# PROOFS OF ERGODICITY OF PIECEWISE MÖBIUS INTERVAL MAPS USING PLANAR EXTENSIONS

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ABSTRACT. We give two results for deducing dynamical properties of piecewise Möbius interval maps from their related planar extensions. First, eventual expansivity and unique ergodicity (with respect to Borel measures absolutely continuous with respect to Lebesgue measure) follow from mild finiteness conditions on the planar extension along with a new property "bounded non-full range" used to relax traditional Markov conditions. Second, the "quilting" operation to appropriately nearby planar systems, introduced by Kraaikamp and co-authors, can be used to prove several key dynamical properties of a piecewise Möbius interval map. As a proof of concept, we apply these results to recover known results about Nakada's $\alpha$-continued fractions; we obtain similar results for a family of interval maps derived from an infinite family of non-commensurable Fuchsian groups.

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# 1. Introduction

1.1. **Motivation.** Recall that the natural extension of a dynamical system was introduced by Rohlin, who showed that the natural extension and the original system have the same Kolmogorov–Sinai entropy (hereafter simply *entropy*). In terse terms, the natural extension is the minimal invertible dynamical system of which the original system is a factor under a surjective map. The natural extension is unique up to metric isomorphism. At least since the work in the late 1970s of Nakada-Ito-Tanaka [NIT], planar maps of the type we study here has been used to give explicit presentations of natural extensions of piecewise Möbius interval maps.

A particularly influential use of planar extensions was that of Bosma-Jager-Wiedijk [BJW], based upon [NIT]. Other works include: applications to study Nakada's $\alpha$-continued fractions in [N, K, LM, NN, CT, KSS, AS] and in various other papers of Tiozzo and co-authors, the related introduction in [K] of $S$-expansions; to related families of continued fraction maps [BKS, KSSm, DKS, AS2, HKLM] and various works of S. Katok and co-authors, see for example [KU]. Arnoux in [Ar] and later with various co-authors stressed the description of planar extensions up to measure zero and their relationship to cross-sections for the geodesic flow on unit tangent bundles; see works of S. Katok for related ideas.

In [CKS], we studied an infinite countable collection of one parameter family of piecewise Möbius interval maps, denoted $T_{m,n,\alpha}$ there (the main focus was on the cases with $m = 3$). Although planar extensions are barely mentioned there, [CKS] was informed by numerous calculations of them. As opposed to say the Nakada $\alpha$-continued fractions, (all but one of) the $T_{m,n,\alpha}$ are *not expansive* maps. A direct proof that each is *eventually expansive* seems tedious at best; this motivated us to seek a general result that can be easily applied to deduce eventual expansitivity. We give such a result here as part of Theorem 2.3.

One expects sufficiently nice continued fraction maps to be ergodic with respect to some measure which is absolutely continuous with respect to Lebesgue measure; the easiest setting to prove such results is when a Markov condition is fulfilled. In the setting of [CKS], and in many cases of continued fraction like maps, Markov properties do not hold. In Definition 2.1 below, we introduce a property that is often fulfilled in these settings. That this property and basic finiteness conditions satisfied by a planar extension for a map then imply ergodicity and more is given in Theorem 2.3. As an application, in § 2.6 we show that each of an infinite collection of maps is ergodic.

We also study a technique used to date for solving for the planar extension of a piecewise Möbius interval map beginning with such a planar extension for a sufficiently "nearby" map. This technique, called *quilting*, was introduced in [KSSm], and has its roots in the discussion of the two-dimensional interpretation of "insertion" and "deletion" in the Ph.D. dissertation [K]. Theorem 3.3, shows that one can use quilting to prove that fundamental dynamical properties are shared between appropriately nearby systems. We give applications of this in the setting of "matching intervals" in § 3.4–3.6.

One can thus pass from a system, say proven to have such properties by use of Theorem 2.3, to nearby systems. In § 4 we show that this approach gives an alternate path to proving properties of the well-studied Nakada $\alpha$-continued fractions.

**Convention** Throughout, we will allow ourselves the minor abuse of using adjectives such as injective, surjective and bijective to mean in each case *up to measure zero*, and thus similarly where we speak of disjointness and the like we again will assume the meaning being taken to include the proviso "up to measure zero" whenever reasonable.

1.2. **Planar extensions for piecewise Möbius maps.** The standard number theoretic planar map associated to a Möbius transformation $M$ is

$$\mathcal{T}_M(x,y) := \Big(M\cdot x, N\cdot y\Big) := \Big(M\cdot x, RMR^{-1}\cdot y\Big) \quad \text{for } x\in \mathbb{I}_M,\ y\in\mathbb{R}\setminus\{(RMR^{-1})^{-1}\cdot\infty\}\,,$$

where $R = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. Thus, $\mathcal{T}_M(x,y) = (\,M\cdot x, -1/(M\cdot(-1/y))\,)$. An elementary Jacobian matrix calculation verifies that the measure $\mu$ on $\mathbb{R}^2$ given by

$$d\mu = \frac{dx\,dy}{(1+xy)^2} \tag{1}$$

is (locally) $\mathcal{T}_M$-invariant.

Suppose that $T$ is a piecewise Möbius interval map whose domain is partitioned $\mathbb{I} = \cup_\beta K_\beta$, such that $T(x) = M_\beta\cdot x$ for all $x\in K_\beta$. We will assume that each $K_\beta$ is an interval and is taken as large as possible. We call these $K_\beta$ the (rank one) cylinders for $T$.

We then set

$$\mathcal{T}(x,y) = \Big(M_\beta\cdot x, RM_\beta R^{-1}\cdot y\Big) \quad \text{for } x\in K_\beta,\ y\in\mathbb{R}\setminus\{N^{-1}\cdot\infty\}\,. \tag{2}$$

Suppose that $\Omega\subset\mathbb{R}^2$ projects onto the interval $\mathbb{I}$ and is a domain of bijectivity of $\mathcal{T}$, that is $\mathcal{T}$ is bijective on $\Omega$ up to $\mu$-measure zero. Let $\mathcal{B}$ be the Borel algebra of $\Omega$, we then call the system $(\mathcal{T},\Omega,\mathcal{B},\mu)$ a *planar extension* for $T$.

## 2. Bounded non-full range and finiteness of $\Omega$ implies ergodicity

We use the term *eventually expansive* to describe an interval map $T$ having some compositional power $r$ that is expansive, thus there is some $c>1$ so that all $x$ in the domain of $T^r$ satisfy $|(T^r)'(x)|\geq c$.

In [CKS], we studied an infinite countable collection of one parameter family of piecewise Möbius interval maps, denoted $T_{m,n,\alpha}$ there (the main focus was on the cases with $m=3$). Although planar extensions are barely mentioned there, [CKS] was informed by numerous calculations of them. As opposed to say the Nakada $\alpha$-continued fractions, (all but one of) the $T_{m,n,\alpha}$ are *not expansive* maps. A direct proof that each is eventually expansive seems tedious at best; this motivated us to seek a general result that can be easily applied to deduce eventual expansitivity. We give such a result here.

One expects sufficiently nice continued fraction maps to be ergodic with respect to some measure which is absolutely continuous with respect to Lebesgue measure; the easiest setting to prove such results is when a Markov condition is fulfilled. In the setting of [CKS], and in many cases of continued fraction like maps, Markov properties do not hold. In Definition 2.1 below, we introduce a property that is often fulfilled in these settings. This property and basic finiteness conditions satisfied by a planar extension for a map then imply ergodicity and more.

### 2.1. Adler's 'Folklore Theorem'.

Making an initial approach of Renyi much more practical, Adler [Ad] gave conditions implying that an interval map $f$ has a unique ergodic measure that is equivalent to Lebesgue measure. In his afterword to [B], Adler sketched how to loosen one of his original conditions, with the resulting result (to which he referred there as a *folklore theorem*).

**Theorem 2.1.** *[Adler] Suppose that $f$ is an interval map such that:*

i.) *All cylinders of $f$ are full;*
ii.) *$f$ is twice differentiable;*
iii.) *$f$ is eventually expansive;*

iv.) *there is a finite bound on* $|f''(x)|/f'(x)^2$ *for* $x$ *in the domain of* $f$.

*Then* $f$ *has a a unique ergodic probability measure that is equivalent to Lebesgue measure.*

### 2.2. Boundedness of fibers of $\Omega$ and a full cylinder implies ergodicity and more.

2.2.1. *Cylinder covering property.* We introduce a condition that can be seen as a weakening of the finite range condition of Ito-Yuri [IY], which itself is a weakening of Adler's condition of having full cylinders.

*Definition* 2.1. We say that an interval map has *bounded non-full range* if there is a full cylinder such that the orbits of the endpoints of all non-full cylinders avoid the interior of this full cylinder.

To illustrate the ease of verification of our property, we show that a large subset of one of the most studied families of continued fraction maps, the Nakada $\alpha$-continued fractions [N], has the bounded non-full range property. As in [CT] let $\mathcal{E} \subset (0,1]$ be the complement of the matching intervals (called synchronization intervals in [KSS]).

**Lemma 2.2.** *Both every rational* $\alpha \in (0,1]$ *and every* $\alpha \in \mathcal{E}$ *is such that Nakada's* $\alpha$*-continued fraction map* $T_\alpha$ *has bounded non-full range.*

*Proof.* Whenever $\alpha \in \mathbb{Q} \cap (0,1]$, the endpoints of $[\alpha-1,\alpha)$ have finite $\alpha$-expansion, both eventually reaching zero. Since for any $\alpha$ the only possible non-full cylinders of $T_\alpha$ are the two extreme end cylinders, these $T_\alpha$ certainly have bounded non-full range.

Recall that $T_1$ is the regular continued fraction map (thus, the classical Gauss map). In the proof of ([KSS], Lemma 6.8) a result of [CT] is verified: $\alpha \in \mathcal{E}$ if and only if $T_1^n(\alpha) \geq \alpha$ for all $n \in \mathbb{N}$. In particular, if $\alpha \in \mathcal{E}$ then its regular continued fraction expansion is of the form $[0; a_1, a_2, \dots]$ with $a_1 \geq a_n$ for all $n > 1$ and in particular the $a_i$ take on only finitely many values. Now, given this expansion of $\alpha \in \mathcal{E}$, from ([KSS], Proposition 4.1) the $\alpha$-expansion of $\alpha - 1$ has digits contained in a finite set and hence from ([KSS] Lemma 6.7) also the $\alpha$-digits of $\alpha$ itself are contained in a finite set. Thus, $T_\alpha$ is of bounded non-full range. □

*Example* 2.2. As recalled above, in [CKS] we study families of maps $T_{3,n,\alpha}, n > 3, \alpha \in [0,1]$. Subsection 4.5 of [CKS] shows that for each $n > 3$ and every non-synchronization $\alpha \in (0, \gamma_{3,n})$ the $T_{3,n,\alpha}$-orbit of $\ell_0(\alpha)$ meets only the two leftmost cylinders, and the $T_{3,n,\alpha}$-orbit of $r_0(\alpha)$ meets only the two leftmost cylinders. In this setting there are only (at most) two non-full cylinders, the leftmost cylinder whose left endpoint is $\ell_0(\alpha)$ and the rightmost with right endpoint $r_0$. These cylinders are right full and left full respectively. From this, we find that each of these maps has bounded non-full range. Indeed, here there are infinitely many full cylinders avoided by the orbits in question.

2.2.2. *Statement of result.*

**Theorem 2.3.** *Suppose that* $T$ *is a piecewise Möbius map on an interval* $I$ *of finite Lebesgue measure and* $\mathcal{T} : \Omega \to \Omega$ *is a planar extension for* $f$ *such that*

a) *the vertical fibers of* $\Omega$ *are of positive Lebesgue measure bounded away from both zero and infinity;*
b) *the vertical fibers are bounded away from the locus of* $y = -1/x$*;*
c) $T$ *has at least one full cylinder for which the set of ratios of the Lebesgue measure of the* $\mathcal{T}$*-image of each vertical fiber above this cylinder to the Lebesgue measure of its receiving fiber is bounded away from zero and one;*

d) *$T$ has bounded non-full range.*

*Then*

i.) $0 < \mu(\Omega) < \infty$*;*
ii.) *$T$ is eventually expansive;*
iii.) *$T$ is ergodic with respect to $\nu$, the marginal measure induced by $\mu'$ the normalization of $\mu$ to a probability measure on $\Omega$;*
iv.) *the system $(\mathcal{T}, \Omega, \mathscr{B}', \mu')$ is the natural extension of $(T, I, \mathscr{B}, \nu)$, where $\mathscr{B}, \mathscr{B}'$ denote the Borel algebras of $I, \Omega$ respectively. In particular, this two dimensional system is also ergodic.*

That $\mu(\Omega)$ is finite is easily seen. We prove the remaining conclusions in three steps.

## 2.3. **Eventual expansivity.**

*Proposition* 2.3. Under the hypotheses (a) –(c) of Theorem 2.3, $T$ is eventually expansive.

We will use the following change of coordinates in the proof. Let

$$\mathcal{Z}(x,y) = (x, y/(1+xy)) \tag{3}$$

and for each $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ as above, let $\widehat{\mathcal{T}}_M = \mathcal{Z} \circ \mathcal{T}_M \circ \mathcal{Z}^{-1}$. Then $\widehat{\mathcal{T}}_M(x,y) = (M\cdot x, (cx+d)^2 y - c(cx+d)\,)$. Clearly, the partial derivative of $(cx+d)^2y - c(cx+d)$ with respect to $y$ equals the multiplicative inverse of the derivative of $M \cdot x$ with respect to $x$. In particular, an elementary Jacobian matrix calculation verifies that Lebesgue measure on $\mathbb{R}^2$ is (locally) $\widehat{\mathcal{T}}_M$-invariant. We let $\widehat{\mathcal{T}}(x,y) = \mathcal{Z} \circ \mathcal{T} \circ \mathcal{Z}^{-1}$, thus it is given piecewise by various $\widehat{\mathcal{T}}_M$. (See [AS] for more about this conjugation.)

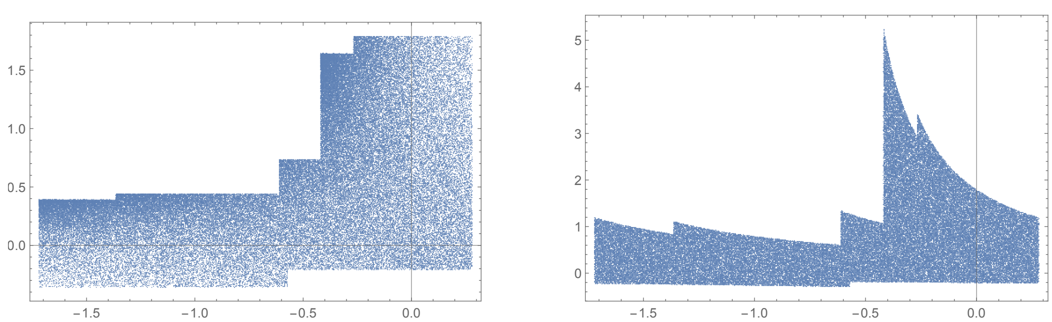


FIGURE 1. Approximate plots of 100,000 points of $\mathcal{T}_{3,0.14}$-orbit map from [CKS] (left), and its image under $\mathcal{Z}(x,y)$ of (3).

*Proof.* Let $\widehat{\mathcal{T}} : \Sigma \to \Sigma$ denote the conjugate two-dimensional system where $\Sigma$ is the image of $\Omega$ under the map $\mathcal{Z}$. Lebesgue measure is invariant for this conjugate system. In particular, for each $x \in I$ the vertical fiber $F_x \subset \Sigma$ projecting to $x$ is mapped by $\widehat{\mathcal{T}}$ into the vertical fiber $F_{Tx}$, with derivative along $F_x$ constantly equal to $(T'(x))^{-1}$. Hypotheses (a) and (b) imply that there are positive finite bounds $0 < b < B$ on the one-dimensional Lebesgue measure of the $F_x$.

The third hypothesis also carries over to the conjugate system. Let us use vertical bars to indicate the one-dimensional Lebesgue measure on vertical fibers of $\Sigma$. Denoting the chosen full cylinder by $\mathcal{C}$, we have that the set of ratios

$$\left\{ \frac{|\widehat{\mathcal{T}}(F_x)|}{|F_{Tx}|} \;:\; x \in \mathcal{C} \right\}$$

is also bounded away from zero and one. We can thus find a $\rho$ with $0 < \rho < 1$ such that $1-\rho$ is a lower bound and $\rho$ an upper bound.

Now, for any $z \in I$, the fiber at $z$ is the union of the images of the fibers over the preimages of $z$; that is, $F_z = \cup_{Tx=z} \widehat{\mathcal{T}}(F_x)$. Given $x \in I \setminus \mathcal{C}$, set $z = Tx$. Since $\mathcal{C}$ is a full cylinder, there is some $x' \in \mathcal{C}$ such that $T(x') = z$ and hence $|\ \widehat{\mathcal{T}}(F_{x'})\ |$ gives at least $1-\rho$ of $|F_{Tx}|$. It follows that $|\widehat{\mathcal{T}}(F_x)| \leq \rho |F_{Tx}|$. Hence, for all $x \in I$, we have $|\widehat{\mathcal{T}}(F_x)| \leq \rho |F_{Tx}|$.

Recall that $\widehat{\mathcal{T}}$ has constant derivative along each vertical fiber. Thus, ratios of measures are preserved; in particular

$$\frac{|\ \widehat{\mathcal{T}}^2(F_x)\ |}{|\ \widehat{\mathcal{T}}(F_{Tx})\ |} = \frac{|\ \widehat{\mathcal{T}}(F_x)\ |}{|\ F_{Tx}\ |}.$$

Using a telescoping expansion and substituting the above, we deduce

$$\frac{|\ \widehat{\mathcal{T}}^2(F_x)\ |}{|\ F_{T^2x}\ |} = \frac{|\ \widehat{\mathcal{T}}(F_x)\ |}{|\ F_{Tx}\ |} \cdot \frac{|\ \widehat{\mathcal{T}}(F_{Tx})\ |}{|\ F_{T^2x}\ |} \leq \rho^2$$

and similarly for higher powers. Now let $r \in \mathbb{N}$ be such that $\rho^{r-1}B < b$. Then for any $x \in I$ we have $b \leq |F_x|$ but $|\widehat{\mathcal{T}}^r(F_x)| \leq \rho^r \mid F_{T^rx} \mid \leq \rho^r B < \rho b$. Thus, $|\widehat{\mathcal{T}}^r(F_x)| < \rho |F_x|$. Since $\widehat{\mathcal{T}}^r$ also preserves two-dimensional Lebesgue measure, we must have that $(T^r)'(x) > \rho^{-1}$. Therefore, $T^r$ is expansive. □

## 2.4. **Ergodicity.**

*Proposition* 2.4. Under the hypotheses of Theorem 2.3, $T$ is ergodic with respect to $\nu$.

*Proof.* Since $T$ has bounded non-full range, there is a largest interval, say $J$, comprised of full cylinders avoided by the orbits of all non-full cylinders. Let $\widetilde{T}$ be the first return map to $J$ of orbits of $T$. We will show that Adler's conditions hold for $f = \widetilde{T}$. The ergodicity of a map induced from $T$ implies that $T$ itself is also ergodic under reasonable hypotheses (see Theorem 17.2.4 of [Schw]). Thus the result will hold.

The cylinders of $\widetilde{T}$ are of the form $Q_\beta$ where $\beta = (a_1, \dots, a_m)$ with $a_1$ an index of a $T$-cylinder within $J$, $[a_1, \dots, a_m]$ a rank $m$ cylinder for $T$, and $Q_\beta = [a_1, \dots, a_m] \cap T^{-m}(J)$. Since $J$ consists of full cylinders for $T$ which the $T$-orbits of the endpoints of the non-full cylinders never enter, each $Q_\beta$ is a full $\widetilde{T}$-cylinder.

The corresponding planar map $\widetilde{\mathcal{T}}$ is bijective up to $\mu$-measure zero on $\widetilde{\Omega}$, the region defined by deleting the portion of $\Omega$ projecting to the complement of $J$. In particular, the arguments in the proof of Theorem 2.3 apply, and thus $\widetilde{T}$ is eventually expansive.

We have ensured that $\widetilde{T}$ has full cylinders, and is eventually expansive. Our construction also preserves the property of being twice differentiable. The crux of the matter is thus to show that Adler's fourth condition holds. Since $T$ is a piecewise Möbius map, certainly for $\nu$-a.e. $x$, there is some matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $\widetilde{T}(x) = M \cdot x$. The first derivative here is $(cx+d)^{-2}$, we must bound $|c(cx+d)|$ over all such $x, M$.

Since the vertical fibers of $\mathcal{Z}(\Omega)$ are bounded, so are those of $\mathcal{Z}(\widetilde{\Omega})$. That is, every vertical fiber has Lebesgue measure in an interval $[b, B]$ bounded away from zero and infinity. To avoid notational unpleasantries, let us now use $\widehat{\mathcal{T}}$ to denote the conjugate of $\widetilde{\mathcal{T}}$ acting on $\mathcal{Z}(\widetilde{\Omega})$. Recall that restricting $\widehat{\mathcal{T}}$ to $F_x$ (the vertical fiber at $x$) defines a map whose derivative equals $(\widetilde{T}'(x))^{-1}$. It follows that $(\widetilde{T}'(x))^{-1} \leq B/b$ for all $x \in I$. Hence, the set of values $(cx+d)^2 y$ is bounded. The boundedness of the fibers directly implies that the values $(cx+d)^2y - c(cx+d)$ are bounded. We conclude that $c(cx+d)$ is bounded throughout $I$. This implies that Adler's fourth condition holds. Therefore, $\widetilde{T}$ is ergodic and, as argued in the first paragraph of this proof, the result holds. □

*Remark* 2.5. Given the above argument, one could ask whether it is always possible to induce past non-full cylinders and be sure that Adler's conditions hold. We strongly doubt this, as in general the return iteration number to the complement of those cylinders will be unbounded. As Zweimüller [Z] states, this in general will cause Adler's condition (4) to fail. In our setting, of course, such "explosion" is impossible due to the boundedness of the vertical fibers of $\mathcal{Z}(\Omega)$.

2.5. **Natural extension.** Arnoux in particular has been a proponent of solving for planar presentations using properties of the interval map. In particular, the main results of [AS3] imply that (1) if a piecewise Möbius interval map $T : I \to I$ is (eventually) expansive then its associated naive two-dimensional map $\mathcal{T} : I \times \mathbb{R} \to I \times \mathbb{R}$ induces a contraction on the complete metric space of compact subsets of this product, where a modified Hausdorff metric is used. (The contraction sends a compact $K$ to the closure of the union of the $\mathcal{T}_\beta(K_\beta)$, where $K_\beta$ is the portion of $K$ projecting to the $\beta$-cylinder.) And, (2) when the fixed point, say $\Omega$, of this contraction has positive measure and $\mathcal{T}$ is bijective on $\Omega$ up to measure zero, then this two-dimensional system is a planar natural extension of $T$.

We thus find the following.

*Proposition* 2.6. Under the hypotheses of Theorem 2.3, $(\mathcal{T}, \Omega, \mathscr{B}', \mu')$ is the natural extension of $(T, I, \mathscr{B}, \nu)$.

2.6. **Application to each of an infinite collection of maps.** Recall that the dynamics of the maps $T_{m,n,1}$ are presented in Section 3 of [CKS]. For ease, we restrict to the case of $m = 3$. We give a planar extension for each of these maps. Each is of infinite mass; by "accelerating" each of the interval maps past the domain of the parabolic element of the underlying group whose fixed point is responsible for the infinitude of the mass, we obtain an interval map of invariant probability measure, as verified by applying Theorem 2.3.

Fix $n \geq 3$, let $T = T_{3,n,1}$ and using the notation of [CKS] let $U = AC(AC^2)^{n-2}$. From [CKS], $U$ is an admissible word for the $T$-orbit of $t = r_0(1)$, and $U \cdot t = t$. Furthermore, all of the cylinders of $T$ are full except for the right most cylinder, $\Delta(1,2)$, of endpoints $\mu + 1/t = 1 + 1/t$ and $t$. We also know that $(AC^2)^{n-2} \cdot t = 1$. Since the interval of definition of $T$ is $(0, t]$, we also have that $\mathfrak{b} = 1$.

Compare the following with Figure 2.

**Proposition 2.4.** *Fix $n \geq 3$ and let $T = T_{n,1}$ and $\mathcal{T}$ be the usual associated two-dimensional map. Let $r_0, r_1, \ldots, r_{n-2}$ be the $T$-orbit of $r_0 = t$. Then $\mathcal{T}$ is bijective up to $\mu$-measure zero on*

$$\Omega = [0,1] \times [-1,0] \bigcup \cup_{i=1}^{n-2} [r_i, r_{i-1}] \times [-1/r_{i-1}, 0].$$

*Proof.* We have that $(0,1]$ is the union of the cylinders $\Delta(i,1)$ with $i \in \mathbb{N}$. Similarly, $(1, 1+1/t]$ is the union of the $\Delta(j,2)$ with $j \geq 2$. Since $1+1/t$ lies between $1 = r_{n-2}$ and $r_{n-3}$, the $y$-fiber of $\Omega$ above each of these $\Delta(j,2)$ is $[-1/r_{n-3}, 0]$, whereas every $\Delta(i,1)$ has $y$-fiber given by $[-1,0]$. Since $RCR^{-1} \cdot 0 = -1$, it follows that $RA^kCR^{-1} \cdot -1 = RA^kC^2R^{-1} \cdot 0$ for any $k$. Hence, each rectangle $\Delta(k,1) \times [-1,0]$ is mapped above the image of $\Delta(k,2) \times [-1/r_{n-3}, 0]$ so as to share exactly a common horizontal line.

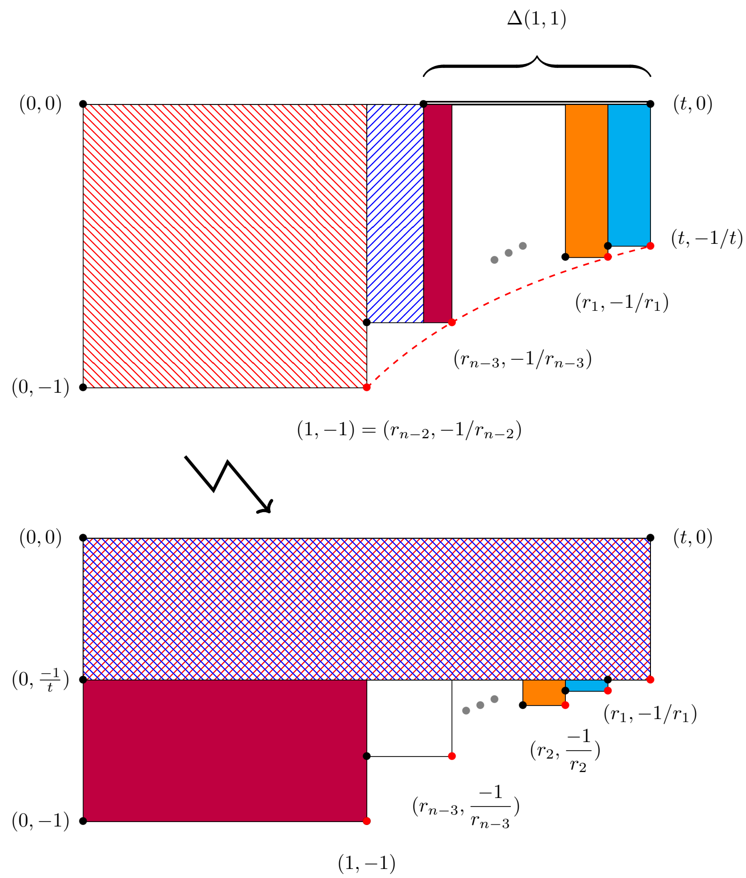


FIGURE 2. Schematic representation of the domain $\Omega_{n,1}$, and its image under $\mathcal{T}_{n,1}, n \geq 3$ as discussed in Proposition 2.4. Compare with Fig. 6 (left hand side) of [CKS]. Here, on the left side, the red dotted curve plots $y = -1/x$ the only points of $\Omega$ on this are the (red) vertices coming from the orbit of $(t, -1/t)$. (Recall that our measure is given by $d\mu = (1 + xy)^{-2}\, dx\, dy$ !) Blocks fibering over intervals whose endpoints are consecutive members of the orbit of $r_0 = t$ under the interval map are filled with solid colors. Red hatching indicates blocks fibering over cylinders indexed by $(i, 1), i \in \mathbb{N}$. Blue hatching indicates blocks fibering over cylinders indexed by $(j, 2), j \geq 2$. Images on the right hand side correspondingly colored, except that the cross-hatching indicates lamination from the hatched portions.

Now, $AC^2 \cdot r_{n-3} = 1$ can be used to show that $RC^2AC^2R^{-1} \cdot -1/r_{n-3} = 0$ and a similar observation implies that each rectangle $\Delta(k, 1) \times [-1, 0]$ is mapped below the image of $\Delta(k + 1, 2) \times [-1/r_{n-3}, 0]$ so as to share exactly a common horizontal line.

Therefore, $\mathcal{T}$ sends $\Omega \cap \{x \leq 1 + 1/t\}$ bijectively up to measure zero to $(0, t] \times [-1/t, 0)$. Furthermore, since every $\mathcal{T}_M$ preserves the locus $y = -1/x$, we easily find that $i = 1, \ldots, n-4$,

$$\mathcal{T}_{AC^2}(\,[r_i, r_{i-1}] \times [-1/r_{i-1}, 0]\,) = [r_{i+1}, r_i] \times [-1/r_i, -1/t].$$

(Of course, when $n \leq 4$ we must make appropriate adjustments.) Furthermore, $\mathcal{T}_{AC^2}$ sends $(1+1/t, r_{n-3}] \times [-1/r_{n-3}, 0]$ to $(0,1] \times [-1, -1/t]$. The result thus holds. $\square$

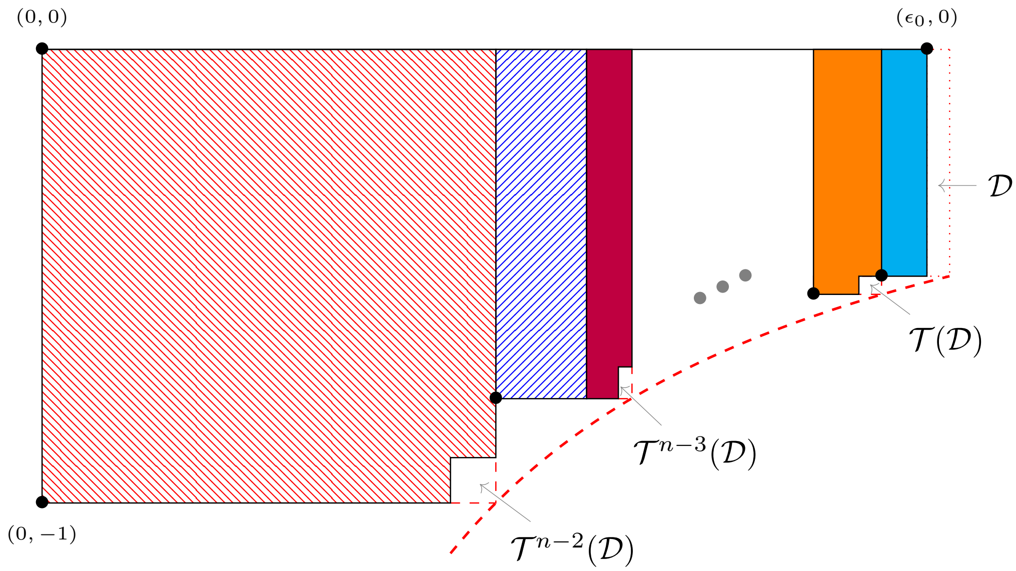


FIGURE 3. Schematic representation of the domain $\Gamma$ for the accelerated two-dimensional map. The domain is given by deleting from $\Omega$ the rectangle $\mathcal{D} = (\epsilon_0, t] \times [-t, 0]$ and its images under $\mathcal{T}, \ldots, \mathcal{T}^{n-2}$. See Lemma 2.5.

Since $U$ is a conjugate (up to sign) of $A^{-1}$ we see that it is a parabolic matrix and thus $t$ is parabolic fixed point under $T$. This in a sense is the cause of the invariant measure $\mu$ being infinite on $\Omega$. Just as in the treatment of $T_{n,0}$ in [CS], we "accelerate" our map by inducing past the cylinder (here of rank $n-1$) related to the parabolic element. There is an easily determined domain of bijectivity for the corresponding two-dimensional map. (Indeed the following result is a specific case of a general phenomenon.) Compare the following with Figure 3.

**Lemma 2.5.** *Fix $n \geq 3$. Let $\epsilon_0 = U^{-1} \cdot 0$ and let $g(x)$ be the first return map of $T_{n,1}$ on $(0, \epsilon_0)$ and $\mathcal{G}$ the corresponding two-dimensional map. Then $\mathcal{G}$ is bijective up to $\mu$-measure zero on*

$$\Gamma = \Omega \setminus \cup_{i=0}^{n-2} \mathcal{T}_{(AC^2)^i}(\mathcal{D}),$$

*where $\mathcal{D} = (\epsilon_0, t] \times [-1/t, 0]$.*

*Proof.* By definition, $g(x) = T^i(x)$ where $i = i(x) \in \mathbb{N}$ is minimal such that $T^i(x) < \epsilon_0$. Since $(\epsilon_0, t] = \Delta(\,(1,1)^{n-2}(1,2)\,)$, we have that $g(x) = U^j \circ T(x)$ where $j \geq 0$ is minimal such that the image is outside of the rank $n-1$ cylinder $\Delta(\,(1,2)^{n-2}(1,1)\,)$.

Now, $(x,y) \in \mathcal{D}$ if and only if $T(x) \in \Delta(\,(1,2)^{n-2}(1,1)\,)$ and therefore the $\mathcal{T}$-orbit of $(x,y)$ includes the initial sequence $\{\mathcal{T}_{(AC^2)^i}(x,y)\}_{i=0}^{n-2}$. Furthermore, due to the bijectivity of $\mathcal{T}$, any $(x,y) \in \cup_{i=0}^{n-2} \mathcal{T}_{(AC^2)^i}(\mathcal{D})$ must belong to a length $n-1$ orbit sequence with an initial orbit element in $\mathcal{D}$.

Suppose now that $(x, y) \in \Gamma$ and $\mathcal{T}(x, y) \notin \mathcal{D}$. From the previous paragraph, we have that $g(x) = T(x) = M \cdot x$ for some matrix $M$ and hence both $\mathcal{G}(x, y) = \mathcal{T}(x, y) = \mathcal{T}_M(x, y)$ and $\mathcal{G}(x, y)$ must indeed belong to $\Gamma$.

On the other hand, if $(x, y) \in \Gamma$ with $\mathcal{T}(x, y) \in \mathcal{D}$, then $T(x) \in (\epsilon_0, t)$ and there is a $j \in \mathbb{N}$ such that $\mathcal{G}(x, y) = \mathcal{T}_{U^j} \circ \mathcal{T}(x, y) = \mathcal{T}_{AC} \circ \mathcal{T}_{(AC^2)^{n-2}} \circ \mathcal{T}_{U^{j-1}} \circ \mathcal{T}(x, y)$. But, $\mathcal{T}_{U^{j-1}} \circ \mathcal{T}(x, y) \in \mathcal{D}$ and hence while $(\mathcal{T}_{AC})^{-1} \circ \mathcal{G}(x, y) \in \cup_{i=0}^{n-2} \mathcal{T}_{(AC^2)^i}(\mathcal{D})$, the application of $\mathcal{T}_{AC}$ must send this value outside of that union. That is, here also $\mathcal{G}(x, y)$ must belong to $\Gamma$.

The bijectivity of $\mathcal{G}$ on $\Gamma$ now follows immediately from that of $\mathcal{T}$ on $\Omega$. □

**Corollary 2.6.** *Fix $n \geq 3$ and let $g(x)$ be induced from $T_{n,1}$ and let $\Gamma$ be as above. Then $g(x)$ is expansive and is ergodic with respect to the probability measure that is the normalized marginal measure from $\mu = (1 + xy)^{-2}\, dx\, dy$ on $\Gamma$.*

*Proof.* From the definition of $\Omega$, it follows that $\Omega$ meets the curve $y = -1/x$ exactly in the $\mathcal{T}$-orbit of $(t, -1/t)$, which is of course a point in $\mathcal{D}$. As well, the remainder of $\Omega$ lies above this curve (with $x \geq 0$). Since $\Gamma = \Omega \setminus \cup_{i=0}^{n-2} \mathcal{T}_{(AC^2)^i}(\mathcal{D})$, it follows that $\Gamma$ not only does not meet the curve, but in fact stays a bounded distance away. From this, hypothesis (b) of Theorem 2.3 is satisfied for the piecewise Möbius map $g(x)$ on the interval $(0, \epsilon_0)$ with $\mathcal{G}$ bijective on $\Gamma$; the other hypotheses are easily verified and hence the result holds. □

## 3. Quilting as a proof tool

We discuss a technique used to date for solving for the planar extension of a piecewise Möbius interval map beginning with such a planar extension for a sufficiently "nearby" map. This technique, called *quilting*, was introduced in [KSSm], and has its roots in the discussion of the two-dimensional interpretation of "insertion" and "deletion" in the Ph.D. dissertation [K]. Theorem 3.3, shows that one can use quilting to prove that certain properties are shared between appropriately nearby systems.

3.1. **Quilting defined, main properties announced.** We give a basic definition.

**Definition 3.1.** Suppose that $f, g$ are piecewise Möbius interval maps on $\mathbb{I}_f, \mathbb{I}_g$ each with uncountably many cylinders and with finite nonzero $\mu$-measure planar two-dimensional domains of bijectivity $\Omega_f, \Omega_g$ for corresponding two-dimensional maps $\mathcal{F}, \mathcal{G}$, respectively. For $x \in \mathbb{I}_f$ let $b_f(x)$ denote the $f$-digit of $x$ (informally, this thus denotes the corresponding Möbius transformation which applied to $x$ gives the value $f(x)$ ), and similarly for $b_g(x)$. We let

$$\Delta = \Delta_{f,g} = \{x \in \mathbb{I}_f \cap \mathbb{I}_g \mid b_f(x) \neq b_g(x)\} \text{ and}$$
$$\mathcal{C} = \mathcal{C}_f = \{(x, y) \in \Omega_f \mid x \in \Delta\}.$$

We then construct a domain on which we will show that $\mathcal{G}$ is bijective (up to sets of measure zero) by deleting the forward $\mathcal{F}$-orbit of $\mathcal{C}$ and adding in the forward $\mathcal{G}$-orbit of $\mathcal{C}$ (here we extend $\mathcal{G}$ to be the piecewise map on $\mathbb{I}_g \times \mathbb{R}$ given by the $\mathcal{T}_M$ where $g$ is given piecewise by $x \mapsto M \cdot x$, recall (2)). In general, each of these orbits is infinite and might even sweep out the respective domains up to measure zero. For a practical version of this approach, we introduce some finiteness conditions.

Recall that we are interested in measure theoretic results, and thus use disjointness of sets to mean that they meet in at most a null set. See Figure 4 for representations of the two cases giving in item (i) below.

**Definition 3.2.** We say that $\Omega_g$ can be *countably quilted* from $\Omega_f$ if $\mathcal{C}$ has positive measure and i.) There is an at most countable partition of $\mathcal{C}$ by $\mathcal{C}_i$ and corresponding integers $d_i, a_i$ such that either: (a) $\mathcal{F}^{1+d_i}_{|\mathcal{C}_i} = \mathcal{G}^{1+a_i}_{|\mathcal{C}_i}$ ; or, (b) both $\mathcal{F}^{d_i}(\,\mathcal{C}_i\,) \subset \mathcal{C}$ and $\mathcal{G}^{1+a_i}_{|\mathcal{C}_i} = \mathcal{G} \circ \mathcal{F}^{d_i}_{|\mathcal{C}_i}$ ;

ii.) The set of the $\mathcal{F}^j(\,\mathcal{C}_i\,)$ indexed over all $i$ and $1 \le j \le d_i$ is pairwise disjoint and has strictly less than full measure in $\Omega_f$, and also the set of all of the $\mathcal{G}^j(\mathcal{C}_i)$ has finite measure; and,

iii.)

$$\Omega_g = \left(\Omega_f \setminus \coprod_{i=1}^{\infty}\coprod_{j=1}^{d_i} \mathcal{F}^j(\,\mathcal{C}_i\,)\right) \amalg \coprod_{i=1}^{\infty}\coprod_{j=1}^{a_i} \mathcal{G}^j(\mathcal{C}_i)\,. \tag{4}$$

Of course, when the partition is finite of cardinality $n$, then $n$ replaces the infinite upper limits appearing in (4). We then speak of *finite quilting*. For simplicity's sake, we will write *quilting* to denote countable quilting.

**Theorem 3.3.** *Assume that $\Omega_f$ is of finite $\mu$-measure. Quilting preserves the properties of: ergodicity of two dimensional maps; the planar extension giving the natural extension of the interval map's system; and, under this second property, allows for an explicit expression of the entropy of $g$ in terms of that of $f$.*

We give precise statements in the propositions of the subsequent subsection, which together prove the theorem.

### 3.2. **Proofs of main properties.**

**Proposition 3.4.** *Suppose that $f, g$ are piecewise Möbius interval maps such that $\Omega_g$ can be quilted from $\Omega_f$, and that $\mu(\Omega_f) < \infty$. If $\mathcal{F} : \Omega_f \to \Omega_f$ is ergodic with respect to the measure $\mu$, then both $\mathcal{G}$ and $g$ are ergodic, with respect to $\mu$ on $\Omega_g$ and its marginal measure on $\mathbb{I}_g$, respectively.*

*Proof.* Since the dynamical system of $g$ is a factor of that of $\mathcal{G}$, its ergodicity will follow from that of this latter system. Now suppose $E \subset \Omega_g$ is $\mu$-measurable, is not of full measure, and is invariant under $\mathcal{G}$. We aim to show that $E$ is a null set. Recall that both $\mathcal{F}, \mathcal{G}$ preserve the measure.

By assumption, $\mathcal{G}$ agrees with $\mathcal{F}$ on $\Omega_g \cap (\Omega_f \setminus \mathcal{C})$. The ergodicity of $\mathcal{F}$ implies that we may assume that $E$ is disjoint from this set. That is, we may assume that $E$ is contained in the set of forward $\mathcal{G}$-orbits of points of $\mathcal{C}$. Each of these has an initial orbit segment corresponding to an initial $\mathcal{F}$-orbit with either (a) eventual equality (which from our assumption must be in $\mathcal{C}$), or (b) such that the $\mathcal{F}$-orbit segment returns to $\mathcal{C}$ and thereafter an application of $\mathcal{G}$ results in equality. If only the first case arises, then by choosing the interpolating forward $\mathcal{F}$-orbit segments until the equalities, we form an $\mathcal{F}$-invariant set. The ergodicity of $\mathcal{F}$ shows that this is a nullset and therefore so must be $E$. If the second case does arise, we then choose the forward $\mathcal{F}$-orbit segments until their return to $\mathcal{C}$, and are in either case (a) or (b) and can continue our process. Again, the result is $\mathcal{F}$-invariant, and we conclude that $E$ is a nullset. □

**Proposition 3.5.** *Suppose that $f, g$ are piecewise Möbius interval maps such that $\Omega_g$ can be quilted from $\Omega_f$, and that the dynamical system of $\mathcal{F}$ is the natural extension of that of $f$. Then the dynamical system of $\mathcal{G}$ is the natural extension of that of $g$.*

*Proof.* The natural extension is the minimal invertible system of which our given system is a factor. Let us call any bi-infinite sequence $(x_i)_{i\in\mathbb{Z}}$ with each $x_i \in \mathbb{I}_g$ satisfying that for all $i$, $g(x_i) = x_{i+1}$ a *bi-infinite $g$-orbit* and similarly for our other maps. To show that the system of $\mathcal{G}$ is the natural extension of the system of $g$, it suffices to show that for every bi-infinite $g$-orbit

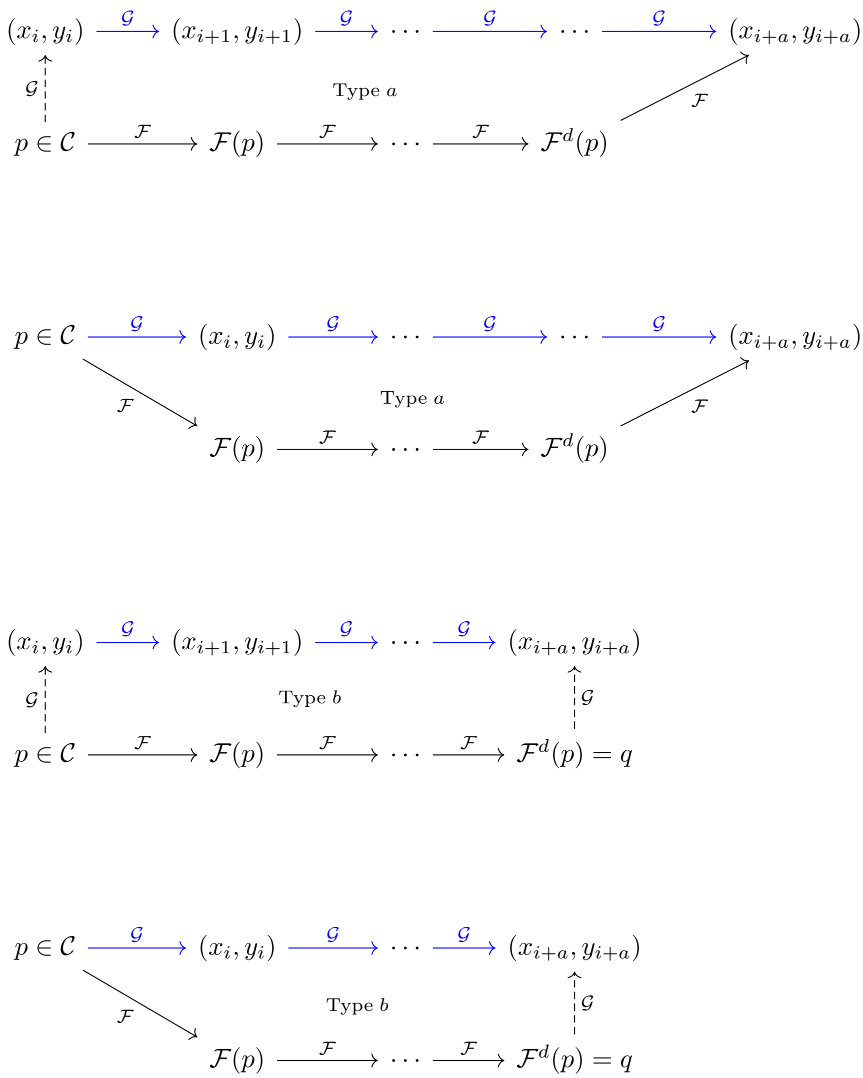


FIGURE 4. The two basic behavior of forward orbits for the hypotheses of quilting are identified by a point at the end of a forward orbit segment either lying in $\Omega_f \cap \Omega_g$ giving type (a); or not, type (b). Such segments begin at some $p \in \mathcal{C}$, which can be either be in both orbits (second and fourth from the top), or only in $\Omega_f$. Here vertical dashed lines indicate applications of $\mathcal{G}$ that are not on points in $\Omega_g$.

there is a unique bi-infinite $\mathcal{G}$-orbit $(x_i, y_i)_{i\in\mathbb{Z}}$ in $\Omega_g$. By hypothesis, the analogous statement is true for the pair $f, \mathcal{F}$.

We claim that there is a function $\delta \mapsto \gamma$ from bi-infinite $f$-orbits to bi-infinite $g$-orbits, up to shifts (that is, up to reindexing the sequence). If $\delta$ remains in $\mathbb{I}_f \setminus \Delta$ then since $g$ agrees with $f$

here, we let $\gamma$ equal $\delta$. Otherwise, $\delta$ must meet $\Delta$ and hence there is some $x_i \in \delta$ and integers $a, d$ with either $f^{d+1}(x_i) = g^{a+1}(x_i)$, or $g \circ f^d(x_i) = g^{1+a}(x_i)$. We then replace the subsequence $x_1, \dots, x_d$ by the values $g(x_i), g^2(x_i), \dots, g^a(x_i)$. The disjointness required in Definition 3.2 shows that none of these values lies in $\Delta$, furthermore the $f, g$ forward orbits agree after this "one-dimensional surgery" until the next encounter with $\Delta$. We can thus repeat our surgery at each such encounter. By taking limits as $i \to -\infty$, this results in a bi-infinite $g$-orbit. Shifting allows the resulting orbit to be independent the choice of our initial $i$.

Now, we can change the direction of quilting, and thus find that to any bi-infinite $g$-orbit we can likewise assign a bi-infinite $f$-orbit, up to shifting. Given two bi-infinite $\mathcal{G}$-orbits $(x_i, y_i)_{i\in\mathbb{Z}}, (x_i, y_i')_{i\in\mathbb{Z}}$ in $\Omega_g$, projecting to the same bi-infinite $g$-orbit, say $\gamma$, we can perform on each "two-dimensional surgeries" corresponding to the one-dimensional surgeries on $\gamma$, resulting in $\mathcal{F}$-orbits that both project to the same bi-infinite $f$-orbit, $\delta$. These two bi-infinite $\mathcal{F}$-orbits must then be equal, up to shifts. Therefore the same is true of the initial bi-infinite $\mathcal{G}$-orbits, but since these are calibrated to have the same $x_0$, they are in fact equal. □

**Proposition 3.6.** *Suppose that $f, g$ are piecewise Möbius interval maps such that $\Omega_g$ can be quilted from $\Omega_f$. Then the entropy of $\mathcal{F}$ and $\mathcal{G}$ are related by*

$$h(\mathcal{G}) = h(\mathcal{F})\,\frac{\mu(\Omega_f)}{\mu(\Omega_g)}\,.$$

*If furthermore the dynamical system of $\mathcal{F}$ is the natural extension of that of $f$, then the entropy of $g$ is given by*

$$h(g) = \left(1 + \sum_{i=1}^{\infty} (a_i - d_i)\,\nu(\,\Delta_i\,)\right)^{-1} h(f), \tag{5}$$

*where for each $i$, $\mathcal{C}_i$ projects to $\Delta_i \subset \mathbb{I}_f$ and $\nu$ is the marginal probability measure induced from $\mu$ on $\Omega_f$.*

*Proof.* We can compute the entropy of $\mathcal{G}$ in terms of that of $\mathcal{F}$ by using Abramov's formula (see [KSS]). Let $\Omega_{f,g} := \Omega_f \setminus \coprod_{i=1}^{\infty} \coprod_{j=1}^{d_i} \mathcal{F}^j(\,\mathcal{C}_i\,)$. The first return maps induced from each of $\mathcal{F}$ and $\mathcal{G}$ to $\Omega_{f,g}$ are equal. Therefore,

$$\frac{h(\mathcal{G})\,\mu(\Omega_g)}{\mu(\Omega_{f,g})} = \frac{h(\mathcal{F})\,\mu(\Omega_f)}{\mu(\Omega_{f,g})}.$$

Since our various maps are $\mu$-measure preserving,

$$h(\mathcal{G}) = \left(1 + \sum_{i=1}^{\infty} (a_i - d_i)\,\frac{\mu(\mathcal{C}_i)}{\mu(\Omega_\alpha)}\right)^{-1} h(\mathcal{F}).$$

If $\mathcal{F}$ gives the natural extension of $f$, then they have the same entropy. Furthermore, from the previous proposition, $\mathcal{G}$ then gives the natural extension of $g$ and thus these also have equal entropy. Since each $\Delta_i$ is the projection of $\mathcal{C}_i$, we have that $\nu(\,\Delta_i\,) = \mu(\mathcal{C}_i)/\mu(\Omega_\alpha)$. Therefore, (5) holds. □

3.3. **Property of first return type is also preserved.** In [AS2] a piecewise Möbius interval map $f$ whose corresponding Möbius transformations are all of determinant one and generate a Fuchsian group $\Gamma_f$ is called *of first return type* if its Lebesgue planar extension (as in the proof of Proposition 2.3 above) $\widehat{\mathcal{F}} : \Sigma_f \to \Sigma_f$ is conjugate to the first return map of the geodesic flow on the unit tangent bundle of $\Gamma_f\backslash\mathbb{H}$ under *Arnoux's map* $(x, y) \mapsto \begin{pmatrix} x & xy-1 \\ 1 & y \end{pmatrix}$. (This statement combines ([AS2] Definition 5.3 and Theorem 5.4); here $\mathbb{H}$ denotes the upper half-plane model of hyperbolic space, with the usual Möbius action on it.) Under the further assumption that $f$ is expansive and ergodic (as always, under the marginal measure of the planar extension) and $\Gamma_f$ is of finite covolume, ([AS2] Theorem 5.5) shows that $f$ is of first return type if and only if the product of the entropy $h(f)$ with the measure of $\Sigma_f$ equals the volume of the unit tangent bundle of $\Gamma_f\backslash\mathbb{H}$.

In general, minor variations of Arnoux's map are necessary; when also negative determinant Möbius transformations are involved, the construction requires a double covering and a factor of two arises in the theorem, as in the earlier [AS]. In all cases, we can suppress the reference to the Lebesgue planar extension, since the map $\mathcal{Z}$ given in (3) takes $\Omega_f$ to $\Sigma_f$, and preserves measures. In particular, the map $f$ is of first return type if and only if $h(f)\mu(\Omega_f) = w_f \cdot \text{vol}(\,T^1(\Gamma_f\backslash\mathbb{H})\,)$, where $w_f$ equals 1 in the determinant 1 case and 2 otherwise.

**Proposition 3.7.** *Suppose that $f, g$ are expansive piecewise Möbius interval maps such that $\Omega_g$ can be quilted from $\Omega_f$. Suppose that $\Gamma_f = \Gamma_g$ is of finite covolume, and $f$ is of first return type. Then $g$ is also of first return type. Furthermore, both maps: are ergodic; have their planar extensions as natural extensions; are factors of the first return map to a section for the geodesic flow on $T^1(\Gamma_f\backslash\mathbb{H})$; and, are Bernoulli systems. Moreover, the two systems are isomorphic if and only if they have equal entropy.*

*Proof.* Since $f$ is of first return type and $\Gamma_f$ is of finite covolume, ([AS2] Corollary 1) gives that $f$ is ergodic. Also, we certainly have that $\mu(\Omega_f) > 0$ which with the expansiveness of $f$ allows [AS3] (as in the discussion of Proposition 2.6 above) to imply that $\mathcal{F}$ gives the natural extension of $f$. Now, Proposition 3.6 gives $h(\mathcal{G})\mu(\Omega_g) = h(\mathcal{F})\mu(\Omega_f)$. Proposition 3.5 and the fact that entropy is shared by a map and its natural extension, then gives that $h(g)\mu(\Omega_g) = h(f)\mu(\Omega_f)$. Hence, $h(g)\mu(\Omega_g)$ equals the volume of the unit tangent bundle of $\Gamma_g\backslash\mathbb{H}$ (or in the mixed determinant case, twice this volume). That is, $g$ is of first return type.

When $f$ is of first return type, using Arnoux's map above shows that $f$ is the factor of the return map to a cross section for the geodesic flow on the unit tangent bundle of $\Gamma_f$. Orstein-Weiss [OW] showed that the geodesic flow on the unit tangent bundle of a finite volume hyperbolic orbifold or surface is a Bernoulli system. Orstein [O] showed that any factor of a Bernouilli system is itself a Bernoulli system, he [O2] then later showed that entropy is a complete isomorphism invariant for Bernoulli systems. The result thus holds. □

3.4. **Finite quilting for close neighbors that match.** We quickly give basic definitions which capture the essence of the *matching* interval phenomenon — also known as: synchronization [CKS], or short cycles [KU] — studied in various of families of continued fraction type maps. Thereafter we show that under reasonable assumptions upon matching intervals there are subintervals on which quilting applies. Our terminology and notation attempts to negotiate between that of [CT] and of [CKS].

3.4.1. *Matching: relations, intervals and their exponents.* Suppose that we are given a one parameter family of piecewise Möbius interval functions, $\{T_\alpha\}_{\alpha\in\mathcal{I}}$, indexed by $\alpha$ ranging over some real interval $\mathcal{I}$, each of whose interval of definition $\mathbb{I}_\alpha = [\,\ell_0(\alpha), r_0(\alpha)\,)$ is of fixed length $\lambda$. (We will always assume that also $|\,\mathcal{I}\,| \leq \lambda$.) In particular, letting $S : x \mapsto x + \lambda$, for all $\alpha$ we have $\ell_0(\alpha) = S^{-1} \cdot r_0(\alpha)$.

A *matching interval* $J \subset \mathcal{I}$ is a subinterval such (1) that there are some $m, n \in \mathbb{N}$ such that for all $\alpha \in J$ we have $T_\alpha^m(\,\ell_0(\alpha)\,) = T_\alpha^n(\,r_0(\alpha)\,)$ (where $m, n$ are minimal except possibly at finitely many $\alpha \in J$, we call the points where this minimality holds *non-exceptional*); (2) the digits of the expansions of the endpoints $\ell_0(\alpha)$ agree in that for all $1 \leq i < m$ there is a Möbius transformation $M_i$ such that $T_\alpha^i(\,\ell_0(\alpha)\,) = M_i \cdot \ell_0(\alpha)$ for all $\alpha \in J$, and similarly for the $r_0(\alpha)$; and, (3) there are Möbius transformations $L_J, R_J$ such that for all $\alpha \in J$ we have $T_\alpha^{n-1}(\, r_0(\alpha)\,) = R_J \cdot r_0(\alpha)$ and $T_\alpha^{m-1}(\, \ell_0(\alpha)\,) = L_J S^{-1} \cdot r_0(\alpha)$. We call $m, n$ the *matching exponents* of $J$. In fact, we need a further property of the family: say that the family has a *matching relation* if (4) there is some Möbius transformation $M$ such that for any matching interval, $M L_J S^{-1} = R_J$.

Many of the well-studied families of continued fractions have matching relations.

*Example* 3.1. We briefly indicate a list of few of these.

- The Nakada $\alpha$-continued fractions has a matching relation: let

$$W = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \tag{6}$$

  then combining ([KSS] Remark 6.9) with ([KSS] Lemmas 6.2, 6.4) shows that $M = W$ gives the relation for each matching interval $J$. (Note that we have adopted the vocabulary of [CT] who also show that there are matching intervals in the Nakada family, but express the matching relations in a different manner than here.)

- In the setting of $\alpha$-continued fraction expansions with odd partial quotients, [HKLM] show a matching relation of the form $M L_J S^{-1} = R_J$, with $M = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}$ see the final line on their p. 28.

- The countably many families of Example 2.2, coming from [CKS], are such that for each $n$ the corresponding family has a matching relation for, in the notation there, $\alpha \in (0, \gamma_{3,n})$. For each of these families, there is a distinct matching relation on $(\gamma_{3,n}, 1]$ (more precisely for those parameter values, one splits each matching interval $J$ into two pieces and find that there is a matching relation for all of the left hand pieces and a nearly identical relation for all of the right hand pieces).

3.4.2. *Close neighbors.* We are interested in applying quilting when $\alpha, \alpha'$ lie within the same synchronization interval; quilting succeeds in the most straightforward manner if we require that $\alpha, \alpha'$ are particularly close.

To lighten notation, let us write $\ell_i$ and $\ell_i'$ for each of $T_\alpha^i(\ell_0(\alpha)\,)$ and $T_{\alpha'}^i(\ell_0(\alpha')\,)$, respectively and similarly for the orbits of the other endpoints. We use the notation of Definition 3.1 in the following.

**Definition 3.8.** Suppose that $J$ is a matching interval with corresponding matching exponents $m, n$. We say that $\alpha, \alpha' \in J$ are *close neighbors* if $\ell_i', \ell_i, r_j', r_j \in \mathbb{I}_{\alpha'} \cap \mathbb{I}_\alpha$ for all $1 \le i \le m$ and $1 \le j \le n$.

Note that the hypothesis on the orbit entries can equivalently be written as: The $\alpha$-digit of $\ell_i'$ equals the $\alpha'$-digit of $\ell_i$ and vice versa for each $1 \le i < m$, and similarly for the various $r_j, r_j'$.

For ease, we will restrict to a setting that holds in the great majority of the aforementioned families.

**Definition 3.9.** Fix a family of piecewise Möbius interval maps, $\mathcal{F} = \{T_\alpha \,|\, \alpha \in \mathcal{I}\}$.

(1) We say that $\mathcal{F}$ is *of purely shift digit changes* if for any $\alpha, \alpha' \in \mathcal{I}$ whenever $x \in \Delta_{T_\alpha, T_{\alpha'}}$ then $T_\alpha(x) = S^{\pm 1} T_{\alpha'}(x)$.

(2) Suppose that $\alpha \in J$ with $J$ a matching interval of matching exponents $m, n$. We say that $\Omega_\alpha$ has *locally constant fibers* if its vertical fibers are constant between the points of the

initial orbits of $\ell_0(\alpha)$ and $r_0(\alpha)$: that is, if the fibers are constant above the connected components of the complement in $\mathbb{I}_\alpha$ of $\{\ell_i(\alpha)\,|\,0 \le i \le m-1\} \cup \{r_j(\alpha)\,|\,0 \le j \le n-1\}$.

**Proposition 3.10.** *Suppose that $\mathcal{F} = \{T_\alpha\,|\,\alpha \in \mathcal{I}\}$ is a family of piecewise Möbius interval maps of purely shift digit changes. Suppose further that $\alpha, \alpha'$ are close neighbors with $\alpha$ non-exceptional for their common matching interval. Then $\Omega_{\alpha'}$ can be finitely quilted from $\Omega_\alpha$. Furthermore, $\Omega_\alpha$ has locally constant fibers if and only if $\Omega_{\alpha'}$ does.*

*Proof.* Note that in terms of Definition 3.1 we have $\mathcal{C} = \mathcal{C}_1$, as there is only one type of digit change possible. For ease, assume that $\alpha' < \alpha$, the other case follows by a symmetric argument. Thus, $\mathcal{T}_\alpha(\mathcal{C})$ fibers over $[r_0', r_0')$ and hence we have $\mathcal{T}_{\alpha'}(\mathcal{C}) = \mathcal{T}_{S^{-1}} \circ \mathcal{T}_\alpha(\mathcal{C})$. Since $\alpha, \alpha'$ are close neighbors, they share a common matching interval $J$, let $m, n$ be its matching exponents. Let $U = U_\alpha$ be the Möbius transformation such that $\ell_m(\alpha) = UL_J \cdot \ell_0(\alpha)$ and $V = V_\alpha$ be such that $r_n(\alpha) = VR_J \cdot r_0(\alpha)$. Since $\mathcal{F}$ has only purely shift digit changes, the condition that $\ell_i', \ell_i, r_j, r_j' \in \mathbb{I}_{\alpha'} \cap \mathbb{I}_\alpha$ for the various $i, j$ implies that when $i < m$ and $j < n$ these values lie outside of $\Delta_{T_\alpha, T_{\alpha'}}$. Hence, $\mathcal{T}_{\alpha'}^{m+1}(\mathcal{C}) = \mathcal{T}_{UL_JS^{-1}} \circ \mathcal{T}_\alpha(\mathcal{C}) = \mathcal{T}_{VR_J} \circ \mathcal{T}_\alpha(\mathcal{C}) = \mathcal{T}_\alpha^{n+1}(\mathcal{C})$.

We claim that $\bigcup_{i=1}^m \mathcal{T}_{\alpha'}^i(\mathcal{C})$ is disjoint from $\Omega_\alpha$. To this end, let $k'$ be minimal such that $\mathcal{T}_{\alpha'}^{k'}(\mathcal{C}) \cap \Omega_\alpha$ has positive measure. (Since $\mathcal{T}_{S^{-1}} \circ \mathcal{T}_\alpha(\mathcal{C})$ projects to $[\ell_0', \ell_0)$ clearly $k' > 1$.) Let $(x, y) \in \mathcal{C}$ such that $\mathcal{T}_{\alpha'}^{k'}(x, y) \in \Omega_\alpha$. Again the close neighbors property gives that thereafter the forward $\mathcal{T}_{\alpha'}$-orbit of this point is given by $\alpha$-admissible Möbius transformations, and thus agrees with its forward $\mathcal{T}_\alpha$-orbit until we reach the $\mathcal{T}_{\alpha'}^{m+1}(x, y) = \mathcal{T}_\alpha^{n+1}(x, y)$. The bijectivity of $\mathcal{T}_\alpha$ then implies that there is some $k$ such that $\mathcal{T}_\alpha^k \circ \mathcal{T}_{\alpha'}^{k'}(x, y) = \mathcal{T}_\alpha^{n+1}(x, y)$ and hence $\mathcal{T}_{\alpha'}^{k'}(x, y) = \mathcal{T}_\alpha^{n+1-k}(x, y)$. If $k' < m + 1$ then by the positivity of the measure of such points, we deduce that there are factorizations $L_J = L_J''U'L_J'$ and $R_J = R_J''V'R_J'$ with $L_J'' = R_J''$ and $U'L_J'S^{-1} = V'R_J'$. Since $\alpha' < \alpha$ are close neighbors, there are other values $\alpha' < \alpha'' < \alpha$ that are also close neighbors of $\alpha$ and hence we find that there is an interval $J' \subsetneq J$ with matching exponents $m', n'$. But, this contradicts the definition of $J$ as the matching interval for its non-exceptional $\alpha, \alpha'$. Therefore, we must have that $k' = n + 1$ and the disjointness of $\bigcup_{i=1}^m \mathcal{T}_{\alpha'}^i(\mathcal{C})$ from $\Omega_\alpha$ does hold.

We next claim that the $\mathcal{T}_{\alpha'}^i(\mathcal{C})$ are pairwise disjoint. To this end, suppose that $\mathcal{T}_{\alpha'}^i(\mathcal{C})$ meets $\mathcal{T}_{\alpha'}^j(\mathcal{C})$ in positive $\mu$-measure for some $1 \le i \le j \le m+1$. Then the same is true for $\mathcal{T}_{\alpha'}^{i+m+1-j}(\mathcal{C})$ and $\mathcal{T}_{\alpha'}^{m+1}(\mathcal{C}) = \mathcal{T}_\alpha^{n+1}(\mathcal{C})$ and arguing as above, we find that $i = j$. The analogous argument shows that the $\mathcal{T}_\alpha^j(\mathcal{C})$ are disjoint.

Finally, due to the disjointness properties which we have shown, it follows that $\Omega_\alpha$ has locally constant fibers if and only if $\Omega_{\alpha'}$ does. □

3.5. **Finite quilting in less restrictive cases.** One can loosen the stringent conditions of the definition of close neighbors and find that (countable infinite) quilting still holds.

3.5.1. *Nearly close neighbors: matched values lie outside intersection.* We now consider the situation of close neighbors with the exception that at least one of the matched values of the endpoints $r_n = \ell_m, r_n' = \ell_m'$ lies outside of $\mathbb{I}_{\alpha'} \cap \mathbb{I}_\alpha$ (and the "except" means that other than the endpoints themselves, these are the first values in the respective orbits for which this is true). We say that $\alpha, \alpha'$ are *nearly close neighbors.*

**Proposition 3.11.** *Assume the hypotheses of Proposition 3.10 except that $\alpha', \alpha$ are nearly close neighbors. Then $\Omega_{\alpha'}$ can be quilted from $\Omega_\alpha$.*

*Proof.* We consider $\alpha' < \alpha$ with $\alpha$ non-exceptional in $J = J_{m,n}$. If $r_n \in \mathbb{I}_\alpha \setminus \mathbb{I}_{\alpha'}$ then of course we have $r_n > r_0' > r_n'$ and similarly the other possibility we are considering is that $r_n' < \ell_0 < r_n$. Since $r_0' < r_0$ in either case since $r_n = VR_J \cdot r_0$ and $r_n' = VR_J \cdot r_0'$ we have that $VR_J$ must

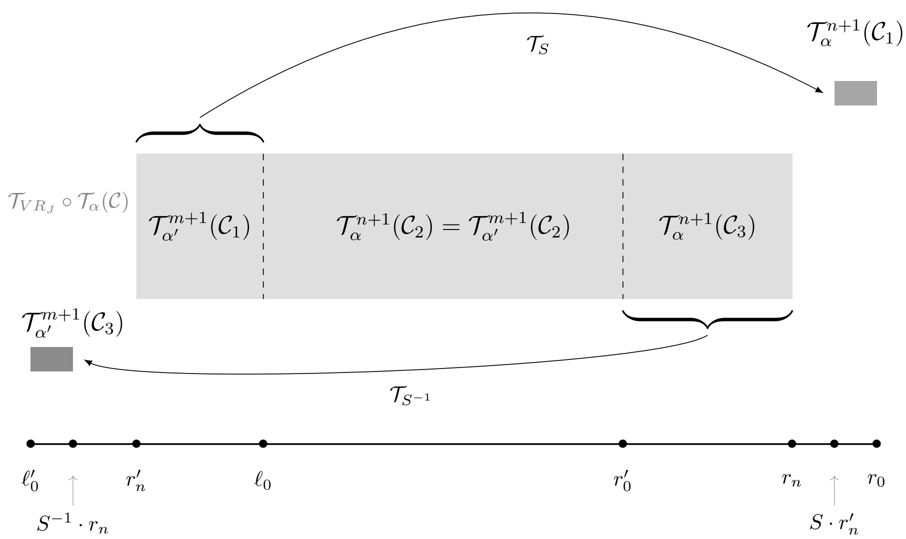


FIGURE 5. Schematic representation of quilting for $\alpha, \alpha'$ being nearly close neighbors: both of the respective initial matched values lie outside of the intersection of their intervals of definition, see Proposition 3.11 . The region $\mathcal{C}_2$ displays behavior of type (a) from Definition 3.2, while $\mathcal{C}_1, \mathcal{C}_3$ are of type (b).

be orientation preserving. In the case that $r_n > r'_0$, let $w = (VR_J)^{-1} \cdot r'_0$, when $r'_n < \ell_0$ let $z = (VR_J)^{-1} \cdot \ell_0$. With the hypotheses of Proposition 3.10 other than our stated exception, we can partition $\mathcal{C} \subset \Omega_\alpha$ by $\mathcal{C}_1 \sqcup \mathcal{C}_2 \sqcup \mathcal{C}_3$, where $\mathcal{T}_\alpha(\mathcal{C}_1)$ projects to $[r'_0, z)$, $\mathcal{T}_\alpha(\mathcal{C}_2)$ projects to $[z, w)$, and $\mathcal{T}_\alpha(\mathcal{C}_3)$ projects to $[w, r_n)$. The cases not necessitating one of $w$ or $z$ are simpler, and we leave them to the reader. For this case with both $z, w$, we show that $\mathcal{C}_2$ is of type (a) from Definition 3.2, while $\mathcal{C}_1, \mathcal{C}_3$ are of type (b); see Figure 5 for a graphical representation of the following.

We have that

$$\mathcal{T}_\alpha^{n+1}(\mathcal{C}_2) = \mathcal{T}_{VR_J} \circ \mathcal{T}_\alpha(\mathcal{C}_2) = \mathcal{T}_{UL_JS^{-1}} \circ \mathcal{T}_\alpha(\mathcal{C}_2) = \mathcal{T}_{\alpha'}^{m+1}(\mathcal{C}_2),$$

exactly as in the proof of Proposition 3.10 for the image of $\mathcal{C}$ there.

Since $T_\alpha^n(\,[r'_0, z)\,) = SVR_J \cdot (\,[r'_0, z)\,) = [S \cdot r'_n, r_0]$, we find $\mathcal{T}_\alpha^{n+1}(\mathcal{C}_1) \subset \mathcal{T}_\alpha(\mathcal{C})$ (note that we are using the fact that our family is of purely shift digit changes). Hence we have

$$\mathcal{T}_{\alpha'}(\mathcal{C}) \supset \mathcal{T}_{S^{-1}} \circ \mathcal{T}_\alpha^{n+1}(\mathcal{C}_1) = \mathcal{T}_{VR_J} \circ \mathcal{T}_\alpha(\mathcal{C}_1) = \mathcal{T}_{UL_JS^{-1}} \circ \mathcal{T}_\alpha(\mathcal{C}_1) = \mathcal{T}_{\alpha'}^{m+1}(\mathcal{C}_1).$$

Therefore, $\mathcal{T}_{\alpha'}^{m+1}(\mathcal{C}_1) = \mathcal{T}_{\alpha'} \circ \mathcal{T}_\alpha^n(\mathcal{C}_1)$. The disjointness of the $\mathcal{T}_{\alpha'}$-orbit of $\mathcal{C}_1$ with $\Omega_\alpha$ follows as in the proof of Proposition 3.10 up to $\mathcal{T}_{\alpha'}^m(\mathcal{C}_1)$. If this were to have a nontrivial intersection with $\Omega_\alpha$, then the $\mathcal{T}_\alpha$-image of this intersection would agree with its $\mathcal{T}_{SU}$-image. However, from the identities and the invertibility of $\mathcal{T}_\alpha$, we would find that on this intersection $SU = SV$ from which a contradiction to the minimality of $m, n$ can be deduced.

Finally, $\mathcal{T}_\alpha^{n+1}(\mathcal{C}_3) = \mathcal{T}_{VR_J} \circ \mathcal{T}_\alpha(\mathcal{C}_3) \subset \mathcal{T}_\alpha(\mathcal{C})$ and $\mathcal{T}_{\alpha'}^{m+1}(\mathcal{C}_3) = \mathcal{T}_{S^{-1}UL_JS^{-1}} \circ \mathcal{T}_\alpha(\mathcal{C}_3) \subset \mathcal{T}_{S^{-1}} \circ \mathcal{T}_\alpha(\mathcal{C})$. Therefore, $\mathcal{T}_{\alpha'}^{m+1}(\mathcal{C}_3) = \mathcal{T}_{\alpha'} \circ \mathcal{T}_\alpha^n(\mathcal{C}_3)$. Disjointness is argued as above. □

3.5.2. *Less then close neighbors: final matching with differing Möbius functions.* We consider $\alpha' < \alpha$ with $\alpha$ non-exceptional in $J = J_{m,n}$ as in the situation of close neighbors, except that

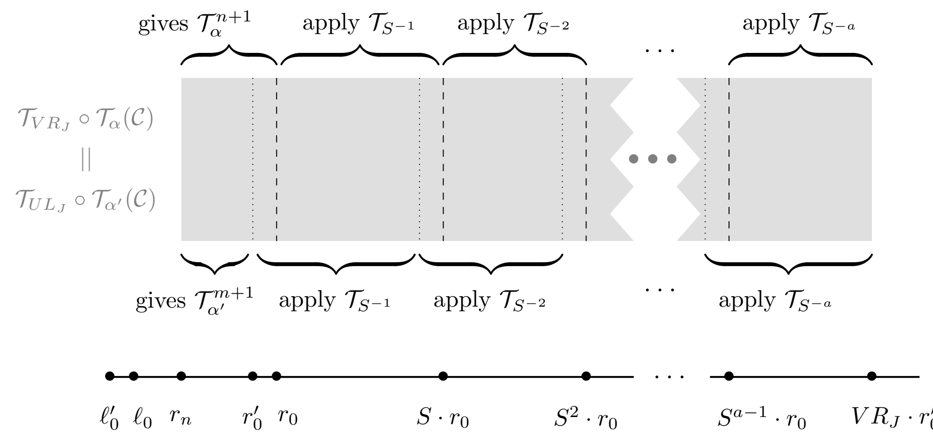


FIGURE 6. A case of less than close neighbors. Schematic representation of $\mathcal{T}_{VR_J} \circ \mathcal{T}_\alpha(\mathcal{C})$ partitioned so applying powers of $\mathcal{T}_{S^{-1}}$ gives agreement with $\mathcal{T}_\alpha^{n+1}(\mathcal{C})$ (top) or with $\mathcal{T}_{\alpha'}^{m+1}(\mathcal{C})$ (bottom). This, in the setting where initial orbit values are in $\mathbb{I}_\alpha \cap \mathbb{I}_{\alpha'}$ but $r_n' = S^{-a}V \cdot r_{n-1}'$ whereas $r_n = V \cdot r_{n-1}$, see Proposition 3.12.

the digits of the final step to reach matching in the orbit is caused by different digits, and these differ only by a power of the basic shift. That is, in the notation of above, whereas $r_n = VR_J \cdot r_0$, we have $r_n' = S^{-a}VR_J \cdot r_0'$ for some nonzero $a \in \mathbb{Z}$. Due to the matching relation, with $\ell_m = UL_J \cdot \ell_0$, we have $\ell_m' = S^{-a}UL_J \cdot \ell_0'$. We say that $\alpha, \alpha'$ are *less than close neighbors.*

With the following, refer to Figure 6.

**Proposition 3.12.** *Assume the hypotheses of Proposition 3.10 except that $\alpha', \alpha$ are less than close neighbors. Then $\Omega_{\alpha'}$ can be quilted from $\Omega_\alpha$.*

*Proof.* For ease, we restrict to the case where $a > 0$, the other case is seen analogously. We further restrict to the case that $VR_j \cdot r_0' > r_0$, with the remaining subcase again seen analogously. We have that $VR_j \cdot [r_0', r_0]$ is partitioned by $[r_n, r_0) \bigcup \cup_{i=1}^{a} S^{-i} \cdot [\ell_0, r_0) \bigcup [\ell_0, VR_j \cdot r_0']$ and similarly with respect to shifted copies of $\mathbb{I}_{\alpha'}$. We again partition $\mathcal{C}$ by $\mathcal{C}_1 \sqcup \mathcal{C}_2 \sqcup \mathcal{C}_3$, where now $\mathcal{T}_\alpha(\mathcal{C}_2)$ projects to those $x \in [r_0', r_0)$ such that $VR_j \cdot x$ lies in a shift of $\mathbb{I}_{\alpha'} \cap \mathbb{I}_\alpha$ by some power of $S$, and $\mathcal{C}_1, \mathcal{C}_3$ correspond to $x \in [r_0', r_0)$ such that such that $VR_j \cdot x$ lies in such a shift of $\mathbb{I}_{\alpha'}$ but not $\mathbb{I}_\alpha$ and *vice versa*, respectively. Since $UL_J S^{-1} = VR_J$, one easily finds that $\mathcal{C}_2$ is of type (a). Also as in the proof of Proposition 3.11, the remaining two partition elements of of type (b) and also the remaining properties hold so that quilting is verified. □

3.6. **Quilting from endpoints of matching intervals: an example.** In general, an endpoint $\alpha$ of a matching interval is announced by *one* of the *ending equalities*: $R_J \cdot r_0(\alpha) \in \{\ell_0(\alpha), r_0(\alpha)\}$ or $L_J \cdot \ell_0(\alpha) \in \{\ell_0(\alpha), r_0(\alpha)\}$. Which of the possibilities arise depend on whether $\alpha$ is the left endpoint or right endpoint, and also upon the determinant of $R_J, L_J$.

We discuss a setting where *two* ending equalities hold: $R_J \cdot r_0(\alpha) = \ell_0(\alpha)$ and $L_J \cdot \ell_0(\alpha) = r_0(\alpha)$. This setting restricts to one treated in [HKLM] using explicit numerical values, see Figure 7. Here, we wish to emphasize this as an application of countable quilting; in particular, we use our usual general notation whenever reasonable. Naturally, the reader is encouraged to verify that our description fully accords with the one given in [HKLM]. (However, their treatment

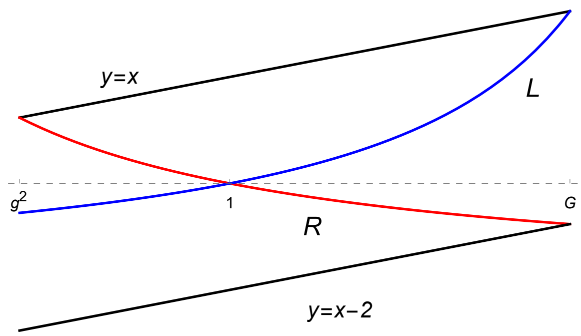


FIGURE 7. A matching interval whose right endpoint satisfies two ending equalities. The hypotheses of Proposition 3.13 are fulfilled when taking the family of $\alpha$-odd continued fractions with its matching matching interval $J = (g^2, G)$, with $G$ the golden ratio and $g = G - 1$, and any $\alpha' \in (1, G)$. Here, the labels $L, R$ mark respectively the curves $y = L_J \cdot r_0(\alpha), y = R_J \cdot r_0(\alpha)$ where $\alpha = x$ runs through $J$. The $x$-axis is shown as a dotted line.

does not accord with our philosophy of only applying the maps $\mathcal{T}_\alpha, \mathcal{T}_{\alpha'}$ when construction $\Omega_{\alpha'}$ from $\Omega_{\alpha'}$.) Confer Figure 8 for the following.

**Proposition 3.13.** *Suppose that $\mathcal{F} = \{T_\alpha \,|\, \alpha \in \mathcal{I}\}$ is a family of piecewise Möbius interval maps of purely shift digit changes and that $J$ is a matching interval for a purely shift digit change family, with matching exponents $m = n = 2$, with its right endpoint $\alpha$ such that $R_J \cdot r_0 = \ell_0$ is $\alpha$-admissible and also $S^{-1}L_J \cdot \ell_0 = \ell_0$ is admissible. Fix $\alpha' < \alpha$ in $J$ and sufficiently close such that there is some $a \in \mathbb{N}$ such that $r'_{m+n-2} = S^{-a}L_J \cdot r'_{n-1} = S^{-a}L_J R_J \cdot r'_0$. Then $\Omega_{\alpha'}$ can be countably quilted from $\Omega_\alpha$.*

*Proof.* The hypotheses defining $\alpha$ imply that $L_J$ preserves orientation whereas $R_J$ reverses orientation; since $ML_JS^{-1} = R_J$, this shows also that $M$ reverses orientation. It further follows that $T^2_\alpha(r_0) = S^{-1}L_JR_J \cdot r_0 = \ell_0$.

With our usual notation we have $V' = S^{-a}L_J$ with $a$ as above. Now, $U'L_JS^{-1} = V'R_J$ gives that $U' = V'M$ and hence $U', V'$ have opposite determinants; since $V' = S^{-a}L_J$ it follows that $U'$ reverses orientation. For ease, let $D = [r'_0, r_0)$ and $\mathcal{D} \subset \Omega_\alpha$ be the set of points fibering over $D$, and similarly, let $A = [\ell'_0, \ell_0)$ and $\mathcal{A} = \mathcal{T}_{S^{-1}}(\mathcal{D})$; that is, $\mathcal{D} = \mathcal{T}_\alpha(\mathcal{C})$ and $\mathcal{A} = \mathcal{T}_{\alpha'}(C)$.

Since $\ell'_{m-1} = L_J \cdot \ell'_0$ is certainly less than $r'_0$ and $L_J \cdot \ell_0 = r_0$, there is some $z_1 \in [\ell'_0, \ell_0)$ such that $r'_0 = L_J \cdot z_1$. That is $\ell'_0 = S^{-1}L_J \cdot z_1$. We now define a sequence: $z_0 = \ell_0$ and $z_i$ such that $S^{-1}L_J \cdot z_i = z_{i-1}$ for $i \geq 0$. Let $A_i = [z_{i-1}, z_i]$ and $D_i = S \cdot A_i$. Note that $A = \cup_{i=1}^\infty A_i$ and similarly for $D$. Let $\mathcal{D}_i \subset \mathcal{D}$ be the subset of points projecting to $D_i$, and similarly define $\mathcal{A}_i$. Then $T^{m-1}_{\alpha'}(A_1) = L_J \cdot A_1 = [\ell'_{m-1}, r'_0)$ and thus $T^{m-1}_{\alpha'}(A \backslash A_1) = S^{-1}L_J \cdot (A \backslash A_1) = [\ell'_0, \ell_0) = A$. It easily follows that for each $i$, we have $T^{i(m-1)}_{\alpha'}(A_i) = L_J(S^{-1}L_J)^{i-1} \cdot A_i = [\ell'_{m-1}, r'_0)$.

Suppose first that $U' = R_J$; equivalently, $\ell'_{m-1}, r'_0$ are in the same $\alpha'$-cylinder. Then each $T^{i(m-1)+1}_{\alpha'}(A_i) = (r'_1, \ell'_m]$. On the other hand, $T^{n-1}_\alpha(D) = R_J \cdot D = (\ell_0, r'_{n-1}]$, giving in particular that $r'_1 = r'_{n-1}$ is an element of $(\ell_0, r_0)$; since $r'_1 < \ell'_m$, also $\ell'_m = r'_n < r'_0$. Therefore, $T^n_\alpha(D_1) = V'R_J \cdot D_1 = (r'_1, r'_n]$. From $S^{-1}L_J \cdot (A \backslash A_1) = A$, we have $L_JS^{-1} \cdot (D \backslash D_1) = D$ and in our case where $U' = R_J$ we find that $V'R_J \cdot (D \backslash D_1) = U'L_JS^{-1} \cdot (D \backslash D_1) = R_JL_JS^{-1} \cdot (D \backslash D_1) =$

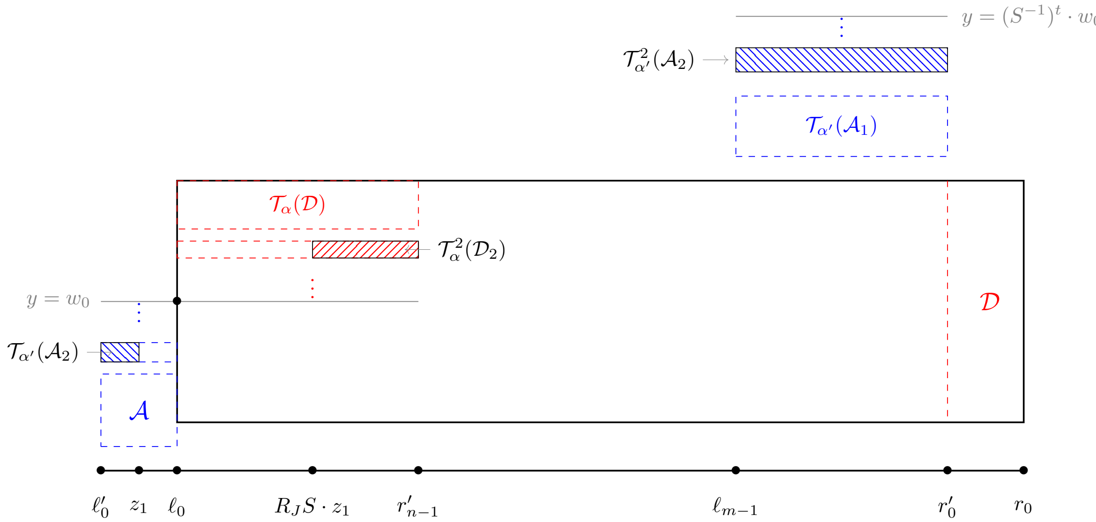


FIGURE 8. Schematic representation of countable quilting to the left from right endpoint $\alpha$ of matching interval of $m = n = 2$, with $R_J \cdot r_0 = \ell_0$ and $S^{-1}L_J \cdot \ell_0 = \ell_0$ admissible (here in the case that $U' = R_J$). From $\Omega_\alpha$, the forward $\mathcal{T}_\alpha$-orbit of $\mathcal{D} = \mathcal{T}_\alpha(\mathcal{C})$ is deleted and the forward $\mathcal{T}_{\alpha'}$-orbit of $\mathcal{A} = \mathcal{T}_{\alpha'}(\mathcal{C})$ is added, until equalities hold. See Proposition 3.13 and its proof. [HKLM] consider the odd partial quotients maps; their setting with $\alpha$ is the golden ratio $G$ and $1 < \alpha' < G$ is a special case of the result here.

$R_J \cdot D = (\ell_0, r'_1]$. (In particular, $V' = S^{-1}L_J$ in this case; this then implies that in general when $V' = S^{-a}L_J$ we have $U' = S^{1-a}R_J$.) One deduces that $T_\alpha^{i(n-1)+1}(D_i) = [r'_1, r'_n]$. It follows that in this case, $\mathcal{T}_\alpha^{i(n-1)+1}(\mathcal{D}_i) = \mathcal{T}_{\alpha'}^{i(m-1)+1}(\mathcal{A}_i)$ for each $i$ and hence letting

$$\Omega_{\alpha'} = \left(\Omega_\alpha \setminus \bigcup_{i=1}^{\infty} \bigcup_{j=0}^{i(n-1)} \mathcal{T}_\alpha^j(\mathcal{D}_i)\right) \cup \bigcup_{i=1}^{\infty} \bigcup_{j=0}^{i(m-1)} \mathcal{T}_{\alpha'}^j(\mathcal{A}_i) \tag{7}$$

does indeed give a planar extension for the system of $T_{\alpha'}$. Note that since $S^{-1}L_J$ fixes $\ell_0$, $\mathcal{T}_{S^{-1}L_J}$ has a fixed point $(\ell_0, w_0)$, and contracts $y$-values to this $w_0$. One finds that from $\Omega_\alpha$, we delete $\mathcal{D}, \mathcal{T}_{R_J}(\mathcal{D})$ and images of subsets of $\mathcal{T}_{R_J}(\mathcal{D})$ under powers of $\mathcal{T}_{S^{-1}L_J}$, while adding $\mathcal{A}$ and images of certain of its subsets under powers of $\mathcal{T}_{S^{-1}L_J}$ and finally the region given by applying $\mathcal{T}_{L_J}$ to all of these.

Finally, if $U' \neq R_J$, then each $T_{\alpha'}^{i(m-1)+1}(A_i)$ is of the form $(r'_1, r'_0) \cup [\ell'_0, r'_0) \cup \cdots \cup [\ell'_0, r'_0) \cup [\ell'_0, \ell'_m]$. Similarly, the $T_\alpha^{i(n-1)+1}(D_i)$ are each of the form $(r'_1, r_0) \cup [\ell_0, r_0) \cup \cdots \cup [\ell_0, r_0) \cup [\ell_0, r'_n]$. One finds that in fact (7) still gives the correct description. In short, $U'L_JS^{-1} = V'R_J$ implies that $S^jU'L_JS^{-1} = S^jV'R_J$ for any $j$, leading to patching in $\Omega_\alpha$ up to $x < r'_0$, but since $(x, y) \in \Omega_\alpha$ with $x \geq r'_0$ lies in $\mathcal{D}$ such remaining points will have already been deleted. Similar words apply to points arising from the $\mathcal{T}_{\alpha'}$-orbit of $\mathcal{A}$ whose $x$-coordinates are less than $\ell_0$.

□

**Remark 3.14.** In the particular setting of [HKLM] with $\alpha = G$ and $1 < \alpha' < G$ in fact both $\Omega_\alpha, \Omega_{\alpha'}$ are connected. The reader can verify that also the determination in [HKLM] of $\Omega_{\alpha'}$ for $G - 1 < \alpha' < 1$ is also a variant of infinite quilting.

## 4. Application: An alternate path to proving properties of Nakada's $\alpha$-continued fractions

To illustrate our techniques, we use an alternate description of the planar extension for each $T_\alpha$ given in [KSS] and certain results of [KSS] (not relying on the ergodicity of the $T_\alpha$) about the planar extensions of Nakada's $\alpha$-continued fractions, to recover a result of Luzzi and Marmi [LM].

**Theorem 4.1.** *For every $0 < \alpha \le 1$, the Nakada $\alpha$-continued fraction is ergodic.*

In fact, we rely on Theorem 2.3 thus in fact reprove more than just ergodicity.

### 4.1. Review of notation and results of [KSS].

Before we give the proof, it would be best to review some notation and results from [KSS]. For $\epsilon \in \{-1, 1\}$ and $d \in \mathbb{N}$, we define $M_{(\epsilon:d)} = \begin{pmatrix} -d & \epsilon \\ 1 & 0 \end{pmatrix}$ and $N_{(\epsilon:d)} = \begin{pmatrix} 0 & 1 \\ \epsilon & d \end{pmatrix}$. Note that projectively, $N_{(\epsilon:d)} = (M^{-1}_{(\epsilon:d)})^t$.

Given $\alpha \in (0,1]$ let $\Delta_\alpha(\epsilon : d)$ be the (rank one) cylinder indexed by the digit $(\epsilon : d)$ and define $\mathcal{T}_{(\epsilon:d)}$ to be the map $(x, y) \mapsto (M_{(\epsilon:d)} \cdot x, N_{(\epsilon:d)} \cdot y)$. Thus for $x \in \Delta_\alpha(\epsilon : d)$ and any $y$ we have $\mathcal{T}_\alpha(x, y) = \mathcal{T}_{(\epsilon:d)}(x, y)$.

It is easily verified both that exactly the digits $(+1 : d)$ and $(-1 : d + 1)$ are such that the image of the open interval $(0, 1)$ under $N_{(\epsilon:d)}$ meets the open interval $(1/(d + 1), 1/d)$, and that for these two digits we have $N_{(\epsilon:d)} \cdot [0, 1] = [1/(d + 1), 1/d]$. In particular, for each $d$ and any $y$ we find that

$$N_{(+1:d)} \cdot y = N_{(-1:d+1)} \cdot (1 - y). \tag{8}$$

Equivalently with $W$ is as in (6), $N_{(+1:d)} \cdot y = N_{(-1:d+1)} W^t \cdot y$. Note that since $W$ is of projective order two, this accords with the easily verified identify: $M_{(+1:d)} = M_{(-1:d+1)} W$.

For $\alpha \in (0, 1]$, let

$$\mathscr{A}_\alpha = \{\, (-1 : d') \mid 2 \le d' \le d_\alpha(\alpha) + 1\} \cup \{(+1 : d_\alpha(\alpha))\},$$

where $d_\alpha(\alpha)$ is the first $\alpha$-digit of $r_0(\alpha) = \alpha$. For $\alpha \in \mathcal{E}$ or $\alpha = \chi_v$ for some $v$, let $\mathscr{L}'_\alpha$ be the words in $\mathscr{A}_\alpha$ which are admissible $\alpha$-expansions (as well as the empty word). Then ([KSS], Lemma 7.11) shows that the region, which we rename for clarity's sake,

$$\Lambda_\alpha = \overline{\bigcup_{w \in \mathscr{L}'_\alpha} T_\alpha^{|w|}(\Delta_\alpha(w)\,) \times N_w \cdot \left[0, \tfrac{1}{d_\alpha(\alpha)+1}\right]}$$

is a bijectivity domain for $\mathcal{T}_\alpha$. In fact, the lemma is stated for all $\alpha$, upon making minor adjustments for the remaining $\alpha$: There is some $v$ such that $\alpha$ is in the same matching interval as $\chi_v$ and one defines $\mathscr{L}'_\alpha = \mathscr{L}'_{\chi_v}$ and replaces $T_\alpha^{|w|}(\Delta_\alpha(w)\,)$ by the $J_w^\alpha$ of ([KSS], (7.2) ) — this last is only a change in the cases that $w$ has a suffix which consists of a prefix of the digits of the $\alpha$-expansion of either $\alpha$ or $\ell_0(\alpha) = \alpha - 1$ extending beyond where matching occurs (in a sense, the adjustment is to keep the digits up to one step before matching). Their proof (in all cases) relies in part on the ergodicity of the $T_\alpha$ and involves showing that the bijectivity domain that they denote $\Omega_\alpha$ is equal to what we have denoted $\Lambda_\alpha$. We now turn this around, and for $\alpha \in (0, 1)$ begin with $\Lambda_\alpha$ to show ergodicity of $T_\alpha$ and more.

4.2. **Reversing the order of argument in** [KSS]**.** We wish to show that $\mathcal{T}_\alpha$ is bijective on $\Lambda_\alpha$ (as always, here and throughout, up to $\mu$-measure zero sets). Since a Möbius transformation is identified by its values on three points, and each $\mathcal{T}_{\epsilon:d)}$ is (locally) measure preserving, that surjectivity implies injectivity can be argued as in ([KSS], Lemma 5.2).

Certainly the image of $\Lambda_\alpha$ under $\mathcal{T}_\alpha$ contains the union over the non-empty words $w \in \mathscr{L}'_\alpha$ of the $T_\alpha^{|w|}(\Delta_\alpha(w)\,) \times N_w \cdot \big[0, \frac{1}{d_\alpha(\alpha)+1}\big]$. The main challenge is to show that all of $\mathbb{I}_\alpha \times \big[0, \frac{1}{d_\alpha(\alpha)+1}\big]$ is in the image, where $\mathbb{I}_\alpha = [\alpha-1, \alpha)$. For this, we introduce notation for the fiber in $\Lambda_\alpha$ over a point $x$: for each $x \in \mathbb{I}_\alpha$, let $\Phi_\alpha(x) = \{y \mid (x,y) \in \Lambda_\alpha\}$.

4.2.1. *Surjectivity follows from fiber symmetry.* We show surjectivity of $\mathcal{T}_\alpha$ by way of an interesting detail that seems not to have been observed in the literature. The fibers over the cylinders of the values not in $\mathscr{A}_\alpha$ satisfy a certain symmetry property. In particular, for any sufficiently large negative $x$, the sets $W^t \cdot \Phi_\alpha(x)$ and $\Phi_\alpha(W \cdot x)$ are disjoint and have union whose closure is $[0,1]$. The reader is encouraged to view the various representations of planar domains $\Omega_\alpha$ given in, say, [KSS] to see that this is reasonable. See also Figure 9.

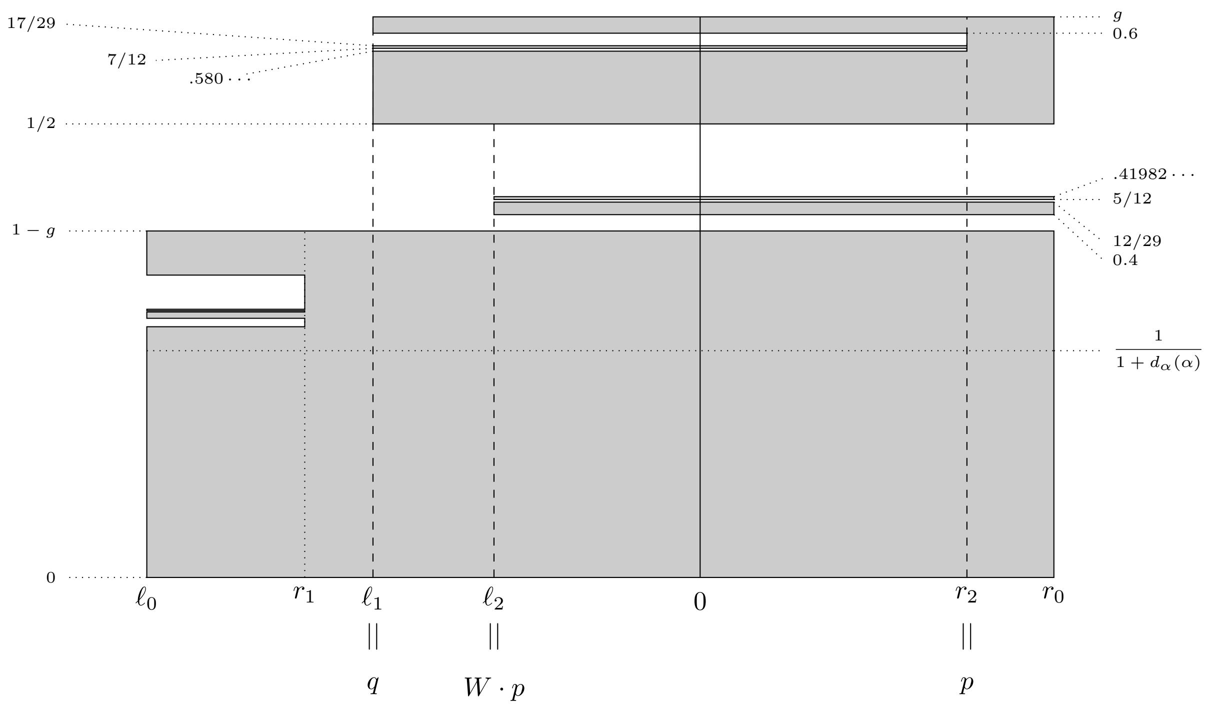


FIGURE 9. The planar domain $\Omega_\alpha$ for Nakada's continued fraction of $\alpha = 0.39$. Here $d_\alpha(\alpha) = 3$. The proof of Proposition 4.2 refers to $p, W \cdot p, q$ in general cases.

**Proposition 4.2.** *For $\alpha \in (0,1)$ the map $\mathcal{T}_\alpha$ is bijective from $\Lambda_\alpha$ to itself.*

*Proof.* Fix $\alpha$. From the definition of $\Lambda_\alpha$, surjectivity onto the complement of $\mathbb{I}_\alpha \times \big[0, \frac{1}{d_\alpha(\alpha)+1}\big]$ is immediate.

The proof of ([KSS], Lemma 5.1) shows, based upon the fact that there is an explicit manner to rewrite $T_\alpha$-orbits in terms of regular continued fraction $T_1$-orbits, that the rectangle $\big[0, \frac{1}{d_\alpha(\alpha)+1}\big]^2$

is contained in the closure of the $\mathcal{T}_\alpha$-orbits of the points contained in this rectangle. The admissible $(\epsilon : d) \notin \mathscr{A}_\alpha$ are exactly those values such that $N_{(\epsilon:d)} \cdot [0,1] \subset \left[0, \frac{1}{d_\alpha(\alpha)+1}\right]$. The $y$-values here show that each $\mathcal{T}_\alpha$-orbit returns to the rectangle only upon an application of some $\mathcal{T}_{(\epsilon:d)}$ with $(\epsilon : d) \notin \mathscr{A}_\alpha$.

It follows that for each $d \geq d_\alpha(\alpha) + 1$, we have that $[1/(d+1), 1/d]$ equals the closure of the union of $N_{(+1:d)} \cdot \Phi_\alpha(x)$ with $N_{(-1:d+1)} \cdot \Phi_\alpha(x')$ for every pair $x \in \Delta_\alpha(+1 : d)$ and $x' \in \Delta_\alpha(-1 : d+1)$ which are sent by $T_\alpha$ to the same value in $\left[0, \frac{1}{d_\alpha(\alpha)+1}\right]$. By (8) (in the equivalent form given in the line directly below it), this implies that $[0,1] = \overline{W^t \cdot \Phi_\alpha(x) \cup \Phi_\alpha(x')}$ for each such pair.

Now, for $\alpha \in \mathcal{E}$ ([KSS], Lemma 7.9) shows that all of the digits of the expansions of both $\alpha - 1$ and $\alpha$ are contained in $\mathscr{A}_\alpha$. Since the only non-full cylinders are associated with prefixes of these expansions, the fibers $\Phi_\alpha(x)$ are constant for all $x \in \left[\frac{-1}{d_\alpha(\alpha)+1}, \frac{1}{d_\alpha(\alpha)+1}\right]$. Therefore, $[0,1] = \overline{W^t \cdot \Phi_\alpha(x) \cup \Phi_\alpha(x')}$ holds for *every* pair $x \in \Delta_\alpha(+1 : d)$ and $x' \in \Delta_\alpha(-1 : d+1)$ that are sent by $T_\alpha$ to the same value. That is, the closure of the union of the images of $\Lambda_\alpha$ under the various $\mathcal{T}_{(+1:d)}$ and $\mathcal{T}_{(-1:d+1)}$ fills out all of $\mathbb{I}_\alpha \times \left[0, \frac{1}{d_\alpha(\alpha)+1}\right]$. Surjectivity holds in this case.

In the case of $\alpha$ of the form $\chi_v$, ([KSS], Lemma 7.9) shows that the digits of the expansions of the two endpoints $\alpha - 1, \alpha$ remain in $\mathscr{A}_\alpha$ until they match at the value zero. One finds that the fibers $\Phi_\alpha(x)$ are constant for all $x > 0$ and also for all $x < 0$ whose $\alpha$-digit is at least $(-1 : d_\alpha(\alpha) + 2)$. We conclude also in this case that $[0,1] = \overline{W^t \cdot \Phi_\alpha(x) \cup \Phi_\alpha(x')}$ holds for *every* pair $x, x'$ as above, and again the result holds.

In the remaining case, $\alpha$ is in the same matching interval as some $\chi_v$, and ([KSS], Lemma 7.9) shows that up to their penultimate digits before matching, digits of the expansions of the two endpoints $\alpha - 1, \alpha$ remain in $\mathscr{A}_\alpha$; ([KSS], Lemma 6.2) implies that the values exactly before matching differ by an application of $W$. Let $p$ denote the larger of these values, thus $W \cdot p$ is the other value; also let $q$ be the maximum value of the rest of $T_\alpha$-orbits of the endpoints $\alpha - 1, \alpha$ up to these index values. The fibers $\Phi_\alpha(x)$ are constant over each of the intervals $[q, W \cdot p), [W \cdot p, p), [p, \alpha)$, with respective values $\Phi_\alpha(q), \Phi_\alpha(W \cdot p), \Phi_\alpha(p)$. Directly related to this is that for any $d > 0$, points $x \in \Delta_\alpha(+1 : d)$ and $x' \in \Delta_\alpha(-1 : d+1)$ are sent by $T_\alpha$ to the same value if and only if $x' = W \cdot x$. (We could have used this in the previous cases, but preferred to minimize notation.)

Since $p > 0$, there exists some $d > d_\alpha(\alpha)$ such that $p$ is strictly greater than all values in $\Delta_\alpha(+1 : d)$. Hence for all $x \in \Delta_\alpha(+1 : d)$ we have $\Phi_\alpha(x) = \Phi_\alpha(W \cdot p)$ and since also $W \cdot p < W \cdot x$ also $\Phi_\alpha(W \cdot x) = \Phi_\alpha(W \cdot p)$. Since there are $x \in \Delta_\alpha(+1 : d)$ such that $[0,1] = \overline{\Phi_\alpha(x) \cup W^t \cdot \Phi_\alpha(W \cdot x)}$, we find that it is always the case that

$$[0,1] = \overline{\Phi_\alpha(W \cdot p) \cup W^t \cdot \Phi_\alpha(W \cdot p)}. \tag{9}$$

From this for any $0 < x < p$ we find that $[0,1] = \overline{\Phi_\alpha(x) \cup W^t \cdot \Phi_\alpha(W \cdot x)}$ and in particular for all $d > 0$ such that $p$ is strictly greater than all values in $\Delta_\alpha(+1 : d)$ we have that $\mathbb{I}_\alpha \times [1/d, 1/(d+1)]$ is contained in $\mathcal{T}_\alpha(\Lambda_\alpha)$.

We next claim that

$$\Phi_\alpha(p) \cup W^t \cdot \Phi_\alpha(q) = \Phi_\alpha(W \cdot p) \cup W^t \cdot \Phi_\alpha(W \cdot p).$$

To prove this, recall that [KSS] use $k', k$ as matching exponents and use $E$ to denote the matrix which acts as shift by $-1$, and show that there is an $M_v$ such that $T_\alpha^{k'-1}(\alpha - 1) = M_v \cdot (\alpha - 1) = M_v E \cdot \alpha$ and $T_\alpha^{k-1}(\alpha) = W M_v E \cdot \alpha$. Thus $\{p, W \cdot p\} = \{M_v E \cdot \alpha, W M_v E \cdot \alpha\}$ and $\Phi_\alpha(W \cdot p)$ is the union of $\Phi_\alpha(q)$ with one of either $N_v \cdot \Phi_\alpha(\alpha - 1)$ or $W^t N_v (E^{-1})^t \cdot \Phi_\alpha(\alpha)$. Similarly, $\Phi_\alpha(p)$

is the union of $\Phi_\alpha(q)$ with both $N_v \cdot \Phi_\alpha(\alpha-1)$ and $W^t N_v (E^{-1})^t \cdot \Phi_\alpha(\alpha)$. Since $W$ and hence $W^t$ is of projective order two, the claim holds.

Using (9) and the claim, we find for all $d > d_\alpha(\alpha)$ that $\mathbb{I}_\alpha \times [1/d, 1/(d+1)]$ is contained in $\mathcal{T}_\alpha(\Lambda_\alpha)$. Thus, our proof of surjectivity is complete.

For completeness, we repeat that injectivity can be now be deduced as in ([KSS], Lemma 5.2). $\square$

We mention that a variant of ([KSS] Lemma 7.5) follows from the above proof.

**Corollary 4.3.** *If $\alpha \neq \chi_v$ is in the same matching interval as $\chi_v$, then $E^t \cdot \Phi_\alpha(\alpha-1) = \Phi_\alpha(\alpha)$.*

*Proof.* Using the notation of the proof of the Proposition, we certainly have that $p < \alpha$ and hence either there is some left portion of $\Delta_\alpha(d_\alpha(\alpha))$ that has fibers $\Phi_\alpha(x) = \Phi_\alpha(W \cdot p)$ with also $\Phi_\alpha(W \cdot x) = \Phi_\alpha(W \cdot p)$, or else such that $\Phi_\alpha(x) = \Phi_\alpha(p)$ and $\Phi_\alpha(W \cdot x) = \Phi_\alpha(q)$. In either case, since $M_{(+1:d)}$ for any $d > 0$ is orientation reversion, we find that $\Phi_\alpha(\alpha) \supset [1/d_\alpha(\alpha), 1/(d_\alpha(\alpha)+1)]$.

Note that since all cylinders are right full, $\Phi_\alpha(\alpha) = \overline{\bigcup_{w \in \mathscr{L}'_\alpha} N_w \cdot \left[0, \frac{1}{d_\alpha(\alpha)+1}\right]}$. Letting $\mathscr{L}_\alpha \subset \mathscr{L}'_\alpha$ consist of the $w$ which index full (rank $|w|$) cylinders (where we include the $w$ of suffix agreeing with expansion of either of $\alpha, \alpha-1$ beyond the matching digits), we have that $\Phi_\alpha(\alpha-1)$ is the subset of $\Phi_\alpha(\alpha)$ by taking the union only over $w \in \mathscr{L}_\alpha$. No word in $\mathscr{L}_\alpha$ ends with the initial digits of the endpoints, thus with $(-1:2)$ or $(+1:d_\alpha(\alpha))$. Thus, changing the final digit of $w \in \mathscr{L}_\alpha$ (of length at least one) by applying $E^{-1}$ results in a word in $\mathscr{L}'_\alpha$. Now, as in ([KSS] Lemma 7.2), we can factor $\mathscr{L}'_\alpha = \mathscr{L}_\alpha \cup \bigcup_{1 \le j < k} \mathscr{L}_\alpha \, \underline{b}^\alpha_{[1,j]} \cup \bigcup_{1 \le j < k'} \mathscr{L}'_\alpha \, \overline{b}^\alpha_{[1,j]}$ where matching occurs at $k', k$. Hence, applying $E^{-1}$ to the final digit of every word of length at least one in $\mathscr{L}_\alpha$ gives all of the words in $\mathscr{L}'_\alpha$ except those ending in $(+1:d_\alpha(\alpha))$ or $(-1:d_\alpha(\alpha)+1)$. Since $E^t$ sends $\left[0, \frac{1}{d_\alpha(\alpha)+1}\right]$ to $\left[0, \frac{1}{d_\alpha(\alpha)}\right]$, we conclude that $E^t \cdot \Phi_\alpha(\alpha-1) = \Phi_\alpha(\alpha)$. $\square$

4.3. **Proof of ergodicity and more.** It is immediate that each $\Lambda_\alpha$ has positive $\mu$-measure. Since $N_a \cdot [0,1] \subset [0,1]$ for every possible digit for any given $\alpha$, it is clear that $\Lambda_\alpha \subset [\alpha-1, \alpha] \times [0,1]$. Therefore, the compact $\Lambda_\alpha$ is bounded away from $y = -1/x$ and has finite vertical fibers. Recall that Lemma 2.2 guarantees that every rational $\alpha \in (0,1]$ and every $\alpha \in \mathcal{E}$ is of bounded non-full range. Furthermore, ([KSS] Theorem 5) shows that if $\alpha$ is an endpoint of a matching interval then both $\alpha - 1$ and $\alpha$ have periodic $T_\alpha$-expansions and thus these maps also are of bounded non-full range. By Theorem 2.3 with $\Omega = \Lambda_\alpha$, we find ergodicity of $T_\alpha$ and the other properties listed in the statement of the theorem. Each of the remaining $\alpha' \in (0,1)$ lies in the interior of some matching interval and the density of the rationals gives that $\alpha'$ has some non-exceptional $\alpha$ as a close neighbor. From Proposition 3.10, combined with Theorem 3.3 we have that each $\mathcal{T}_\alpha : \Lambda_\alpha \to \Lambda_\alpha$ gives the natural extension to the system of $T_\alpha$, which is in particular ergodic.

## References

[Ad] R. Adler, *Continued fractions and Bernoulli trials*, in: Ergodic Theory (A Seminar), J. Moser, E. Phillips and S. Varadhan, eds., Courant Inst. of Math. Sci. (Lect. Notes 110), 1975, New York.

[Ar] P. Arnoux, *Le codage du flot géodésique sur la surface modulaire.* Enseign. Math. (2) 40 (1994), no. 1-2, 29–48.

[AS] P. Arnoux and T. A. Schmidt, *Cross sections for geodesic flows and α-continued fractions*, Nonlinearity 26 (2013), 711–726.

[AS2] ———, *Commensurable continued fractions*, Discrete and Continuous Dynamical Systems - Series A (DCDS-A) Vol. 34, no. 11, (2014) 4389–4418.

[AS3] ———, *Natural extensions and Gauss measures for piecewise homographic continued fractions*, Bull. Soc. Math. France 147 (2019) 515–544.

[BJW] W. Bosma, H. Jager, and F. Wiedijk, *Some metrical observations on the approximation by continued fractions*, Nederl. Akad. Wetensch. Indag. Math. **45** (1983), no. 3, 281–299.

[B] R. Bowen, *Invariant measures for Markov maps of the interval*, With an afterword by Roy L. Adler and additional comments by Caroline Series. Comm. Math. Phys. 69 (1979), no. 1, 1–17.

[BKS] R. M. Burton, C. Kraaikamp and T. A. Schmidt, *Natural extensions for the Rosen fractions*, Trans. Amer. Math. Soc. 352 (1999), 1277–1298.

[CKS] K. Calta, C. Kraaikamp and T. A. Schmidt, *Synchronization is full measure for all $\alpha$-deformations of an infinite class of continued fraction transformations*, Annali della Scuola Normale Superiore di Pisa, Classe di Scienze (5) Vol. XX (2020), 951–1008.

[CS] K. Calta and T. A. Schmidt, *Continued fractions for a class of triangle groups*, J. Austral. Math. Soc., 93 (2012) 21–42.

[CT] C. Carminati and G. Tiozzo, *A canonical thickening of $\mathbb{Q}$ and the entropy of $\alpha$-continued fraction transformations*, Ergodic Theory Dynam. Systems 32 (2012), no. 4, 1249–1269.

[DKS] K. Dajani, C. Kraaikamp, W. Steiner, *Metrical theory for $\alpha$-Rosen fractions*, J. Eur. Math. Soc. (JEMS) 11 (2009), no. 6, 1259–1283.

[HKLM] Y. Hartano, C. Kraaikamp, N. Langeveld and C. Merriman, *Natural extensions and entropy of $\alpha$-continued fraction expansions with odd partial quotients*, `arXiv:2106.08789v2`

[IY] S. Ito and M. Yuri, *Number theoretical transformations with finite range structure and their ergodic properties*, Tokyo J. Math. 10 (1987), no. 1, 1–32.

[KU] S. Katok, I. Ugarcovici, *Structure of attractors for (a,b)-continued fraction transformations*, J. Mod. Dyn. 4 (2010), no. 4, 637–691.

[K] C. Kraaikamp, *A new class of continued fraction expansions*, Acta Arith. 57 (1991), no. 1, 1–39.

[KSSm] C. Kraaikamp, T. A. Schmidt and I. Smeets, *Natural extensions for $\lambda$-Rosen continued fractions*, J. Math. Soc. Japan 62, No. 2 (2010), 649–671

[KSS] C. Kraaikamp, T. A. Schmidt and W. Steiner, *Natural extensions and entropy of $\alpha$-continued fractions*, Nonlinearity 25 (2012), no. 8, 2207–2243.

[LM] L. Luzzi and S. Marmi, *On the entropy of Japanese continued fractions*, Discrete Contin. Dyn. Syst. **20** (2008), no. 3, 673–711.

[N] H. Nakada, *Metrical theory for a class of continued fraction transformations and their natural extensions*, Tokyo J. Math. 4 (1981), 399-426.

[NIT] H. Nakada, S. Ito, and S. Tanaka, *On the invariant measure for the transformations associated with some real continued-fractions*, Keio Engrg. Rep. 30 (1977), no. 13, 159–175.

[NN] H. Nakada, and R. Natsui, *The non-monotonicity of the entropy of $\alpha$-continued fraction transformations*, Nonlinearity **21** (2008), no. 6, 1207–1225.

[O] D. Ornstein, *Factors of a Bernoulli shifts are Bernoulli shifts*, Advances in Math., 5 (1970), 349–364.

[O2] ———, *The isomorphism theorem for Bernoulli flows*, Advances in Math. 10 (1973), 124–142.

[OW] D. Ornstein and B. Weiss, *Geodesic flows are Bernouillian*, Israel. J. Math. 14 (1987), 184–198.

[Schw] F. Schweiger, *Ergodic theory of fibred systems and metric number theory.* Oxford: Clarendon Press, 1995.

[Z] R. Zweimüller, *Ergodic structure and invariant densities of non-Markovian interval maps with indifferent fixed points*, Nonlinearity 11 (1998) 1263–1276.

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