§7 The Itô/Stratonovich dilemma

- The dilemma: what does the idealization of delta-function-correlated noise mean?

$$\begin{aligned} \dot{x} &= f(x) + g(x)\eta(t) \\ \langle \eta(t)\eta(t') \rangle &= 2\kappa\delta(t-t'). \end{aligned} \tag{1}$$

- Previously, we argued by a limiting procedure: taking noise with a finite correlation time τ, then letting τ go to zero.

- This is a physical procedure, and results in the so-called Stratonovich interpretation of the SDE (1).

- As we shall see, an alternative interpretation (due to Itô) is also commonly used, mostly in mathematics and in financial mathematics. The Stratonovich interpretation is more common in physics and engineering.

- Both interpretations lead to the same Fokker-Planck equation if g is constant (i.e., for additive noise).

- However, for multiplicative noise ($g \neq$ constant) the Itô and Stratonovich interpretations can give different results: this is the source of much confusion among modellers of noisy phenomena.

- Following [Risken section 3.3.3], we consider the noise integral

  $$W(t) = \int_0^t \eta(t')dt'.$$

  Note we have already seen that this (non-stationary) process has

  $$\langle W(t) \rangle = 0$$

  and

  $$\left\langle W(t)^2 \right\rangle = 2\kappa t.$$

- The Langevin equation (1) can be recast as the integral equation

  $$x(t+\tau) = x(t) + \int_t^{t+\tau} f(x(t'))dt' + \int_t^{t+\tau} g(x(t'))dW(t'),$$

  where the Stieltjes integral is used in the final term, as

  $$dW = \dot{W}dt = \eta(t)dt$$

  is not well-defined for a standard (Riemann) integral.

- The process $W(t)$ is called a Wiener process, and it is used to rigorously prove some of the heuristic limiting arguments we have used.
- The **increment** of the Wiener process is defined as

$$\omega(\tau) = W(t+\tau) - W(t) = \int_t^{t+\tau} \eta(t')dt'.$$

- The distribution of $\omega(\tau)$ is Gaussian, since $\eta(t)$ is Gaussian. It is easy to show [Risken p.51] that

$$\begin{aligned} \omega(0) &= 0 \\ \langle \omega(\tau) \rangle &= 0 \\ \langle \omega(\tau_2)\omega(\tau_1) \rangle &= 2 \min \left[\tau_1, \tau_2\right]. \end{aligned}$$

- We will need to calculate stochastic integrals of the form

$$A = \int_0^\tau \Phi\left[\omega(\tau'), \tau'\right] dW(\tau'), \tag{2}$$

  and the Itô and Stratonovich interpretations give different rules (and hence different values) for these.
- We use the subscript I for Itô and S for Stratonovich.
- Itô's definition of the stochastic integral (2) is

$$A_I = \lim_{\Delta \to 0} \sum_{i=0}^{N-1} \Phi\left[\omega(\tau_i), \tau_i\right]\left[\omega(\tau_{i+1}) - \omega(\tau_i)\right],$$

  where

$$\Delta = \max\left(\tau_{i+1} - \tau_i\right), \qquad 0 = \tau_0 < \tau_1 < \ldots < \tau_N = \tau.$$

- Note Itô uses $\Phi\left[\omega(\tau_i), \tau_i\right]$ in the summation, this is independent of the increment $\omega(\tau_{i+1}) - \omega(\tau_i)$.
- Stratonovich defined a different (but equally valid) interpretation of (2):

$$A_S = \lim_{\Delta \to 0} \sum_{i=0}^{N-1} \Phi\left[\frac{\omega(\tau_i) + \omega(\tau_{i+1})}{2}, \frac{\tau_i + \tau_{i+1}}{2}\right]\left[\omega(\tau_{i+1}) - \omega(\tau_i)\right],$$

- Here the Φ arguments in the summation depend on the values of ω at τ_i and at τ_{i+1} in a symmetrical way: note the definitions agree if Φ is independent of ω.

- Now we'll use the Itô and Stratonovich interpretations to evaluate the average of the stochastic integral (which we'll need later):

$$B = \left\langle \int_0^\tau \omega(\tau')dW(\tau') \right\rangle .$$

- Itô:

$$\begin{aligned} B_I &= \left\langle \sum_{i=0}^{N-1} \omega(\tau_i) \left[\omega(\tau_{i+1}) - \omega(\tau_i)\right] \right\rangle \\ &= \sum_{i=0}^{N-1} \left[\langle \omega(\tau_i)\omega(\tau_{i+1})\rangle - \langle \omega(\tau_i)\omega(\tau_i)\rangle\right] \\ &= \sum_{i=0}^{N-1} (2\kappa\tau_i - 2\kappa\tau_i) \\ &= 0. \end{aligned}$$

- Thus $B_I = 0$.
- Stratonovich:

$$\begin{aligned} B_S &= \left\langle \sum_{i=0}^{N-1} \left[\frac{\omega(t_i) + \omega(t_{i+1})}{2}\right] \left[\omega(\tau_{i+1}) - \omega(\tau_i)\right] \right\rangle \\ &= \frac{1}{2} \sum_{i=0}^{N-1} \left[\langle \omega(\tau_i)\omega(\tau_{i+1})\rangle + \langle \omega(\tau_{i+1})\omega(\tau_{i+1})\rangle - \langle \omega(\tau_i)\omega(\tau_i)\rangle - \langle \omega(\tau_{i+1})\omega(\tau_i)\rangle\right] \\ &= \frac{1}{2} \sum_{i=0}^{N-1} \left[2\kappa\tau_i + 2\kappa\tau_{i+1} - 2\kappa\tau_i - 2\kappa\tau_i\right] \\ &= \kappa \sum_{i=0}^{N-1} (\tau_{i+1} - \tau_i) \\ &= \kappa\tau. \end{aligned}$$

- So $B_S = \kappa\tau$.
- This affects how we calculated the drift and diffusion coefficients for insertion into the Kramers-Moyal expansion when deriving the FPE.
- Back to our integral equation:

$$x(t+\tau) = x(t) + \int_t^{t+\tau} f(x(t'))dt' + \int_t^{t+\tau} g(x(t'))dW(t').$$

- Writing $\xi(t)$ for $x(t)$, and let $x = \xi(t)$:

$$
\begin{aligned}
\xi(t+\tau) - x &= \int_t^{t+\tau} f(\xi(t'))dt' + \int_t^{t+\tau} g(\xi(t'))dW(t') \\
&= \int_0^{\tau} f(\xi(t+\tau'))d\tau' + \int_0^{\tau} g(\xi(\tau+\tau'))dW(\tau'),
\end{aligned}
$$

  using $\tau' = t' - t$.

- Now iterate, get first iteration (using $\xi = x$ in the integrals):

$$
\begin{aligned}
\xi^{(1)}(t+\tau) - x &= \int_0^{\tau} f(x)d\tau' + \int_0^{\tau} g(x)dW(\tau') + \ldots \\
&= \tau f(x) + g(x)\omega(\tau) + \ldots .
\end{aligned}
$$

- Iterating again, using $\langle \omega(\tau) \rangle = 0$, and retaining only terms proportional to τ, we get after averaging

$$
\begin{aligned}
\left\langle \xi^{(2)}(t+\tau) - x \right\rangle &= \tau f(x) + \left\langle \int_0^{\tau} \frac{\partial g}{\partial x}(x)g(x)\omega(\tau')dW(\tau') \right\rangle + \ldots \\
&= \tau f(x) + \frac{\partial g}{\partial x}(x)g(x) \left\langle \int_0^{\tau} \omega(\tau')dW(\tau') \right\rangle + \ldots
\end{aligned}
$$

  [Note the expressions here are simpler than Risken's because we assume f and g are time-independent].

- The drift coefficient is defined as

$$
D^{(1)} = \lim_{\tau \to 0} \frac{1}{\tau} \left. \langle \xi(t+\tau) - x \rangle \right|_{\xi(t)=x}.
$$

  Thus the second iterate above gives a drift coefficient of

$$
D_S^{(1)} = f(x) + \kappa \frac{\partial g}{\partial x}(x)g(x)
$$

  in the Stratonovich interpretation, or

$$
D_I^{(1)} = f(x) + 0
$$

  in the Itô interpretation.

- For the diffusion coefficient $D^{(2)}$, the expansion of $\left\langle (\xi(t+\tau) - x)^2 \right\rangle$ to first order in τ does not involve stochastic integrals, and both Itô and Stratonovich coefficients are therefore

$$
\begin{aligned}
D^{(2)} &= \frac{1}{2} \lim_{\tau \to 0} \frac{1}{\tau} \left. \left\langle (\xi(t+\tau) - x)^2 \right\rangle \right|_{\xi(t)=x} \\
&= \frac{1}{2} \lim_{\tau \to 0} \frac{1}{\tau} g^2(x) \left\langle \omega(\tau)^2 \right\rangle \\
&= \kappa\, g^2(x).
\end{aligned}
$$

- Referring back to the Kramers-Moyal expansion, which gives the FPE in the form

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}\left[D^{(1)}P\right] + \frac{\partial^2}{\partial x^2}\left[D^{(2)}P\right],$$

we see that the Itô interpretation of the Langevin equation

$$\dot{x} = f(x) + g(x)\eta(t)$$

leads to the "Itô FPE":

$$\frac{\partial P}{\partial t} = -\frac{\partial}{\partial x}[fP] + \kappa\frac{\partial^2}{\partial x^2}\left[g^2 P\right],$$

while the Stratonovich gives the familiar "Stratonovich FPE":

$$\begin{aligned}\frac{\partial P}{\partial t} &= -\frac{\partial}{\partial x}\left[\left(f+\kappa g' g\right)P\right] + \kappa\frac{\partial^2}{\partial x^2}\left[g^2 P\right] \\ &= -\frac{\partial}{\partial x}[fP] + \kappa\frac{\partial}{\partial x}\left[g\frac{\partial}{\partial x}(gP)\right],\end{aligned}$$

as seen before.

- Note that if g is constant (independent of x), then both interpretations give the same results.

- **Example:** Langevin equation with multiplicative noise:

$$\frac{dx}{dt} = x\,\eta(t)$$

with $x(0) = x_0$ and $\langle\eta(t)\eta(t')\rangle = 2\kappa\delta(t-t')$.

- Interpreting as (i) a Stratonovich SDE and (ii) an Itô SDE, calculate the mean $\langle x(t)\rangle$.

- (i) Stratonovich interpretation leads to the FPE

$$\begin{aligned}\frac{\partial P}{\partial t} &= -\frac{\partial}{\partial x}[fP] + \kappa\frac{\partial}{\partial x}\left[g\frac{\partial}{\partial x}(gP)\right] \\ \Rightarrow \frac{\partial P}{\partial t} &= \kappa\frac{\partial}{\partial x}\left[x\frac{\partial}{\partial x}(xP)\right].\end{aligned}$$

- We want to find $\langle x(t)\rangle$ and noting

$$\langle x(t)\rangle = \int_{-\infty}^{\infty} xP(x,t)dx,$$

we multiply the FPE by x and integrate:

$$\begin{aligned}
\frac{\partial}{\partial t}\int_{-\infty}^{\infty} xP dx &= \kappa\int_{-\infty}^{\infty} x\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(xP)\right]dx \\
\Rightarrow \frac{d}{dt}\langle x\rangle &= -\kappa\int_{-\infty}^{\infty} x\frac{\partial}{\partial x}(xP)\,dx \\
\Rightarrow \frac{d}{dt}\langle x\rangle &= \kappa\langle x\rangle \\
\Rightarrow \langle x(t)\rangle &= x_0 e^{\kappa t},
\end{aligned}$$

since $x(0) = x_0$.

- (ii) Itô interpretation leads to the FPE

$$\begin{aligned}
\frac{\partial P}{\partial t} &= -\frac{\partial}{\partial x}[fP] + \kappa\frac{\partial^2}{\partial x^2}\left[g^2 P\right] \\
\Rightarrow \frac{\partial P}{\partial t} &= \kappa\frac{\partial^2}{\partial x^2}\left[x^2 P\right].
\end{aligned}$$

- Multiply by x and integrate:

$$\begin{aligned}
\frac{d}{dt}\langle x\rangle &= \kappa\int_{-\infty}^{\infty} x\frac{\partial^2}{\partial x^2}\left(x^2 P\right) \\
\Rightarrow \frac{d\langle x\rangle}{dt} &= 0 \\
\Rightarrow \langle x\rangle &= x_0.
\end{aligned}$$

- Note: If we average the original Langevin equation we get

$$\frac{d\langle x\rangle}{dt} = \langle x\,\eta(t)\rangle.$$

- The Itô interpretation sets $\langle x(t)\eta(t)\rangle$ to zero. This is the "non-anticipating" character of the Itô interpretation: $x(t)$ depends $\eta(t')$ only for $t' < t$, and so is independent of $\eta(t)$.

- The Stratonovich has $\langle x\,\eta(t)\rangle \neq 0$, essentially because taking the white-noise limit via continuous processes implies a correlation between $x(t)$ and $\eta(t)$.

- The Itô interpretation has a number of advantages:
  - The drift coefficient is equal to the noise-free velocity in the Langevin, $D^{(1)} = f(x)$.

  - The non-anticipating character (e.g. $\langle x(t)\eta(t)\rangle$ above) is crucial for rigorous proofs.

- However, the Itô interpretation requires the use of a "new" calculus, the Itô calculus. Itô's rule for changing variables is a good example.

- By contrast, Stratonovich's interpretation is based on the limit of coloured noises as the correlation time limits to zero, and it allows the use of the ordinary rules of calculus. Rigorous proofs are made difficult by the anticipating nature, and in practice Stratonovich SDEs are usually translated to equivalent Itô SDEs for mathematical analysis of their properties.

- Chief usages:

  - Stratonovich: Physics and engineering.
  - Itô: Mathematical analysis, financial mathematics.