HW5 Solutions

PART I Find $\mathcal{F}(e^{-ax^2}) = \mathcal{F}(u(x))$

$$\hat{u}(k) = \int_{-\infty}^{\infty} e^{-ax^2} e^{ikx}\, dx$$

$$\frac{d\hat{u}}{dk} = \cancel{i\int_{-\infty}^{\infty} e^{-ax^2}} = i\int_{-\infty}^{\infty} e^{-ax^2} e^{ikx}\, x\, dx$$

integrate by parts:

let $v = e^{ikx}$ $\qquad w = -\frac{1}{2a}\frac{d}{dx}(e^{-ax^2})$

$dv = ike^{ikx}dx \qquad dw = xe^{-ax^2}$

$$\frac{d\hat{u}}{dk} = I e^{ikx}\left(-\frac{1}{2a}\frac{d}{dx}e^{-ax^2}\right)\Bigg|_{-\infty}^{\infty} + \frac{(-i)}{2a}\int_{-\infty}^{\infty}\frac{d}{dx}(e^{-ax^2})e^{ikx}\,dx$$

$= 0$ ~~since~~

$$(*) \quad \frac{d\hat{u}}{dk} = \frac{-i}{2a}\int_{-\infty}^{\infty}\frac{d}{dx}(e^{-ax^2})e^{ikx}\,dx - \frac{k}{2a}\hat{u}(k)$$

since $\mathcal{F}\left(\frac{df}{dx}\right) = i\hat{F}(k)$

Solving $(*)$ $\quad \frac{d\hat{u}}{dk} + \frac{k}{2a}\hat{u} = 0$ yields

$$\hat{u} = Ce^{-k^2/4a}$$

$$u(k=0) = \int_{-\infty}^{\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}} \quad \therefore \quad \hat{u}(k) = \sqrt{\frac{\pi}{a}}\, e^{-k^2/4a} \;//$$

Part II

(a) $\dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} = u_{xx} \qquad x\in\mathbb{R} \quad t>0$

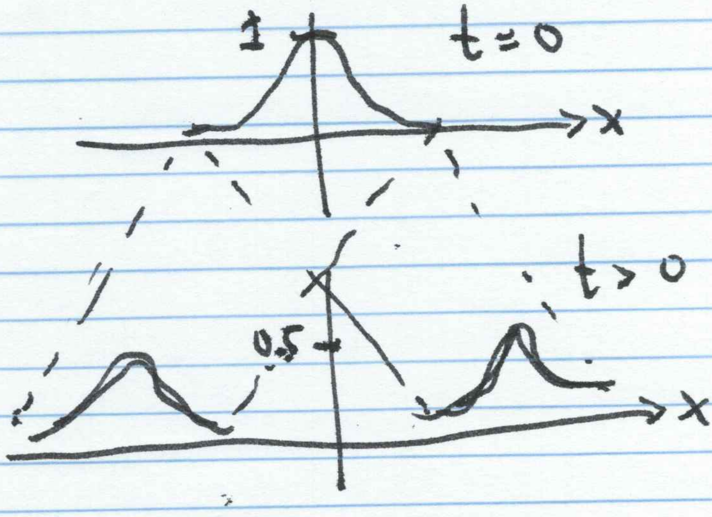
$u(x,0) = f(x) \qquad u_t(x,0) = 0$

$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right)$

(c) if $f(x) = e^{-x^2/2a}$

the solution is

$$u(x,t) = \frac{1}{2}\left(e^{-(x+ct)^2/2a} + e^{-(x-ct)^2/2a}\right)$$



(b) now $\dfrac{\partial u}{\partial t}(x,0) = g(x)$

$\mathcal{F}\left(\dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}\right)$ and $\mathcal{F}(f(x)) = \hat{F}(k)$

$\mathcal{F}(g(x)) = \hat{G}(k)$

$$\left(\frac{d^2}{dt^2} + k^2c^2\right)\hat{u}(k,t) = 0$$

$$\hat{u}(k,t) = A\cos kct + B\sin kct$$

$$\frac{d}{dt}\hat{u}(k,t) = -kcA\sin kct + Bck\cos kct$$

$$\hat{u}(k,0) = \hat{F}(k) = A$$

$$\frac{d\hat{u}}{dt}(k,0) = \hat{G}(k) = ckB$$

$$\therefore \quad \hat{u} = \hat{F}(k)\cos kct + \frac{G(k)}{ck}\sin kct$$

$$u = \mathcal{F}^{-1}(\hat{u}) = \mathcal{F}^{-1}(\text{products of functions of } k)$$

~~but~~ can use convolution

$$u(x,t) = f(x) * \mathcal{F}^{-1}(\cos kct) + g(x) * \mathcal{F}^{-1}\left(\frac{\sin kct}{ck}\right)$$

using identities supplied as hints

$$\mathcal{F}^{-1}(\cos kct) = \frac{1}{2}\left[\delta(x+ct) + \delta(x-ct)\right]$$

$$\mathcal{F}^{-1}\left(\frac{\sin kct}{kc}\right) = \frac{1}{2c}\left[H(x+ct) - H(x-ct)\right]$$

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right]$$

$$+ \frac{1}{2c}\int_{-\infty}^{\infty} dy\, g(x-y)\left[H(y+ct) - H(y-ct)\right]$$

$$= \frac{1}{2}\left[f(x-ct) + f(x+ct)\right]$$

$$+ \frac{1}{2c}\int_{x-ct}^{x+ct} g(x')\,dx' \qquad //$$

(2) Show $\mathcal{F}(g(y-x)) = e^{-iky}\hat{G}(k)$

(a) $$\int_{-\infty}^{\infty} g(y-x)e^{ikx}dx = \int_{-\infty}^{\infty} e^{-iky} g(z) e^{ikz} dz$$

$$= e^{-iky}\int_{-\infty}^{\infty} g(x)e^{ikx}dx = e^{-iky}\hat{G}(k)$$

//

Show that $\mathcal{F}(u*v) = \hat{U}(k)\hat{G}(k)$

(b) $$\int_{-\infty}^{\infty}dx \int_{-\infty}^{\infty} u(x-y)v(y)dy\, e^{ikx}$$

$$= \int_{-\infty}^{\infty}dy\left[\int_{-\infty}^{\infty}dx\, u(x-y)e^{ikx}\right]v(y)$$

$$= \int_{-\infty}^{\infty}dy\, e^{iky}v(y)\hat{U}(k) = \hat{V}(k)\hat{U}(k)$$ //

(2) (a) $\mathcal{F}^{-1}\left(e^{-a|k|}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-a|k|} e^{-ikx}\,dk$

$= \frac{1}{2\pi}\int_{-\infty}^{0} e^{ak-ikx}\,dx + \frac{1}{2\pi}\int_{0}^{\infty} e^{-ak-ikx}\,dk$

$= \frac{1}{2\pi}\,\frac{1}{a-ix}\, e^{(a-ix)k}\Big|_{-\infty}^{0} + \frac{1}{2\pi}\,\frac{1}{-(a+ix}\Big|_{0}^{\infty} =$

$\frac{1}{2\pi}\,\frac{1}{a-ix} + \frac{1}{2\pi}\,\frac{1}{(a+ix)} = \frac{a}{\pi}\,\frac{1}{x^2+a^2}$ $\quad a>0$

(b) $\Delta u = 0 \quad -\infty<x<\infty \quad 0<y<\infty$

u bounded $\quad u(x,0)=f(x)$

$\mathcal{F}(\Delta u) = \hat{u}_{yy}(k) - k^2\hat{u}(k) = 0$

which has the general solution

$\hat{u}(k,y) = \hat{c}(k)e^{ky} + \hat{a}(k)e^{-ky}$

the boundedness condition forces $c(k)=0$ for $k>0$ and $a(k)=0$ for $k<0$ $\therefore$

$\hat{u}(k,y) = \hat{b}(k)\,e^{-|k|y}$

$\hat{f}(k) = \mathcal{F}(f(x)) = \hat{b}(k) = \hat{u}(k,0)$

$\Rightarrow \hat{u}(k,y) = \hat{f}(k)e^{-|k|y}$

$$\therefore \quad u = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(k)\, e^{-|k|y}\, dk$$

$$\text{but } w = \mathcal{F}^{-1}\left(e^{-y|k|}\right) = \frac{y}{\pi}\,\frac{1}{x^2+y^2}$$

i.e. $a = y$ in the result of part (a).

$$\therefore\ u = \frac{y}{\pi}\,\frac{1}{x^2+y^2} * f(x) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(\tau)\,d\tau}{(x-\tau)^2+y^2}$$

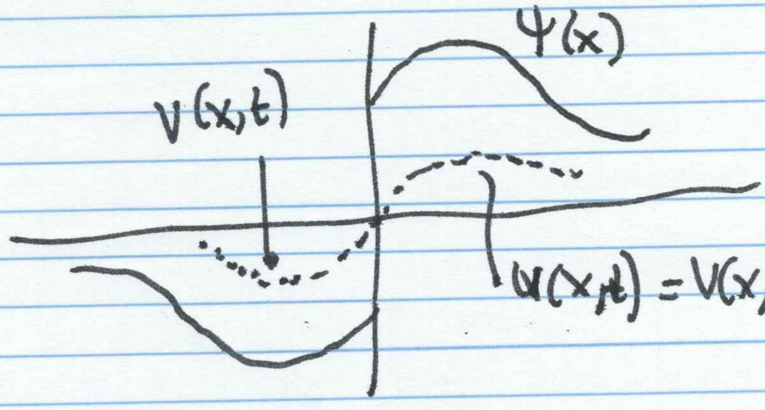
(3) $u_t = \nu u_{xx} \qquad x>0 \quad t>0$

$u(0,t) = 0$

$u(x,0) = \phi(x) \qquad x>0$

let $\psi(x) = \phi(x)$ $x>0$; $\psi(x) = -\phi(-x)$ $x<0$; $\psi(0) = 0$

"ODD extension"



$$v_t = \nu v_{xx} \qquad x \in \mathbb{R},\ t > 0$$

$$v(x,0) = \psi(x) \qquad x \in \mathbb{R}$$

$\mathcal{F}(v_t = \nu v_{xx})$ gives

$$\frac{d}{dt}\hat{v}(k,t) - \nu k^2 \hat{v}(k,t) = 0$$

$$\mathcal{F}(v(x,0)) = \mathcal{F}(\psi(x)) = \hat{\psi}(k)$$

$$\hat{v}(k,t) = \hat{\psi}(k) e^{-\nu k^2 t}$$

in class we showed that

$$W(x,t) \equiv \mathcal{F}^{-1}\left(e^{-k^2 \nu t}\right) = \frac{1}{\sqrt{4\pi\nu t}} e^{-x^2/4\nu t}$$

so $u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\psi}(k) e^{-\nu k^2 t} e^{-ikx}\, dk$

can be written as

$$v = \int_{-\infty}^{\infty} W(x-y,t)\psi(y)\, dy$$

Breaking up the integral

$$v(x,t) = \int_{-\infty}^{0} \omega(x-y,t)\,\psi(y)\,dy + \int_{0}^{\infty} w(x-y,t)\,\psi(y)\,dy$$

$$v(x,t) = +\int_{+\infty}^{0} \omega(x+y,t)\,\psi(y)\,dy + \int_{0}^{\infty} w(x-y,t)\,\psi(y)\,dy$$

replacing $\psi(y) = \phi(y)$ for $y > 0$, where $v = u$

$$u(x,t) = \int_{0}^{\infty} \left[w(x-y,t) - w(x+y,t) \right] \phi(y)\,dy$$

$$= \int_{0}^{\infty} \frac{1}{\sqrt{4\pi\nu t}} \left[e^{-(x-y)^2/4\nu t} - e^{-(x+y)^2/4\nu t} \right] \phi(y)\,dy$$

$$= \operatorname{erf}\left(\frac{x}{\sqrt{4\nu t}}\right)$$ (using Table of integrals)