(1) Solve (a) $u_t + 3u_x = 0$, $u(x,0) = \sin x$

$x \in \mathbb{R}, t > 0$

$$u(x,t) = f(x-3t) = \sin(x-3t)$$

(b) $u_t - x^2 t\, u_x = 0 \quad t > 0, x \in \mathbb{R}$

$u(x,0) = x+1 \quad t=0$

$$\frac{du}{dt} = u_t + \frac{\partial x}{\partial t} u_x = 0 \quad \therefore \quad \frac{\partial x}{\partial t} = -x^2 t$$

$$\frac{dx}{x^2} = -t\,dt \Rightarrow -x^{-1} = -\frac{1}{2}t^2 + c$$

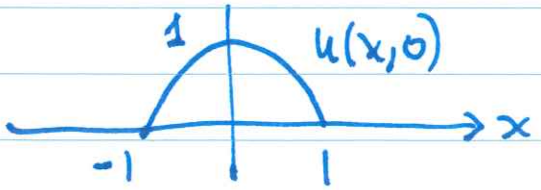
$$\frac{1}{x} = \frac{1}{2}t^2 + c' \left(\begin{array}{l}\text{but at } t=0 \\ x = \xi\end{array}\right) \frac{1}{x} = \frac{1}{2}t^2 + \frac{1}{\xi}$$

since u is constant along

$$u(x,t) = \xi + 1 = \frac{2x}{2 - xt^2} + 1$$

(2) Find shock time $u_t + uu_x = 0$

$$u(x,0) = \begin{cases} 1 - x^2 & |x| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$



$$\frac{dx}{dt} = u$$

$$x = u(\phi(\xi))\,t + \xi = (1-\xi^2)t + \xi$$

$$u_x = \frac{\phi'(\xi)}{1 + c'(\phi)\phi'(\xi)t}$$

the denominator $\uparrow \infty$ when

$$t_b = \min_{\xi} \frac{-1}{\phi'(\xi)c'(\phi)}$$

in this case $c(u) = u \quad \therefore c'(u) = 1$

$$\phi = 1 - \xi^2 \qquad \phi' = -2\xi$$

$$t_b = \min_{\xi} \frac{-1}{1 \cdot (-2\xi)} = \min_{\xi} \frac{1}{2\xi}$$

$$t_b = 1/2$$

(3) $\begin{cases} u_t + uu_x + au = 0 & t > 0 \quad x \in \mathbb{R}^1 \\ u(x,0) = f(x) \end{cases}$

show that $u = f(\xi)e^{-at} \qquad x = \frac{f(\xi)}{a}(1 - e^{-at}) + \xi$

$$\frac{du}{dt} = u_t + uu_x = -au$$

$$\therefore u = Ke^{-at} \qquad u(t=0) = f(\xi)$$

$$\therefore u = f(\xi)e^{-at}$$

(3)

$$\begin{cases} u_t + u u_x + a u = 0 \\ u(x,0) = f(x) \end{cases}$$

$$\frac{du}{dt} = u_t + \frac{dx}{dt} u_x = -au$$

$$\frac{du}{dt} = -au \implies u = Ke^{-at}$$

$$u(t=0) = f(\xi) \quad \therefore \; u(\xi,t) = f(\xi)e^{-at}$$

$$\frac{dx}{dt} = u = f(\xi)e^{-at}$$

$$x = \int^t f(\xi)e^{-as}\,ds + x_0$$

$$x(0) = \xi$$

$$x = -\frac{1}{a} f(\xi) e^{-at} + x_0$$

$$x_0 = \xi + \frac{1}{a} f(\xi)$$

$$\therefore \; x = -\frac{1}{a} f(\xi) e^{-at} + \xi + \frac{1}{a} f(\xi)$$

$$\implies x = \frac{f(\xi)}{a}\left(1 - e^{-at}\right) + \xi$$

(4) Find solution to $\begin{cases} 2xuu_x + 2tuu_t = tu^2 - tx^2 - t^3 \\ u(x,0) = f(x) \end{cases}$ $\begin{cases} t>0 \\ x \in \mathbb{R}^1 \end{cases}$

let $V = u^2(x,t)$ then $xV_x + tV_t = tV - tx^2 - t^3$

$$V_t + \frac{x}{t} V_x = V - x^2 - t^2$$

$$\frac{dx}{dt} = \frac{x}{t} \Rightarrow x^2 = t^2 + \xi^2$$

$$\frac{dv}{dt} = V - x^2 - t^2 = V - \xi^2$$

$$\frac{dv}{dt} - V = -\xi^2 \Rightarrow V = -\xi^2 + (\xi^2 + f^2(\xi))e^t$$

$$u^2(x,t) = -x^2 + (x^2 + f^2(x))e^t$$