The FPE Stationary Solutions

FPE $$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x}\left[A(x,t)\,\rho(x,t)\right] + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[B(x,t)\,\rho(x,t)\right]$$

$$\rho(x,t) = p(x,t\,|\,x_0,t_0)$$

$$\text{I.C.}\quad p(x,t_0\,|\,x_0,t_0) = \delta(x-x_0)$$

$$dx = A(x,t)\,dt + \sqrt{B(x,t)}\,dW_t \qquad \text{SDE}$$

$$\text{let } J = A\rho - \frac{1}{2}\partial_x(B\rho)$$

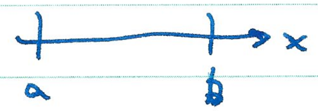
"probability current"

FPE $$\frac{\partial \rho}{\partial t} + \partial_x J = 0$$

Stationary solutions

$$\partial_x J = 0 \quad \Rightarrow J = \text{const}$$

S'pse $J(a) = J(x) = J(b) \equiv J$



and $\hat{n}\cdot J(x,t) = 0$ $\hat{n}$ is normal (at a, or b)

"Reflecting Boundary" (if one or both are reflecting

$\Rightarrow J = 0$

$J=0$ also if periodic.

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Zero Current (Potential Solution)

(a) $J=0$ $\qquad A\,p_s(x) = \frac{1}{2}\frac{d}{dx}\left[B(x)p_s(x)\right]$

$$\Rightarrow p_s(x) = \frac{1}{Z}\frac{1}{B(x)} e^{2\int_a^x dx'\,A(x')/B(x')}$$ "potential sol"

$$\int_a^b dx\, p_s(x) = 1$$ yields the constant Z.

(b) Periodic B.C.

$$A(x)\,p_s(x) - \frac{1}{2}\partial_x\left[B p_s\right] = J \qquad (\ddagger)$$

J cannot be arbitrary ; it has to be determined by normalization

$(\star)$ $p_s(a) = p_s(b) \Rightarrow J(a) = J(b)$

let $\psi(x) = e^{2\int_a^x dx'\,A(x')/B(x')}$

then $(\ddagger)$ integrates to

$$p_s(x)\,B(x)/\psi(x) = p_s(a)\,B(a)/\psi(a) - 2J\int_a^x dx'/\psi(x')$$

by imposing B.C. $(\star)$ we find that

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$$J = \left[B(b)/\psi(b) - B(a)/\psi(a) \right] p_s(a) \Big/ \int_a^b dx' \psi^{-1}(x')$$

so that

$$p_s(x) = p_s(a) \left[\frac{\int_a^x \frac{dx'}{\psi(x')} \frac{B(b)}{\psi(b)} + \int_x^b \frac{dx'}{\psi(x')} \frac{B(a)}{\psi(a)}}{\frac{B(x)}{\psi(x)} \int_a^b \frac{dx'}{\psi(x')}} \right]$$

(c) Infinite Range or Singular Boundaries

Some examples that make sense:

(i) Diffusion in gravitational field:

$$dx = -g\,dt + \sqrt{D}\,dW \Rightarrow \frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left(gp + \frac{1}{2} D \frac{\partial p}{\partial x}\right)$$

on an interval $x \in [a, b)$

$$p_s = \frac{1}{Z} e^{-2gx/D}$$

if $a = a_0$ and $b \to \infty$ then particles diffuse, falling down & will never stop falling.

if periodic on $a, b \Rightarrow p_s(x) = p(a)$, constant $\Rightarrow$ particles pass freely from a to b and back.

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(ii) OU Process

$$\frac{\partial p}{\partial t} = \frac{\partial}{\partial x}\left(kxp + \frac{1}{2} D \frac{\partial p}{\partial x}\right)$$

$$p_s(x) = \frac{1}{Z} e^{-kx^2/D}$$

provided $k>0$ p_s is normalizable on $(-\infty, \infty)$. If $k<0$ can only make sense on finite interval:

take $-a \quad 0 \quad a \rightarrow x$

$$\psi(x) = e^{-\frac{k}{D}(x^2 - c^2)} \qquad \psi(x=a) = \psi(x=-a)$$

$$\Rightarrow p_s(x) = p_s(a)\,\psi(x)/\psi(a) = p_s(a)\, e^{-\frac{k}{D}(x^2-c^2)}$$

(yields same solution as reflecting B.C.)

EIGENFUNCTION METHOD (homogeneous case, only)

(a) Reflecting Boundaries case

$$\frac{\partial p}{\partial t} = -\partial_x [Ap] + \frac{1}{2}\partial_x^2 [Bp]$$

here $A = A(x)$ and $B = B(x)$

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Let $\overset{(*)}{p(x,t) = P_s(x)\, q(x,t)}$, substitute

$$\frac{\partial q}{\partial t} = A\,\partial_x q + \frac{1}{2} B\,\partial_x^2 q$$ "Backward FPE" BFPE

solutions: $\begin{cases} p(x,t) = P_\lambda(x)\, e^{-\lambda t} \\ q(x,t) = Q_\lambda(x)\, e^{-\lambda t} \end{cases}$

$$\begin{cases} -\partial_x\left[A(x) P_\lambda(x)\right] + \frac{1}{2}\partial_x^2\left[B(x) P_\lambda(x)\right] = -\lambda P_\lambda(x) \\ \cancel{A(x)\partial_x Q}\ A(x)\,\partial_x Q_{\lambda'} + \frac{1}{2} B(x)\,\partial_x^2 Q_{\lambda'} = -\lambda' Q_{\lambda'}(x) \end{cases}$$

by integrating

$$(\lambda' - \lambda)\int_a^b dx\, P_\lambda(x) Q_{\lambda'}(x) = \Big[\overbrace{Q_{\lambda'}(x)\Big\{-A(x) P_\lambda(x) + \frac{1}{2}\partial_x\left[B(x) P_\lambda(x)\right]\Big\}}^{(\#)}$$
$$- \frac{1}{2} B(x) P_\lambda(x)\,\partial_x Q_{\lambda'}(x)\Big]_a^b$$

Solve for $q(x,t)$ in $(*)$ and substitute in BFPE to get

$$\frac{1}{2} B(x)\,\partial_x Q_{\lambda'}(x) = -A(x) P_{\lambda'}(x) + \frac{1}{2}\partial_x\left[B(x) P_{\lambda'}(x)\right]$$

and use this in $(\#)$

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if we use reflecting b.c. $\partial_x Q|_{a,b} = 0$

hence $(\lambda'-\lambda)\int_a^b dx\, P_\lambda(x) Q_{\lambda'}(x) = \delta_{\lambda\lambda'}(\lambda'-\lambda)$

Bi-orthogonal system

i.e. $\begin{cases} \int_a^b dx\, p_s(x) Q_\lambda(x) Q_{\lambda'}(x) = \delta_{\lambda\lambda'} \\ \text{and} \\ \int_a^b dx\, [p_s(x)]^{-1} P_\lambda(x) P_{\lambda'}(x) = \delta_{\lambda\lambda'} \end{cases}$

where $\lambda = \lambda' = 0$ we can get the normalization of the stationary solution $p_s(x)$, since

$$P_{\lambda=0}(x) = p_s(x) \qquad Q_{\lambda=0} = 1$$

if $p(x,\lambda) = \sum_\lambda A_\lambda P_\lambda(x) e^{-\lambda t}$

then $\int_a^b dx\, Q_\lambda(x)\, p(x,0) = A_\lambda$

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ex) OU Process $\partial_t p = \partial_x (kxp) + \frac{1}{2} D \partial_x^2 p$

$$P_s = \frac{1}{Z} e^{-kx^2/D} \quad \text{let } p(x,t) = p_s(x)\, g(x,t)$$

$$p(x,t) = P_\lambda(x) e^{-\lambda t} \qquad g(x,t) = Q_\lambda(x) e^{-\lambda t}$$

$$\Rightarrow \frac{d^2}{dx^2} Q_\lambda - \frac{2kx}{1} \frac{d}{dx} Q_\lambda + \frac{2\lambda}{D} Q_\lambda = 0$$

let $y = x\sqrt{k/D}$

$$\frac{d^2}{dy^2} Q_\lambda - 2y \frac{d}{dy} Q_\lambda + \left(\frac{2\lambda}{k}\right) Q_\lambda = 0$$

$$Q_\lambda = \frac{1}{\sqrt{2^n n!}} H_n\left(x\sqrt{k/D}\right) \qquad \lambda = nk,\ n = 0, 1, 2 \ldots$$

$$H_0 = 1 \qquad H_1 = 2x,\ H_2 = 4x^2 - 2,\ H_3 = 8x^3 - 12x,\ \cancel{\text{etc}}$$

$$\int_{-\infty}^{\infty} H_m H_n e^{-x^2} dx = \delta_{mn}$$

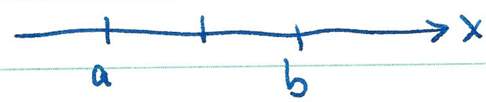
$$p_s = \left(\frac{k}{\pi D}\right)^{1/2} e^{-kx^2/D}$$

$$p(x,t) = \sum \sqrt{\frac{k}{2^n n! \pi D}}\, e^{-kx^2/D} H_n(x\sqrt{k/D})\, e^{-nkt} A_n$$

$$A_n = \int_{-\infty}^{\infty} dx\, p(x,0)\, H_n(x\sqrt{k/D})\, (2^n n!)^{-1/2}$$

First Passage Time for Homogeneous Process

How long will a particle whose position is described by FPE remains in a certain region in x?



$a \le x \le b$ at $t=0$

How long will it stay there?

Use absorbing B.C. $p(x=a,t) = 0 = p(x=b,t)$. Once particle reaches a/b it is <u>removed</u> from the system.

$\int_a^b dx'\, p(x',t|x,0) = G(x,t)$ is prob that at time t it is still in (a,b).

want to find T the time when particle leaves region.

$$\text{Prob}(T \geq t) = \int_a^b dx'\, p(x',t|x,0) = G(x,t),$$

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Since time-homogeneous

$$p(x',t|x,0) = p(x',0|x,-t) \quad \text{and use}$$

~~B~~FPE $\quad \partial_t p(x',t|x,0) = A p_x(x',t|x,0) + \frac{1}{2} B p_{xx}(x',t|x,0)$

$$\Rightarrow \quad \partial_t G = A G_x + \frac{B}{2} G_{xx}$$

the B.C. $p(x',0|x,0) = \delta(x-x') \Rightarrow \begin{cases} G(x,0)=1 & a\le x\le b \\ \quad\;\; =0 & \text{otherwise} \end{cases}$

& if $x=a$, or $x=b$ the particle is absorbed, so

$\text{prob}(T\ge t)=0$ when $x=a$ or $x=b$

$G(a,t) = G(b,t) = 0$

Since $G(x,t)$ is $\text{Prob}(T\ge t)$, the mean for any function of T is

$$\langle f(T)\rangle = -\int_0^\infty f(t)\,\underbrace{dG(x,t)}_{dG}$$

$\therefore$ The "first passage time" $\quad T(x) = \langle T\rangle = -\int_0^\infty t\,G_t(x,t)\,dt = \int_0^\infty G(x,t)\,dt$

after integrating by parts.

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$$\int_0^{\infty} \partial_t G \, dt = G(x,\infty) - G(x,0) = -1$$

$$(\exists) \begin{cases} A(x)\partial_x T(x) + \frac{1}{2}B(x)\partial_x^2 T(x) = -1 \\ T(a) = T(b) = 0 \end{cases}$$

Integrating with $\psi = e^{\int_a^x dx' [2A/B]}$

$$T(x) = \frac{2}{\int_a^b \frac{dy}{\psi(y)}} \left[\int_a^x \frac{dy}{\psi(y)} \int_x^b \frac{dy'}{\psi(y')} \int_a^{y'} \frac{dz\,\psi(z)}{B(z)} - \int_x^b \frac{dy}{\psi(y)} \int_a^x \frac{dy'}{\psi(y')} \int_a^{y'} \frac{dz\,\psi(z)}{B(z)} \right]$$

Take $(\exists)$ with $x=a$ reflecting $\to G_x(a,t)=0$, b absorbing $\to G(b,t)=0$ $\quad a<b$

$$T(x) = 2\int_x^b \frac{dy}{\psi(y)} \int_a^y \frac{\psi(z)}{B(z)} dz$$