diffeomorphism of $\mathcal{C} = \mathcal{C}(\ell, w)$ is the Dehn twist $T = T_{\mathcal{C}}$. In the coordinates above we have $T(s,t) = (s + t\frac{\ell}{w} \mod \ell\mathbb{Z},\, t)$. Since T fixes the boundary $\partial\mathcal{C}(\ell, w)$ pointwise, it defines the Dehn twist for any cylinder, $\mathcal{C} \subset \mathcal{S}$, of length ℓ and width w. A direction θ is *periodic* for $\mathcal{S}$ if every geodesic in this direction is closed. A periodic direction defines a decomposition of $\mathcal{S}$ as a finite union of cylinders $\mathcal{C}_i$, $1 \le i \le k(\theta)$. Let w_i, ℓ_i, μ_i be the respective parameters, and let $T_i : \mathcal{C}_i \to \mathcal{C}_i$ be the respective Dehn twists. There exist $N_i \in \mathbb{N}$ such that the iterates $T_i^{N_i}$, $1 \le i \le k(\theta)$, fit together, yielding an affine diffeomorphism $\phi_\theta : \mathcal{S} \to \mathcal{S}$, if and only if the moduli μ_i are commensurable. In this case θ is a *parabolic direction*. The smallest positive $\mu = \mu(\theta)$ such that $\mu = N_i\mu_i, 1 \le i \le k(\theta)$, is the *modulus of the parabolic direction* θ. The diffeomorphism $\phi_\theta \in \mathrm{Aff}(\mathcal{S})$ is the *principal parabolic diffeomorphism* corresponding to θ. We use the same notation for its differential, which belongs to $\Gamma(\mathcal{S})$. In appropriate coordinates, ϕ_θ is given by the parabolic upper triangular 2×2 matrix with $\mu(\theta)$ in the corner.

2.2. **Affine Equivalence and Coverings.** There is a natural action of $\mathrm{GL}(2, \mathbb{R})$ on the space of translation surfaces, which is simple to describe in terms of the coordinate charts, [Vch84], [Vch86], [GJ00]. If $\mathcal{S}$ is a translation surface, and $g \in \mathrm{SL}(2, \mathbb{R})$, we denote by $g \cdot \mathcal{S}$ the new translation surface. The translation surfaces $\mathcal{S}$ and $g \cdot \mathcal{S}$ are *affinely equivalent*, and $\Gamma(g \cdot \mathcal{S}) = g\Gamma(\mathcal{S})g^{-1}$. Hence, this action preserves arithmeticity and the (pre)lattice property. In particular, if α, β is a pair of transversal parabolic directions for $\mathcal{S}$, then $g \cdot \alpha, g \cdot \beta$ is the corresponding transversal pair for $g \cdot \mathcal{S}$. The statements announced in the Introduction are each appropriately either invariant or equivariant under the affine equivalence.

We use this observation for two purposes: 1) To *normalize* a pair of parabolic directions; 2) To replace an *affine covering* by a *translation covering*. Let $\mathcal{S}$ be a translation surface, and let α, β be a transversal pair of parabolic directions for $\mathcal{S}$. Replacing $\mathcal{S}$ by an affinely equivalent surface, if need be, we can assume without loss of generality that α, β are the positive x, y-directions respectively

Natural mappings of translation surfaces are the affine coverings [GJ00]. Let $p : \mathcal{X} \to \mathcal{Y}$ be one. Then p defines a (possibly branched) covering of the corresponding closed topological surfaces. Furthermore, p is affine outside of the cone sets. The differential, $Dp(x) \in \mathrm{GL}(2, \mathbb{R})$, is a constant matrix. Translation coverings are the affine coverings whose differential is the identity matrix. Hence, replacing either $\mathcal{X}$ or $\mathcal{Y}$ by an affinely equivalent surface (in general, in the extended sense), we can assume that $p : \mathcal{X} \to \mathcal{Y}$ is a translation covering [GJ96, Vo96, GJ00, HS01].

Definition 4. Let $p : \mathcal{X} \to \mathcal{Y}$ be an affine covering of translation surfaces. Then p is *balanced* if $p(C(\mathcal{X})) = C(\mathcal{Y})$ and $p^{-1}(C(\mathcal{Y})) = C(\mathcal{X})$.

The following theorem was proved independently by E. Gutkin and C. Judge, and by Ya. Vorobets.

Theorem 6. *(*[GJ96, GJ00] *and* [Vo96]*.) Let $p : \mathcal{X} \to \mathcal{Y}$ be a balanced affine covering of translation surfaces. Then the groups $\Gamma(\mathcal{X})$ and $\Gamma(\mathcal{Y})$ are commensurable in the wide sense. If, besides, p is a translation covering, then $\Gamma(\mathcal{X})$ and $\Gamma(\mathcal{Y})$ are commensurable.*

3. Periodic Points of Translation Surfaces

Let $\mathcal{C}$ be a flat cylinder, and let $T : \mathcal{C} \to \mathcal{C}$ be the Dehn twist. A point $z \in \mathcal{C}$ is periodic if $T^n z = z$, for some $n > 0$. The smallest such n is the period of z.

We do the computations for the standard cylinder $\mathcal{C} = \mathcal{C}(1, 1)$. It is straightforward to extend them to arbitrary $\mathcal{C}(\ell, w)$. Thus, $T : (x, y) \mapsto (x + y \mod 1, y)$. The restriction of T to the closed geodesic $L_y = \{y = \text{const}\} \subset \mathcal{C}$ is the rotation by y. Hence, a point $z \in \mathcal{C}$ is periodic if and only if $z \in L_y$, where y is rational. Moreover, the set of points of period n is the union of $L_{k/n}$, with k and n relatively prime. Thus, we have $\phi(n)$ closed geodesics consisting of the points of period n, where ϕ is Euler's totient function.

The number of geodesics in $\mathcal{C}$, consisting of the points of period at most n is $\Phi(n) := \sum_{m=1}^{n} \phi(m) = (3/\pi^2) \cdot n^2 + O(n \log n)$. See [HW38], Theorem 330.

We consider the subgroups of affine diffeomorphisms of $\mathcal{C}$, generated by powers of T. For $n \in \mathbb{N}$ let $\mathcal{F}_n$ be the set of rational rotation numbers with denominator at most n. Thus, $\mathcal{F}_n := \{\, (k, l) \in \mathbb{N}^2 \,|\, \gcd(k, l) = 1, k < l \leq n \,\}$, and $|\mathcal{F}_n| = \Phi(n) \leq n^2$. The map of the unit interval to itself, $x \mapsto \{Nx\}$, is N-to-1 and sends $\mathcal{F}_n$ to itself. In particular, the points of period at most n under T^N lie on $N\Phi(n)$ closed geodesics in $\mathcal{C}$.

The translation surface $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ with the marked point $(0, 0)$ is the *standard torus*. Any flat torus is affinely equivalent to $\mathbb{T}$, we thus restrict our considerations to the standard torus. The group of affine diffeomorphisms of $\mathbb{T}$ is $\mathrm{SL}(2, \mathbb{Z})$, generated by the horizontal and the vertical Dehn twists, T_h and T_v respectively. We have $T_h : (x, y) \mapsto (x + y \mod 1, y)$ and $T_v : (x, y) \mapsto (x, y + x \mod 1)$. The points $(x, y) \in \mathbb{T}$ which are periodic with respect to $\mathrm{SL}(2, \mathbb{Z})$ are the rational points $(x, y) \in \mathbb{Q}^2/\mathbb{Z}^2$. The set of points which are periodic of period at most n under T_v and of period at most m under T_h is the intersection of the horizontal and vertical closed geodesics that we have just considered. The cardinality of this set is asymptotic to $(9/\pi^4) \cdot m^2 n^2$, as $m, n \to \infty$.

Let θ be a parabolic direction on a translation surface $\mathcal{S}$. Using the preceding material, we speak of *rational closed geodesics*, their *periods* and their *rotation numbers*. Note that the periodic points of period n under the restriction of ϕ_θ to the cylinder $\mathcal{C}_i$ lie on $N_i \phi(n)$ rational geodesics of $\mathcal{C}_i$. The set of rotation numbers of these geodesics is $\mathcal{F}_n$.

Theorem 7. *Let $\mathcal{S}$ be a translation surface, and let α, β be a pair of transversal parabolic directions. Then there exist positive integers M and N, depending only on the ratios of the parameters of the cylinder decompositions, so that the following statements hold.*

(i) *If $\mathcal{S}$ has more than M periodic points with respect to $\mathit{Aff}_{\alpha,\beta}(\mathcal{S})$, then $\mathcal{S}$ is arithmetic.*

(ii) *If $\mathcal{S}$ has an $\mathit{Aff}_{\alpha,\beta}(\mathcal{S})$-periodic point of period greater than N, then $\mathcal{S}$ is arithmetic.*

Theorem 7 follows from several technical lemmas and propositions. By the remarks in § 2.2, we assume without loss of generality that α, β are the coordinate directions. We use labels v and h to refer to the *vertical* and the *horizontal* directions respectively. From now until Proposition 4 the standing assumption is that both coordinate directions are parabolic. A *rectangle* in $\mathcal{S}$ is a connected component of the intersection $\mathcal{C}_i^h \cap \mathcal{C}_j^v$. The interior of any rectangle is isometric to the Euclidean rectangle $(0, w_j^v) \times (0, w_i^h)$. Let $\mu_{i,j}$ be the number of rectangles formed by this intersection. We denote the rectangles by $\mathcal{R}_{i,j}^l, 1 \le l \le \mu_{i,j}$. The (essentially disjoint) decomposition

$$\mathcal{S} = \cup_{i=1}^{k(h)} \cup_{j=1}^{k(v)} \cup_{l=1}^{\mu_{i,j}} \mathcal{R}_{i,j}^l \tag{1}$$

implies

$$\sum_{i=1}^{k(h)} \sum_{j=1}^{k(v)} \mu_{i,j} w_i^h w_j^v = \mathrm{Area}(\mathcal{S})\,.$$

Lemma 1. *For $1 \le i \le k(h)$ (resp. $1 \le j \le k(v)$) let H_i (resp. V_j) be a finite set of closed geodesics in $\mathcal{C}_i^h$ (resp. $\mathcal{C}_j^v$). Then*

$$|(\cup_{i=1}^{k(h)} H_i) \cap (\cup_{j=1}^{k(v)} V_j)| = \sum_{i=1}^{k(h)} \sum_{j=1}^{k(v)} \mu_{i,j} |H_i||V_j|. \tag{2}$$

Proof. The intersection of a longitude in $\mathcal{C}_i^h$ with a longitude in $\mathcal{C}_j^v$ consists of $\mu_{i,j}$ points. ■

To simplify the notation, we denote the subgroups of $\mathrm{Aff}(\mathcal{S})$ generated by the diffeomorphisms ϕ_h and ϕ_v by A and B, respectively. Let $\langle A, B\rangle$ be the subgroup generated by A and B.

If f and g are functions of natural argument, we use the notation $f \le\sim g$ to indicate that $f(n) \le g(n)$ for n sufficiently large, and $f \sim g$ means that $f(n)/g(n) \to 1$ as n goes to infinity. The proposition below is immediate from Lemma 1 and the preceding remarks.

Proposition 1. *For any subgroup $G \subset Aff(\mathcal{S})$ let $P^G \subset \mathcal{S}$ be the set of G-periodic points. Denote by $P_n^G \subset P^G$ the subset of points of periods at most n. Then*

(i) *For any m and n we have*

$$|P_m^A \cap P_n^B| = \Phi(m)\Phi(n) \sum_{i=1}^{k(h)} \sum_{j=1}^{k(v)} \mu_{i,j} N_i^h N_j^v \,. \tag{3}$$

(ii) *We have*

$$|P_m^A \cap P_n^B| \sim \frac{9}{\pi^4} \left(\sum_{i=1}^{k(h)} \sum_{j=1}^{k(v)} \mu_{i,j} N_i^h N_j^v \right) m^2 n^2. \tag{4}$$

Corollary 3. *We have the asymptotic inequality*

$$|P_n^{\langle A,B\rangle}| \leq\sim \frac{9}{\pi^4} \left(\sum_{i=1}^{k(h)} \sum_{j=1}^{k(v)} m_{i,j} N_i^h N_j^v \right) n^4 \,. \tag{5}$$

Proof. Use equation (4) and the inclusion $P_n^{\langle A,B\rangle} \subset P_n^A \cap P_n^B$. ■

We state a few immediate consequences of the propositions above, then formulate and prove a few technical lemmas.

If α, β is a pair of transversal parabolic directions on $\mathcal{S}$, we denote by $\mathrm{Aff}_{\alpha,\beta}(\mathcal{S}) \subset \mathrm{Aff}(\mathcal{S})$ the subgroup generated by the diffeomorphisms ϕ_α and ϕ_β. A subgroup $G \subset \mathrm{Aff}(\mathcal{S})$ is *basic* if its intersection with some $\mathrm{Aff}_{\alpha,\beta}(\mathcal{S})$ has finite index in the latter.

Corollary 4. *Let $\mathcal{S}$ be a prelattice translation surface. Let $G \subset Aff(\mathcal{S})$ be any basic subgroup. Then*

(i) *The sets P_n^G are finite.*

(ii) *The cardinality $|P_n^G|$ grows at most quartically in n, as n tends to infinity.*

(iii) *The set P^G is infinite if and only if it contains periodic points of arbitrarily large periods.*

Lemma 2. *There exist constants c_0 and n_0, depending only on the parameters of the transversal pair of parabolic decompositions of $\mathcal{S}$, such that any finite orbit of $\langle A, B\rangle$ of cardinality $n > n_0$ contains points of periods at least $c_0\sqrt[4]{n}$ with respect to each of A and B.*

Proof. We choose $c_0 > 0$ so that

$$c_0^4 = (\frac{3}{\pi^2} + 1)^{-2} \left(\sum_{i=1}^{k(h)} \sum_{j=1}^{k(v)} \mu_{i,j} N_i^h N_j^v \right)^{-1} .$$

By equations (4) and (5), there exists $m_0 \in \mathbb{N}$ such that for $m > m_0$ one has

$$|P_m^{\langle A,B \rangle}| < c_0^{-1} m^4. \tag{6}$$

Rewriting this inequality as $m^4 > c_0 |P_m^{\langle A,B \rangle}|$ and setting $n_0 = c_0 m_0^4$, we obtain the claim. ∎

If x_i^α, $1 \le i \le k(\alpha)$, are any parameters of the cylinders of a parabolic direction α, we denote by $x_{\min}^\alpha$ and $x_{\max}^\alpha$ the smallest and the biggest ones.

Lemma 3. *There is $m_0 \in \mathbb{N}$, depending only on the parameters of the horizontal and vertical decompositions of $\mathcal{S}$, such that the following holds:*
If a finite $\langle A, B \rangle$-orbit contains a point of A-period $m > m_0$, then the A-orbit of this point contains a point whose B-period is at least $[2(m \frac{w_{\min}^v}{\ell_{\max}^h} - 1)/N_{\max}^v]^{\frac{1}{2}}$.

Proof. Suppose that $\mathcal{O}$ is a finite $\langle A, B \rangle$-orbit, and $s \in \mathcal{O}$ is of A-period m. We assume, without loss of generality, that $s \in \mathcal{C}_1^h$, and let $L \subset \mathcal{C}_1^h$ be the closed geodesic containing s. It intersects at least one vertical cylinder. Again, we can assume that L intersects $\mathcal{C}_1^v$. Let $\mathcal{R} \subset \mathcal{C}_1^h \cap \mathcal{C}_1^v$ be one of the rectangles.

The distance between consecutive points of $A \cdot s$ is ℓ_1^h/m. Hence the number of points of the orbit $A \cdot s$ in the interval $L \cap R$ is at least $\lfloor w_1^v/(\ell_1^h/m) \rfloor \ge (m w_1^v/\ell_1^h) - 1$. The interval $L \cap \mathcal{R}$ intersects each closed geodesic of $\mathcal{C}_1^v$ exactly once. Hence $\{A \cdot s\} \cap \mathcal{R}$ intersects at least $(m w_1^v/\ell_1^h) - 1$ distinct closed geodesics of $\mathcal{C}_1^v$.

Let $X \subset [0,1] \cap \mathbb{Q}$ be the set of rotation numbers of these geodesics with respect to the basic Dehn twist of $\mathcal{C}_1^v$. Recall that the closed geodesics in a cylinder are parametrized by their rotation numbers. Set $N := N_1^v$ and $Y := \{ \{Nx\} \,|\, x \in X \}$. Then Y is the set of rotation numbers of these geodesics with respect to the diffeomorphism ϕ_v of $\mathcal{S}$. Let n be the smallest positive integer such that $Y \subset \mathcal{F}_n$. Then n is the largest B-period of the geodesics in question. Using that $|Y| \ge |X|/N$ and the obvious upper bound for $|\mathcal{F}_n|$, we have

$$\frac{m \frac{w_{\min}^v}{l_{\max}^h} - 1}{N_{\max}^v} < \frac{n^2}{2}. \tag{7}$$

Taking $m > l_{\max}^h / w_{\min}^v$, we obtain the claim. ∎

The following two lemmas put the statements above into a more suitable form. The proofs are straightforward, and we leave them to the reader.

Lemma 4. *There exist $c_1 > 0$ and $n_0 \in \mathbb{N}$ depending only on the parameters of the two decompositions of $\mathcal{S}$, and such that the following holds:*
Let $n > n_0$, and let $\mathcal{O} \subset \mathcal{S}$ be an $\langle A, B\rangle$-periodic orbit of cardinality at least $c_1 n^8$. Then $\mathcal{O}$ contains a point, s, with the following properties:

- (i) *The A-period of s is at least n;*
- (ii) *Every vertical cylinder which intersects nontrivially the horizontal cylinder containing s contains a point of $B \cdot \{A \cdot s\}$, whose B-period is at least n.*

Lemma 5. *There exist $c_2, c_3 > 0$ and $n_0 \in \mathbb{N}$ so that the following holds:*

- (i) *Let $n \geq n_0$, and let $\mathcal{O} \subset \mathcal{S}$ be a finite $\langle A, B\rangle$-orbit of cardinality greater than $c_2 n^4$. Then $\mathcal{O}$ contains a point of A-period at least n, and a point of B-period at least n.*
- (ii) *Suppose that an $\langle A, B\rangle$-periodic orbit $\mathcal{O}$ contains an A-periodic point, s, of period at least $c_3 n^2$ with $n \geq n_0$. Then every vertical cylinder which intersects nontrivially the horizontal cylinder containing s contains a point of $A \cdot s$ whose B-period is greater than or equal to n.*

Note that in the lemmas above A and B are interchangeable. The following proposition is the main technical result.

Proposition 2. *Let the assumptions be as above. There exist $c_4 > 0$, $n_0 \in \mathbb{N}$ and $d \in \mathbb{N}$, so that the following holds:*
Let $\mathcal{O} \subset \mathcal{S}$ be a finite $\langle A, B\rangle$-orbit of cardinality greater than $c_4 n^{2^{d+2}}$ with $n \geq n_0$. Then in every horizontal (resp. vertical) cylinder there is a point of $\mathcal{O}$ whose A-period (resp. B-period) is at least n.

Proof. We sketch the proof, leaving the details to the reader. In particular, we will pretend that in the lemmas above the constants c_i are equal to one and that all the thresholds n_0 are the same. The latter can always be achieved by taking the biggest threshold of them all. The former can be arranged by (for instance) increasing the exponents in the lemmas by an arbitrarily small, but positive amount, and raising the threshold. By the first claim of Lemma 5, there is a horizontal cylinder, $\mathcal{C}_1^h$, such that $\mathcal{O} \cap \mathcal{C}_1^h$ contains a finite A-orbit of cardinality at least n^{2^d}. Then every vertical cylinder intersecting $\mathcal{C}_1^h$ contains a B-periodic point of $\mathcal{O}$, whose period is greater than or equal to $n^{2^{d-1}}$. See the second claim of Lemma 5. If the union of these vertical cylinders with $\mathcal{C}_1^h$ covers $\mathcal{S}$, then we proved the claim. Otherwise, we continue the inductive argument. At each consecutive iteration of the argument we just lose a factor of 2 in the exponent. Since $\mathcal{S}$ is connected, after a finite number of steps we exhaust the surface. ∎

4. Large Periodic Orbits Imply Arithmeticity

We need a few more technical propositions about transversal pairs of parabolic directions. We continue to use the convention of § 2, and restrict the exposition to the pair of coordinate directions.

4.1. **Commensurability of Parameters.**

Lemma 6. *Let $\mathcal{C}_i^v$ and $\mathcal{C}_j^h$ be two cylinders such that $\mathcal{C}_i^h \cap \mathcal{C}_j^v \neq \emptyset$. Let $\mathcal{R} \subset \mathcal{C}_i^v \cap \mathcal{C}_j^h$ be one of the rectangles in the intersection. Suppose that two distinct points of $\mathcal{R}$ lie in the same A-orbit and in a finite $\langle A, B\rangle$-orbit. Then $w_j^v/\ell_i^h \in \mathbb{Q}$.*

Proof. We denote by (x, y) the natural coordinates in $\mathcal{R}$. Then $0 \leq x \leq w_j^v$, $0 \leq y \leq w_i^h$. Let $s = (x, y)$ and $s' = (x', y')$ be the two points in question. By assumption, there is $0 \neq n \in \mathbb{Z}$ so that

$$x' = x + n\frac{y}{w_i^h}\ell_i^h, \quad y' = y\,. \tag{8}$$

Since s is A-periodic, $\frac{y}{w_i^h} \in \mathbb{Q}$. On the other hand, since s and s' are both B-periodic, they belong to rational closed geodesics in $\mathcal{C}_j^v$. Thus, both x/w_j^v and x'/w_j^v are rational numbers. Hence

$$\frac{x' - x}{w_j^v} = n\frac{y}{w_i^h}\frac{\ell_i^h}{w_j^v} \in \mathbb{Q}. \tag{9}$$

Since, as we already noted, $\frac{y}{w_i^h} \in \mathbb{Q}$, we obtain the claim. ∎

Remark 1. The interchange of A and B in the assumptions of the preceding Lemma yields $w_i^h/\ell_j^v \in \mathbb{Q}$.

The following technical proposition is crucial. It is also of independent interest.

Proposition 3. *Let the notation be as in Proposition 2. Set*

$$m = m(A, B) = \max\{\frac{\ell_{\max}^h}{w_{\min}^v}, \frac{\ell_{\max}^v}{w_{\min}^h}\}. \tag{10}$$

Suppose that $\mathcal{S}$ has an $\langle A, B\rangle$-periodic point of period greater than or equal to $c_4 m^{2^{d+2}}$. Then

(i) *All numbers w_j^v/ℓ_i^h and w_i^h/ℓ_j^v are rational;*

(ii) *The lengths ℓ_i^h, $1 \leq i \leq k(h)$, are commensurate, and the lengths ℓ_j^v, $1 \leq j \leq k(v)$, are commensurate, as well.*

(iii) *The widths w_i^h, $1 \leq i \leq k(h)$, are commensurate, and the widths w_j^v, $1 \leq j \leq k(v)$, are commensurate, as well.*

Proof. Let $\mathcal{O}$ be the $\langle A, B\rangle$-orbit in question. By Proposition 2, every horizontal (resp. vertical) cylinder contains a point of $\mathcal{O}$ of A-period (resp. B-period) greater than m. In view of equation (10), every rectangle $\mathcal{R} \subset \mathcal{C}_i^h \cap \mathcal{C}_j^v$ contains (at least) two points, s and s' of $\mathcal{O}$, such that $s' = \phi_h \cdot s$ (resp. $s' = \phi_v \cdot s$). Lemma 6 and Remark 1 imply our first claim.

Suppose that $\mathcal{C}_i^h$ and $\mathcal{C}_{i'}^h$ intersect the same vertical cylinder, $\mathcal{C}_j^v$. We have already proved that w_j^v/ℓ_i^h and $w_j^v/\ell_{i'}^h$ are rational. Thus ℓ_i^h and $\ell_{i'}^h$ are commensurate. In view of the connectedness of $\mathcal{S}$, for any pair $\mathcal{C}, \mathcal{C}'$ of horizontal cylinders, there is a sequence $\mathcal{C}_1, \dots, \mathcal{C}_k$ of horizontal cylinders such that $\mathcal{C} = \mathcal{C}_1^h$, $\mathcal{C}' = \mathcal{C}_k^h$, and every two consecutive cylinders of the sequence intersect a common vertical cylinder. Thus $\ell(\mathcal{C})/\ell(\mathcal{C}')$ is rational. The same argument works for vertical cylinders, proving our second claim. The proof of the last claim is essentially identical, and we leave it to the reader. ∎

From now on we drop our convention that our transversal parabolic directions is the pair x, y, and explicitly formulate all of our assumptions. The following proposition is of independent interest.

Proposition 4. *Let $\mathcal{S}$ be a translation surface. Let α and β be transversal parabolic directions. Let w_i^α, $1 \leq i \leq k(\alpha)$, and w_j^β, $1 \leq j \leq k(\beta)$, be the widths of the respective cylinders. Suppose that the numbers w_i^α are all commensurate, and the numbers w_j^β are commensurate, as well. Then $\mathcal{S}$ is an arithmetic translation surface.*

Proof. Replacing $\mathcal{S}$ by an affinely equivalent surface, we assume without loss of generality that α and β are the coordinate directions. In what follows we use h for α and v for β.

Changing the translation structure $\mathcal{S}$ by a diagonal transformation, if need be, we ensure that all the widths w_i^h and w_i^v are rational. Now we use the relations

$$\ell_j^v = \sum_{i=1}^{k(h)} \mu_{i,j} w_i^h \ , \quad \ell_i^h = \sum_{j=1}^{k(v)} \mu_{i,j} w_j^v . \tag{11}$$

Thus, all the lengths ℓ_i^h, ℓ_j^v are rational, as well. Applying a homothety to $\mathcal{S}$, we make all these parameters integral. By Theorem 5.5 of [GJ00], $\mathcal{S}$ is arithmetic. ∎

Proposition 4 is a special case of a more general statement: A translation surface all of whose parameters are commensurate is arithmetic [GJ00].

4.2. **Proofs of Theorems 1, 8, 7.** First, we prove the main quantitative result.

Proof of Theorem 7. We begin with the latter claim. Let $m = m(\alpha, \beta)$ be given by equation (10). By Proposition 3, if $N \geq m(\alpha, \beta)$, then the assumptions of Proposition 4 are satisfied. Hence, $\mathcal{S}$ is arithmetic.

By Corollary 3, the existence of M periodic points implies the existence of a periodic point of period at least $N = \text{const}\sqrt[4]{M}$. This holds only for M greater than a certain threshold, depending on the data, which also determines the constant in question. Therefore, if $\mathcal{S}$ satisfies the assumption of claim (i), then it satisfies the assumption of claim (ii), as well. ∎

The hypothesis of the first claim of Theorem 7 implies that of the second, with $M = N$. By this observation and the preceding argument, we reformulate Theorem 7 as follows.

Corollary 5. *Let $\mathcal{S}$ be a prelattice translation surface. Then there exists $n \in \mathbb{N}$, determined from any pair of transversal parabolic directions, so that the following holds:*
If $\mathcal{S}$ has at least n periodic points, then it is arithmetic.

Theorem 8. *Let $\mathcal{S}$ be a translation surface, and let $G \subset \mathrm{Aff}(\mathcal{S})$ be a basic subgroup. If the set of G-periodic points of $\mathcal{S}$ is infinite, then $\mathcal{S}$ is an arithmetic translation surface.*

Proof. If $H \subset G \subset \mathrm{Aff}(\mathcal{S})$ is a tower of subgroups, then $P \subset P^G \subset P^H$. The claim now follows directly from Theorem 7. ∎

Proof of Theorem 1. We have already proved that a nonarithmetic (pre)lattice translation surface has a finite number of periodic points. Note that the cone points are necessarily periodic! Now let $\mathcal{S}$ be an arithmetic translation surface. Replacing $\mathcal{S}$ by an equivalent translation surface, if need be, we can assume that $\mathcal{S}$ admits a balanced translation covering of the standard torus $\mathbb{T}$. By Theorem 6, $\mathrm{Aff}(\mathcal{S})$ is commensurable with $\mathrm{Aff}(\mathbb{T}) = \mathrm{SL}(2, \mathbb{Z})$.

The set $\mathbb{Q}^2/\mathbb{Z}^2$ of rational points is dense in $\mathbb{T}$. But it is also the set of $\mathrm{SL}(2, \mathbb{Z})$-periodic points. The set of periodic points in $\mathcal{S}$ is the preimage of $\mathbb{Q}^2/\mathbb{Z}^2$ under the covering, hence it is dense in $\mathcal{S}$. ∎

4.3. **Proof of Theorem 2.** It suffices to prove the claim under the convention that the coordinate directions are parabolic. Let $X \subset \mathcal{S}$ be an infinite closed $\langle A, B\rangle$-invariant subset. Suppose that X contains a 'coordinate' closed geodesic, L. We can assume without loss of generality that L is vertical. Let $\mathcal{R}$ be one of the rectangles intersecting L. The set of ϕ_h-rotation numbers of the points in the vertical interval $\mathcal{R} \cap L$ is $(0, 1)$. For every point $z \in \mathcal{R} \cap L$ of irrational rotation number, the ϕ_h-orbit of z is dense in the horizontal geodesic containing z. Since X is closed, it contains this geodesic. Since irrational numbers are dense in $(0, 1)$, the horizontal cylinder containing $\mathcal{R} \cap L$ belongs to X. Since $\mathcal{R}$ was chosen arbitrarily, X contains the union, X_1, of the horizontal cylinders intersecting L. Replacing L by a horizontal closed geodesic in X_1, we conclude that X contains the union, X_2, of the vertical cylinders intersecting X_1. This inductive process produces a sequence $L \subset X_1 \subset X_2 \subset \cdots \subset X$, where either $X_{i+1} \setminus X_i$ contains at least one coordinate cylinder, or $X_i = \mathcal{S}$. Since the number of cylinders is finite, $X = \mathcal{S}$.

It remains to prove that X contains a vertical or a horizontal closed geodesic. Let $\mathcal{R}$ be a coordinate rectangle, and let $z = (x, y) \in \mathcal{R}$ be an arbitrary point. Denote by $r_h(z)$ and $r_v(z)$ the ϕ_h and ϕ_v rotation numbers respectively. Note that r_h is a locally linear function of y alone; similarly for r_v with respect to x. Since X is infinite, there is at least one $\mathcal{R}$ such that the set $X \cap \mathcal{R}$ is infinite. Denote by $R_h(X)$ and $R_v(X)$ the sets of horizontal and vertical rotation numbers of the points in $X \cap \mathcal{R}$. Since $X \cap \mathcal{R}$ is closed, both $R_h(X)$ and $R_v(X)$ are closed subsets of $[0, 1]$.

If $R_h(X) \cup R_v(X)$ contains an irrational number, then there exists a closed (vertical, without loss of generality) geodesic, L, with an irrational rotation number, containing a point of X. Then, by minimality of irrational rotations, $L \subset X$. Assume then that $R_h(X) \cup R_v(X) \subset \mathbb{Q}$. There are two possibilities: the set $R_h(X) \cup R_v(X)$ is infinite or finite.

Assume first that it is finite. Then there is a closed (horizontal, without loss of generality) geodesic, L, with a rational rotation number which contains infinitely many points of X. Since $X \cap L \cap \mathcal{R}$ is infinite, we have infinitely many vertical rotation numbers, contrary to the assumption.

Suppose now that $R_h(X) \cup R_v(X)$ is infinite. Assume, without loss of generality, that $|R_h(X)| = \infty$. Let $r \in \mathbb{Q}$ be an accumulation point of $R_h(X)$. Then there is an infinite sequence of points $z_n \in X \cap \mathcal{R}$ converging to $z \in X \cap \mathcal{R}$, and $r = r_h(z)$. Set $r_h(z_n) = p_n/q_n$. Since $p_n/q_n \to r$, as $n \to \infty$, the sequence q_n is unbounded. Let L_n (resp. L) be the horizontal closed geodesic containing z_n (resp. z). The distance between the consecutive points of the orbit $A \cdot z_n \subset L_n$ is of the order of q_n^{-1}. Since L_n converges to L, we conclude that L consists of accumulation points of X. Since X is closed, $L \subset X$. ∎

5. Prelattice Surfaces

5.1. **Rational Points.** Let $\mathcal{S}$ be a prelattice translation surface, and let α, β be a transversal pair of parabolic directions. Let $\mathcal{R} \subset \mathcal{C}_i^h \cap \mathcal{C}_j^v$ be one of the *parallelograms* $\mathcal{R}_{i,j}^l$ of the associated decomposition. We change the affine structure of $\mathcal{S}$ by any $g \in \mathrm{SL}(2, \mathbb{R})$ which sends α and β to the coordinate directions. Let x, y be the Euclidean coordinates such that the interior of $\mathcal{R}$ is parametrized by $(0 < x < w_v, 0 < y < w_h)$. In view of possible identifications on the boundary, $\mathcal{R}$ itself may not be isometric to the Euclidean rectangle $[0, w_v] \times [0, w_h]$. However, there is a mapping $[0, w_v] \times [0, w_h] \to \mathcal{R}$, inducing an isometry of $(0, w_v) \times (0, w_h)$ onto $\mathrm{Int}(\mathcal{R})$.

Reversing the affine equivalence above, we return to the original directions α, β. This construction yields an affine mapping $f_{\mathcal{R}} : [0, w_v] \times [0, w_h] \to \mathcal{R}$, which is onto, preserves orientation and area, and is an affine isomorphism of $(0, w_v) \times (0, w_h)$ and $\mathrm{Int}(\mathcal{R})$.

Definition 5. Let $\mathcal{S}$ be a translation surface, and let α, β be a transversal pair of parabolic directions. Let $z \in \mathcal{S}$ be an arbitrary point, let $\mathcal{R}$ be a parallelogram of the decomposition (1), containing z, and let $f_{\mathcal{R}} : [0, w_v] \times [0, w_h] \to \mathcal{R}$ be the corresponding affine mapping. Then z is *rational with respect to the pair* α, β if $z = f_{\mathcal{R}}(x, y)$, where $x/w_v, y/w_h \in \mathbb{Q}$. A point $z \in \mathcal{S}$ is *rational*, if there is a pair of transversal parabolic directions such that z is rational with respect to it.

We use *irrational* for all points that are not rational in the sense of Definition 5. If $\mathcal{R}$ is a parallelogram of the decomposition (1), we denote by $\mathcal{R}_{\mathbb{Q}}$ the set of its rational points. We use the notation $\mathcal{S}_{\mathbb{Q}}^{\alpha,\beta}$ for the set of rational points with respect to the pair α, β, and $\mathcal{S}_{\mathbb{Q}}$ for the set of rational points of $\mathcal{S}$. Note that the concepts of rational and irrational points are applicable only to *prelattice* surfaces.

We leave the straightforward proof of the following proposition to the reader.

Proposition 5. *Let $\mathcal{S}$ be a prelattice translation surface, and let α, β be a pair of transversal parabolic directions for $\mathcal{S}$. Let $s \in \mathcal{S} \setminus C(\mathcal{S})$. Then the following statements are equivalent.*

(i) *The point s is rational with respect to α, β.*
(ii) *The directions α, β are parabolic for the punctured surface $(\mathcal{S}; s)$.*
(iii) *The point s is periodic with respect to both ϕ_α and ϕ_β.*
(iv) *The point s is an intersection point of two rational geodesics, with directions α and β respectively.*

5.2. **Marking Points.** We continue with the proofs of the claims of the Introduction.

Proof of Theorem 3. The set $\mathcal{S}_{\mathbb{Q}}$ is the union of $\mathcal{S}_{\mathbb{Q}}^{\alpha,\beta}$ over all transversal pairs of parabolic directions. Each set $\mathcal{S}_{\mathbb{Q}}^{\alpha,\beta}$ is countable and dense in $\mathcal{S}$. A parabolic direction is periodic, thus the set of parabolic directions of any translation surface is at most countable. Any $s \in P(\mathcal{S})$ is periodic with respect to every $\mathrm{Aff}_{\alpha,\beta}(\mathcal{S}) \subset \mathrm{Aff}(\mathcal{S})$, hence $P(\mathcal{S}) \subset \cap_{\alpha,\beta} \mathcal{S}_{\mathbb{Q}}^{\alpha,\beta} \subset \cup_{\alpha,\beta} \mathcal{S}_{\mathbb{Q}}^{\alpha,\beta} = \mathcal{S}_{\mathbb{Q}}$. Claim (a) follows.

If $\mathcal{S}$ is arithmetic, then, by the proof of Theorem 1, $\mathcal{S}_{\mathbb{Q}} \subset \mathcal{S}_{\mathbb{Q}}^{\alpha,\beta} \subset P(\mathcal{S})$ for any transversal pair α, β, hence $\mathcal{S}_{\mathbb{Q}} \subset P(\mathcal{S})$. If $\mathcal{S}_{\mathbb{Q}} \subset P(\mathcal{S})$, then the set $P(\mathcal{S})$ is infinite, hence, by Theorem 8, $\mathcal{S}$ is arithmetic. This proves claim (b).

It remains to prove claim (c). If $s \in \mathcal{S}_{\mathbb{Q}}$, then $(\mathcal{S}, s)$ is a prelattice surface, by Proposition 5. By the same Proposition, $\Gamma(\mathcal{S}; s)$ is not a prelattice if and only if for any transversal parabolic pair α, β we have $s \in \mathcal{S} \setminus \mathcal{S}_{\mathbb{Q}}^{\alpha,\beta}$, i.e., $s \in \mathcal{S} \setminus \mathcal{S}_{\mathbb{Q}}$. ■

Proof of Corollary 1 The first claim is in [HS00]. The second claim follows from the first and Theorem 3. The third is contained in part (c) of Theorem 3. ■

As a byproduct of Corollary 1, we obtain a new characterization of arithmetic translation surfaces. See [GJ00] for other characterizations.

Corollary 6. *Let $\mathcal{S}$ be a lattice translation surface. Then $\mathcal{S}$ is arithmetic if and only if the following dichotomy holds:*

For any $s \in \mathcal{S}$ the surface $(\mathcal{S}; s)$ is either a (necessarily arithmetic) lattice surface, or it is not a prelattice surface.

Theorem 9. *A. Let $\mathcal{S}$ be a prelattice translation surface, and let $s \in \mathcal{S}$. Let $p : \mathcal{R} \to (\mathcal{S}; s)$ be a balanced affine covering. Then the following trichotomy holds.*

(i) *The surface $\mathcal{R}$ is a prelattice surface, and the groups $\Gamma(\mathcal{R}), \Gamma(\mathcal{S})$ are commensurable in the wide sense if and only if $s \in P(\mathcal{S})$.*

(ii) *The surface $\mathcal{R}$ is a prelattice surface, and the group $\Gamma(\mathcal{R})$ is commensurable in the wide sense to a prelattice of infinite index in $\Gamma(\mathcal{S})$ if and only if $s \in \mathcal{S}_{\mathbb{Q}} \setminus P(\mathcal{S})$.*

(iii) *The surface $\mathcal{R}$ is not a prelattice surface if and only if $s \in \mathcal{S} \setminus \mathcal{S}_{\mathbb{Q}}$.*

B. Suppose further that p is a balanced translation covering. Then the groups in question are commensurable (in the "narrow" sense).

Proof. It suffices to prove the theorem under the assumption that $p : \mathcal{R} \to (\mathcal{S}; s)$ is a balanced translation covering. By Theorem 6, the groups $\Gamma(\mathcal{R})$ and $\Gamma(\mathcal{S}; s)$ are commensurable. Hence, all but one our claims follow from Theorem 3. The remaining claim concerns $\Gamma(\mathcal{S}; s)$ for $s \in \mathcal{S}_{\mathbb{Q}} \setminus P(\mathcal{S})$. Let α, β be a transversal pair of parabolic directions such that $s \in \mathcal{S}_{\mathbb{Q}}^{\alpha,\beta}$. Then the stabilizer $\Gamma_s^{\alpha,\beta} \subset \Gamma(\mathcal{S}; s)$ is a prelattice. Therefore $\Gamma(\mathcal{S}; s) \subset \Gamma(\mathcal{S})$ is a prelattice as well. But, since the orbit $\mathrm{Aff}(\mathcal{S}) \cdot s$ is infinite, the index of $\Gamma(\mathcal{S}; s)$ in $\Gamma(\mathcal{S})$ is infinite. ∎

A translation surface, $\mathcal{S}$, can be viewed as a closed Riemann surface, S, equipped with a holomorphic 1-form, say ω, see say [MT01]; we write $\mathcal{S} = (S, \omega)$. The cone points $C(\mathcal{S})$ is the set of zeros of ω. Let $p : R \to S$ be a branched covering of Riemann surfaces, and let α be the pull-back of ω. Let $\mathcal{R}$ be the translation surface corresponding to (R, α). Then $C(\mathcal{R})$ is the union of $p^{-1}(C(\mathcal{S}))$ and the set of the ramification points of $p : R \to S$.

Proposition 6. *Let $\mathcal{S}$ be a translation surface without marked points, and let $s \in \mathcal{S} \setminus C(\mathcal{S})$. For any $n > 1$ there exists a translation surface $\mathcal{R}$ without marked points, and a balanced m-to-1, $m \geq n$, translation covering $p : \mathcal{R} \to (\mathcal{S}; s)$.*

Proof. Let (S, ω) be the Riemann surface with 1-form corresponding to $\mathcal{S}$. It suffices to exhibit branched coverings of Riemann surfaces, $p : R \to S$, of arbitrarily high degrees such that the branch locus of p is contained in $C(\mathcal{S}) \cup \{s\}$ and the set $p^{-1}(s) \subset \mathcal{R}$ belongs to the ramification locus of p. It is well-known that such coverings exist [FK92]. ∎

Proof of Theorem 4. Let $\mathcal{S}$ be a nonarithmetic lattice surface, and let α, β be a transversal pair of periodic directions. Let $s \in \mathcal{S}_{\mathbb{Q}}^{\alpha,\beta} \setminus P(\mathcal{S})$, which is nonempty, by Theorems 1 and 3. Set $\Gamma' = \Gamma((S; s))$. Then Γ' is a prelattice of infinite index. By Theorem 6, each one of the infinitely many balanced translation coverings, $p : \mathcal{R} \to (\mathcal{S}; s)$, provided by Proposition 6 gives an almost-realization of Γ'.

Let now $\mathcal{S}$ be arithmetic, and let $\Gamma' \subset \Gamma(\mathcal{S})$ be a prelattice subgroup of infinite index. Suppose that Γ' is almost realizable, and let $\mathcal{R}$ provide an almost-realization of Γ'. Thus, $\Gamma'' = \Gamma(\mathcal{R}) \cap \Gamma'$ has finite index in both groups. In view of arithmeticity, the trace of any $g \in \Gamma$ is rational. Since Γ'' contains hyperbolic elements, by Theorem 28 of [KS00], the holonomy field of $\mathcal{R}$ is $\mathbb{Q}$. Therefore, by Theorem 5.5 of [GJ00], $\mathcal{R}$ is arithmetic. But $\Gamma(\mathcal{R})$ is not a lattice! ∎

Corollary 7. *Let $\mathcal{S}$ be a nonarithmetic lattice translation surface. Let α, β be a transversal pair of parabolic directions for $\mathcal{S}$. Then there exists a translation covering $p : \mathcal{R} \to \mathcal{S}$ where $\mathcal{R}$ is a nonlattice, prelattice translation surface, and α, β are parabolic directions for $\mathcal{R}$.*

Corollary 8. *Let $\mathcal{S}$ be a prelattice, but nonlattice translation surface. Then $\Gamma(\mathcal{S})$ is not commensurable (in the wide sense) with any subgroup of $SL(2, \mathbb{Z})$.*

Proof of Corollaries 7,8, 2. Corollary 7 follows from the proof of Theorem 4; Corollary 8 is immediate from the statement of Theorem 4. The nontrivial implication of Corollary 2 follows from Corollary 8. ∎

5.3. **Examples and Applications.** In this subsection we illustrate and augment the preceding material, and apply it to polygonal billiards. We begin with an infinite family of prelattice subgroups of $\mathrm{SL}(2, \mathbb{Z})$.

Example 1. For $m, n \in \mathbb{N}$, let $G_{m,n} \subset \mathrm{SL}(2, \mathbb{Z})$ be the group generated by the parabolic matrices $\mu = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$ and $\nu = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}$.

Let $\Gamma \subset \mathrm{SL}(2, \mathbb{R})$ be any Fuchsian group. Denote by $K_1(\Gamma)$ (resp. $K_2(\Gamma)$) the smallest field extension of $\mathbb{Q}$ containing $\mathrm{tr}(g)$ (resp. $\mathrm{tr}(g^2)$) for all $g \in \Gamma$. The condition $K_1(\Gamma) = K_2(\Gamma)$ is necessary for Γ to be realizable as a Veech group [HS01]. The groups $G_{m,n}$ obviously satisfy this condition. For $mn > 4$ (resp. $mn \leq 4$) the group $G_{m,n}$ has a fundamental domain in $\mathbb{H}^2$ of infinite (resp. finite) area [B83]. By Corollary 2, $G_{m,n}$ is almost realizable as a Veech group if and only if $mn \leq 4$.

Recall that a polygon, P, is *rational* if its angles are commensurate with π. In the subject of mathematical billiards there is a well known construction that replaces a *rational polygon*, P, by a translation surface, $\mathcal{S} = \mathcal{S}(P)$ and reduces the billiard flow in P to the geodesic flow in $\mathcal{S}$, see [St06, FoKe36, KZ75, Gut84, Gut96, MT01].

Definition 6. Let P be a rational polygon, and let $\mathcal{S}$ be the corresponding translation surface. We say that P is a *lattice polygon* (resp. a *prelattice polygon*) if $\mathcal{S}$ is a lattice (resp. a prelattice) translation surface.

Remark 2. Let P and $\mathcal{S}$ be as above. If $\Gamma(\mathcal{S})$ has a parabolic element, then it also has a hyperbolic element [KS00]. It then follows that P is a prelattice polygon if and only if its translation surface has a parabolic direction.

We will denote by $\Gamma(P)$ the Veech group of the translation surface $\mathcal{S}(P)$, and say that $\Gamma(P)$ is the *Veech group of the rational polygon* P. *Arithmetic polygons* are the polygons P such that $\Gamma(P)$ is an arithmetic lattice. They were investigated in [Gut84] and [GJ00]. In particular, if P tiles the plane under reflections, it is arithmetic; $\mathcal{S}(P)$ is then a flat torus. Veech showed that the right triangle whose smallest angle is π/n is a *nonarithmetic lattice polygon* if $n \neq 4, 6$ [Vch89].

Let $p, q, r \in \mathbb{N}$ be relatively prime. We denote by $T(p, q, r)$ the Euclidean triangle with angles $p\pi/(p+q+r)$, $q\pi/(p+q+r)$, $r\pi/(p+q+r)$. In this notation, the right triangle above is $T(2, n-2, n)$ if n is odd and $T(1, m-1, m)$ if $n = 2m$.

Example 2. Set $T_1 = T(2, 3, 5)$ and $T_2 = T(3, 3, 4)$. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be the corresponding translation surfaces, and let Γ_1 and Γ_2 be the respective Veech groups. By [Vch89], T_1 is a lattice triangle. We will show that T_2 is a prelattice but nonlattice triangle. The surface $\mathcal{S}_1$ is obtained by glueing along parallel sides two copies of the regular pentagon; their vertices are glued into a single point, $C(\mathcal{S}_1)$. The isosceles triangle T_2, with angles $2\pi/5, 3\pi/10, 3\pi/10$, is the "doubling" of T_1 along a side. Accordingly, there is a two-to-one translation covering $p : \mathcal{S}_2 \to \mathcal{S}_1$.

Let o_1, o_2 be the centers of the two pentagons. The covering $p : \mathcal{S}_2 \to (\mathcal{S}_1; o_1, o_2)$ is balanced. By Theorem 6, Γ_2 and $\Gamma((\mathcal{S}_1; o_1, o_2))$ are commensurable. By Proposition 3 of [HS00], Γ_2 is not a lattice. Hence T_2 is not a lattice triangle. By Theorem 9, o_1, o_2 are not periodic points of the lattice surface $\mathcal{S}_1$. Let α, β be the directions of two distinct diagonals of the regular pentagon. They are parabolic [Vch89]. Thus, o_1, o_2 are intersection points of parabolic geodesics (saddle connections) for a transversal pair of parabolic directions. Hence, they are rational points of $\mathcal{S}_1$. Therefore, $(\mathcal{S}_1; o_1, o_2)$ is a prelattice translation surface, and T_2 is hence a prelattice triangle.

6. Weierstrass Points versus Periodic Points

Definition 7. Let $\mathcal{S}$ be a translation surface without marked points. We say that $\mathcal{S}$ is a *hyperelliptic translation surface* if the corresponding Riemann surface is hyperelliptic.

Under certain conditions, the Weierstrass points of a hyperelliptic translation surface are periodic and can even be the only periodic points of the surface.

6.1. **Periodicity of Hyperelliptic Weierstrass Points.**

Recall that the nonarithmetic lattice surfaces of [Vch89] are hyperelliptic. Their Veech groups are either generated by elliptic elements, or by an elliptic and a parabolic element.

Theorem 10. *Let $\mathcal{S}$ be a hyperelliptic translation surface such that $\Gamma(\mathcal{S})$ is generated by elliptic elements. Then the set of Weierstrass points of $\mathcal{S}$ is Aff($\mathcal{S}$)-invariant.*

Since the set of Weierstrass points is always finite, these points are all periodic under the above hypotheses.

Our proof relies on the following simple lemma.

Lemma 7. *Let $\mathcal{S}$ be a translation surface, and let $\phi \in$ Aff($\mathcal{S}$) be an elliptic element. Then there is an affinely equivalent translation surface $\mathcal{T}$ such that the induced diffeomorphism $\psi \in$ Aff($\mathcal{T}$) is an isometry.*

Proof. A diffeomorphism $f \in \mathrm{Aff}(\mathcal{S})$ is an isometry if $Df \in \mathrm{SO}(2)$. Since $D\phi \in \mathrm{SL}(2,\mathbb{R})$ is elliptic, there is an element $g \in \mathrm{SL}(2,\mathbb{R})$ such that $g \cdot D\phi \cdot g^{-1} \in \mathrm{SO}(2)$. Set $\mathcal{T} = g \cdot \mathcal{S}$. The induced diffeomorphism $\psi \in \mathrm{Aff}(\mathcal{T})$ satisfies $D\psi = g \cdot D\phi \cdot g^{-1}$. ∎

Proof of Theorem 10. To simplify the exposition, we will not notationally distinguish between a translation surface and its underlying Riemann surface. By the results of Veech [Vch93b], any hyperelliptic translation surface is obtained by identifying the opposite sides of a centrally symmetric, planar polygon. Any such polygon, P, yields a hyperelliptic translation surface, $\mathcal{S}$. Without loss of generality, the center of symmetry of P is the origin o. The hyperelliptic involution is then induced by the map $z \mapsto -z$. Denote by $W(\mathcal{S})$ the set of Weierstrass points. The points of the cone set $C(\mathcal{S})$ come from the vertices of P, and $C(\mathcal{S}) \subset W(\mathcal{S})$. Furthermore, $W(\mathcal{S})$ contains the points arising from o and the midpoints of the sides of P.

Let $g \in \mathrm{SL}(2,\mathbb{R})$, and let $\mathcal{T} = g \cdot \mathcal{S}$. Then $\mathcal{T}$ is represented by the polygon $Q = g \cdot P$. Hence, $\mathcal{T}$ is hyperelliptic as well, and g induces a bijection of $W(\mathcal{S})$ and $W(\mathcal{T})$.

Let now $\phi \in \mathrm{Aff}(\mathcal{S})$ be an elliptic diffeomorphism. Let g and $\mathcal{T} = g \cdot \mathcal{S}$ be as in Lemma 7. Since the induced diffeomorphism $\psi : \mathcal{T} \to \mathcal{T}$ is conformal, it preserves the Weierstrass set $W(\mathcal{T})$ [FK92]. Since $\psi = g \cdot \phi \cdot g^{-1}$, ϕ preserves $W(\mathcal{S})$. Thus, $W(\mathcal{S})$ is invariant under $\mathrm{Aff}(\mathcal{S})$. ∎

Denote by T_n the isosceles triangle with base angle π/n, $n \geq 3$, and let $\mathcal{S}_n$ be the corresponding translation surface. By results of Veech [Vch89], T_n is a lattice polygon, which is nonarithmetic if $n \neq 3, 4, 6$. The surface $\mathcal{S}_n$ is hyperelliptic.

Corollary 9. *Let $\mathcal{S}_n$ be the hyperelliptic translation surface corresponding to the isosceles triangle T_n, for $n \geq 3$. Then the set of Weierstrass points of $\mathcal{S}_n$ is Aff($\mathcal{S}_n$)-invariant.*

Proof. By [Vch89], $\mathrm{Aff}(\mathcal{S}_n)$ is generated by an elliptic element and a parabolic element that preserves the set $W(\mathcal{S}_n)$. The claim follows, by the preceding argument. ∎

6.2. **Examples Proving Theorem 5.** The examples below led to the present work. In particular, they prove Theorem 5.

Example 3: The golden mean gnomon. Let P be the polygon modeling the "Swiss cross" with the golden ratio parameter $\lambda = (1+\sqrt{5})/2$. See Lemma 2 of [HS01]. Identifying opposite sides of P by translation, we obtain a translation surface, $\mathcal{S}$, of genus 2 (and thus certainly hyperelliptic). The hyperelliptic involution is induced by inversion of P with respect to its center. The six Weierstrass points of $\mathcal{S}$ thus arise from: the center of P; the exterior corners — giving two points; the interior corners — identified to the single cone point; the boundary points of the figure which lie on the axes of either horizontal or vertical symmetry.

By a cut-and-paste operation, we transform P into a "gnomon" (i.e., an "L" shape), see Figure 1. We denote the gnomon by P, as well. The surface $\mathcal{S}$ is obtained from it by the natural identifications. The 8 points marked by black circles in Figure 1 are identified to the cone point of $\mathcal{S}$. Let O, $A, \dots, D$ be the remaining five Weierstrass points, marked by open circles in Figure 1. The coordinate directions are periodic. Since P is symmetric about the diagonal, it suffices to study the vertical cylinders. Their parameters are: $w_1^v = 1$, $\ell_1^v = \lambda$, $w_2^v = \lambda - 1$ and $\ell_2^v = 1$. Hence $\mu_1 = 1/\lambda$, $\mu_2 = \lambda - 1$. Since $1/\lambda = \lambda - 1$, these moduli are equal; thus the coordinate directions form a transversal parabolic pair.

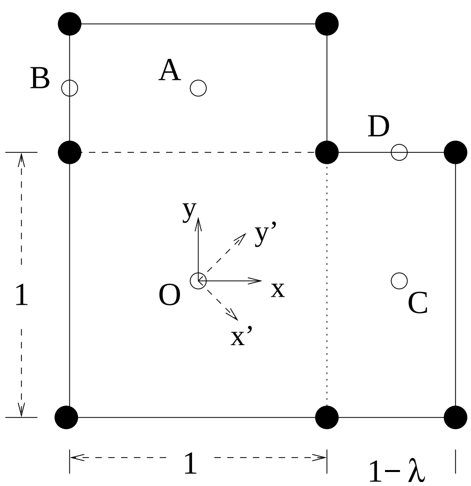

Figure 1. The golden ratio gnomon.

The directions $\pi/4$ and $3\pi/4$ are also periodic. Their cylinder decompositions have the same parameters: $w_1^{\pi/4} = (\lambda-1)/\sqrt{2}$, $\ell_1^{\pi/4} = (\lambda+1)\sqrt{2}$, $w_2^{\pi/4} = (2-\lambda)/\sqrt{2}$, $\ell_2^{\pi/4} = \lambda\sqrt{2}$, see Figure 2. Hence $\mu_1 = \frac{\lambda-1}{\lambda+1}$, $\mu_2 = \frac{2-\lambda}{\lambda}$, and $\mu_1/\mu_2 = 1$. Thus, $\pi/4, 3\pi/4$ is also a transversal pair of parabolic directions

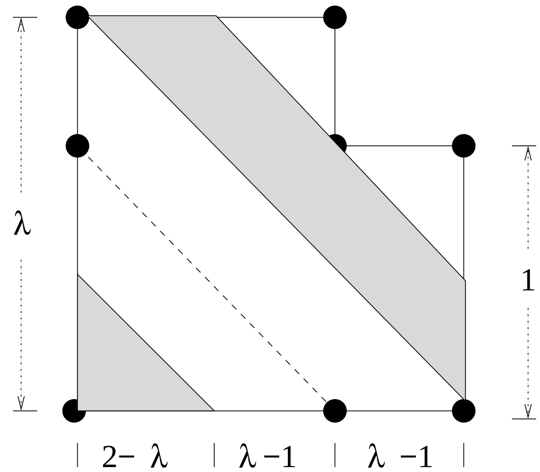

Figure 2. The cylinders in the direction $3\pi/4$.

Let U be the unit square contained in P. Denote by O the center of U, and let x, y and x', y' be the standard coordinate system and its rotation by $-\pi/4$, respectively. See Figure 1. By the preceding material, if $s(x, y) \in U$ is a periodic point, then x, y are rational. Denote by $I \subset U$ the intersection of the two cylinders of width $(\lambda - 1)/\sqrt{2}$. If $s = (x, y) \in I$ is a periodic point, then $(x, y) = (\,(x' + y')/\sqrt{2}, (-x' + y')/\sqrt{2}\,)$. Thus, $\sqrt{2}y' = x - y \in \mathbb{Q}$. Since y' is a rational multiple of the width of the cylinders, $\sqrt{2}y' \in \mathbb{Q} \cap (\lambda - 1)\mathbb{Q}$. Hence, $y' = 0$. Analogously, $x' = 0$.

Let now $s = (x, y) \in U \setminus I$ be periodic. By symmetry, it suffices to consider the bottom left corner of U. The same rationality argument as above yields $x' = 0$. Analogous considerations show that $y' \in \sqrt{2}\mathbb{Q} \cap \frac{2-\lambda}{\sqrt{2}}\mathbb{Q}$. Hence, $y' = 0$, and $U \setminus I$ contains no periodic points. Therefore, the only periodic point in U is the center.

Let s be a periodic, regular point. We show that the orbit of s meets U. Suppose that s belongs to the interior of the first vertical cylinder (i.e, the cylinder of width one). The vertical closed geodesic upon which s lies clearly must meet U in at least half of its length. Therefore, there is some power of the basic vertical affine map which takes s into U. By symmetry, if s belongs to the interior of the first horizontal cylinder, the orbit of s also meets U. But, the horizontal Dehn twist sends the boundary of the first vertical cylinder into the union of the interiors of the two "first cylinders".

We conclude that the orbit of any periodic point meets the set of Weierstrass points. By Theorem 10, the set of periodic points is exactly the Weierstrass points.

Example 4: The regular octagon. Denote by P the regular octagon, inscribed in the unit circle, and let $\mathcal{S}$ be the translation surface obtained by identifying the opposite sides of P. It is a hyperelliptic surface of genus 2. Furthermore, $\mathcal{S}$ is a nonarithmetic lattice surface and $\Gamma(\mathcal{S})$ is generated by elliptic and parabolic elements [Vch89]. As in the preceding example, the six Weierstrass points of $\mathcal{S}$ come from: the center of P, the midpoints of its edges, and the vertices. The parabolic generator of $\Gamma(\mathcal{S})$, referred to above, stabilizes $W(\mathcal{S})$ [AH00].

We claim that the Weierstrass and periodic points of $\mathcal{S}$ coincide.

The coordinate directions form a transversal parabolic pair in $\mathcal{S}$. The $3\pi/8, 7\pi/8$ pair is also parabolic. There are two cylinders in each decomposition. By the 8-fold symmetry of the regular octagon, it suffices to determine the parameters of two of the four decompositions. In the notation of Figure 3, we have $w_1 = \sqrt{2}/2$, $w_2 = (2-\sqrt{2})/2$, and $w_{1'} = 2\sin\pi/8$, $w_{2'} = \cos\pi/8 - \sin\pi/8$.

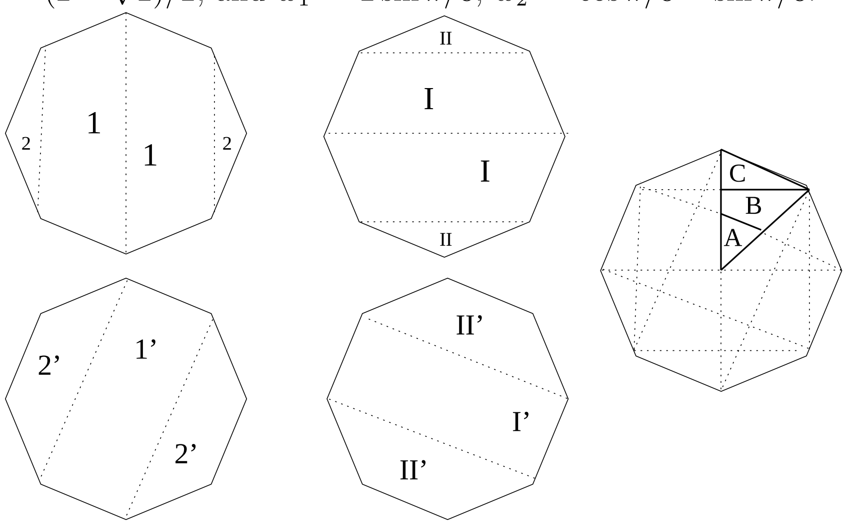

Figure 3. Cylinder decompositions for parabolic directions; The induced partition of the triangle.

Let $s \in \mathcal{S}$ be a periodic point. By symmetry, we can assume that s belongs to the triangle T with vertices $0, e^{i\pi/4}, i$. Intersecting T with the cylinders above, we obtain the decomposition $T = A \cup B \cup C$. See Figure 3. The triangle A intersects the cylinders 1, I, $1'$ and I'. The quadrilateral B intersects the cylinders 1, I, $1'$ and II'. The triangle C intersects the cylinders 1, II, $1'$ and II'. We denote by x, y and x', y' the standard coordinate system about the center of P and its rotation by $-\pi/8$, respectively.

Let $s \in A$ be a periodic point, of respective coordinates (x, y), (x', y'). Then: $x, y \in \sqrt{2}\mathbb{Q}$, $x', y' \in (\sin\pi/8)\mathbb{Q}$, and

$$x = x'\cos\pi/8 + y'\sin\pi/8,\ y = -x'\sin\pi/8 + y'\cos\pi/8. \tag{12}$$

Set $x' = \frac{p}{q}\sin\pi/8$. By trigonometry, $x'\cos\pi/8 \in \sqrt{2}\mathbb{Q}$, hence $y'\sin\pi/8 \in \sqrt{2}\mathbb{Q}$. Since $y' = \frac{u}{v}\sin\pi/8$, we conclude that $x' = y' = 0$. Thus, s is the center of P.

Let $s \in B \cup C$ be a periodic point, of respective coordinates (x, y), (x', y'). Applying the preceding argument, we obtain: $x \in \sqrt{2}\mathbb{Q}$, $x' \in (\sin\pi/8)\mathbb{Q}$, and $y' - \sin\pi/8 \in (\cos\pi/8 - \sin\pi/8)\mathbb{Q}$. The same argument yields $x'\cos\pi/8 \in \sqrt{2}\mathbb{Q}$.

By eq. (12), $y' \sin \pi/8 \in \sqrt{2}\mathbb{Q}$. Set $y' = \sin \pi/8 + (\cos \pi/8 - \sin \pi/8)\frac{u}{v}$. Then $(2-\sqrt{2})/4 + [\sqrt{2}/2 - (2-\sqrt{2})/4]\frac{u}{v} \in \sqrt{2}\mathbb{Q}$, implying that $u = v$, and $y' = \cos \pi/8$. Thus s belongs to the outer edge of C. The Dehn twist of cylinder $1'$ fixes the endpoints and sends the midpoint of the edge into the center of P. The rest of the edge is sent into the interior of P, avoiding the center.

We have shown above that the interior of P contains no periodic points, with the possible exception of the center. Therefore $s \in W(\mathcal{S})$. By Corollary 9, $W(\mathcal{S}) \subset P(\mathcal{S})$, hence the claim.

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