Numerical Solution of the Fokker Plank Equation

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1 Fokker-Planck Equation

The specific form of the Fokker-Planck Equation considered here is

$$\frac{\partial P}{\partial t} + \frac{\partial [f(x)P]}{\partial x} = \nu \frac{\partial}{\partial x}\frac{\partial P}{\partial x}, \tag{1}$$

where $P = P(x,t)$, and $\nu = \kappa^2/2$ is a positive constant. The function $f(x) = 4x(1-x^2)$. The numerical approximation of the solution of this advection-diffusion equation poses special challenges when $f(x)$ is nonlinear. Here we will describe the numerical technique we adopted. First, we complete the statement of the problem. We seek solutions to (2) for $t > 0$ such that

$$\lim_{x\to\pm\infty} \frac{\partial P(x,t)}{\partial x} = 0$$

for all t, given $P(x,t) = P_0(x,0)$.

The standard numerical approach is composed of two parts: (1) make a straightforward substitution that recasts (1) as a diffusion equation, which effectively incorporates upwinding of the advection term; (2) choose a discretization that preserves certain desired properties, namely conservation of P with respect to a weight (in our specific problem the weight is 1), and total non-negativity. One such discretization is due to Larsen *et al.* citeyearparlars. This technique accomplishes both goals, regardless of whether $f(x)$ is linear or nonlinear. In addition, this scheme will also effectively control serious under and overflow problems in the computation, which are the result of the nonlinear nature of $f(x)$ for the problem at hand.

Briefly, (1) is rewritten as

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial x}\left\{F\frac{\partial}{\partial e^{\psi}}[\nu e^{\psi} G P]\right\}, \tag{2}$$

where it is understood that the derivatives with respect of e^{ψ} are taken holding t constant. This equation, in turn can be written formally as the conservation law

$$\frac{\partial P}{\partial t} = \frac{\partial S}{\partial x}.$$

We define $P_m(t)$, where $m = 1 \ldots M+1$, as the semi-discrete approximation of $P(x,t)$ on a uniform grid in x with spacing $\Delta x = 2L/M$ over $-L \leq x \leq L$. A second order flux-preserving scheme in space is then used to discretize the conservation law and the boundary conditions. Let

$$S_m(t) = \left\{F_m \frac{\partial}{\partial e^{\psi}_m}[\nu e^{\psi_m} G_m P_m(t)]\right\}$$

denote the semi-discrete approximation of the flux S.

Then, the conservation law for the semi-discretization becomes

$$\frac{\partial P_m}{\partial t} = \left[\frac{S_{m+1/2} - S_{m-1/2}}{x_{m+1/2} - x_{m-1/2}}\right],$$

for $m = 2 \ldots M$. At $m = 1$ and $m = M+1$ discretized zero flux boundary conditions are imposed using the same flux-preserving scheme. The fully discrete scheme is obtained by a suitable choice of time integrator. A trapezoidal scheme in time preserves mass and circumvents the serious stability constraints imposed on the time step Δt, should we have chosen an explicit mass preserving scheme instead. In fact, it is easy to show that the resulting fully discrete equations conserve mass for any Δt. The scheme is second-order accurate in time and space. Both the conservation estimate and the convergence rate estimates were checked in our computer implementation of the scheme.