HW5 Cauchy Problems

PART I FOURIER TRANSFORMS.

$$\begin{cases} \mathcal{F}(u(x)) = \hat{u}(k) = \int_{-\infty}^{\infty} u(x) e^{ikx}\, dx & \text{(A)} \\ \mathcal{F}^{-1}(\hat{u}(k)) = u(x,t) = \int_{-\infty}^{\infty} \hat{u}(k)\, e^{-ikx} \frac{dk}{2\pi} & \text{(B)} \end{cases}$$

(1) Find $\mathcal{F}(e^{-ax^2}) = \sqrt{\frac{\pi}{a}}\, e^{-k^2/4a}$

(a) Write (A) for e^{-ax^2}

(b) differentiate the expression in (a) with respect to k to get a differential equation for $\hat{u}(k)$

(c) Solve the differential equation in (b): show it has a solution $\hat{u}(k) = Ce^{-k^2/4a}$

Use the fact that $\hat{u}(k) = \int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$ (from a table of integrals) to find final expression for $\hat{u}(k)$

(2) Prove that $\mathcal{F}(u * v) = \hat{U}(k)\hat{V}(k)$

where $u * v = \int_{-\infty}^{\infty} u(x-y)\, v(y)\, dy = \int_{-\infty}^{\infty} v(x-y)\, u(y)\, dy$

is the "convolution" of u & v.

(a) first show (by a change of variable in the integration) that
$$\mathcal{F}\left(g(y-x)\right) = e^{-iky}\,\hat{G}(k)$$

(b) use this result to compute
$$\mathcal{F}\left(\int_{-\infty}^{\infty} u(x-y)\,v(y)\,dy\right) = \hat{U}(k)\,\hat{V}(k)$$

PART II Cauchy Problems: initial value PDE's posed on a semi-infinite or infinite domain are called "Cauchy Problems".

(1) In class (see notes) we showed that

$$(\ast)\begin{cases} \frac{1}{c^2}\,u_{tt} = u_{xx} & x \in \mathbb{R} \quad t>0 \\ u(x,0) = f(x) & u_t(x,0) = 0 \end{cases}$$

has a "D'Alembert Solution"
$$u(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right)$$

(a) Find the general solution to $\ast$ with
$$f(x) = e^{-x^2/2a}$$

(b) Using class notes, now find solution to

$$\frac{1}{c^2} u_{tt} = u_{xx} \qquad x \in \mathbb{R} \quad t>0$$

$$u(x,0) = f(x) \qquad \frac{\partial u}{\partial t}(x,0) = g(x)$$

Hint: use convolution.

<u>Hint:</u> $\mathcal{F}^{-1}\left(\cos kct\right) = \frac{1}{2}\left[\delta(x+ct) + \delta(x-ct)\right]$

$$\mathcal{F}^{-1}\left(\frac{\sin kct}{kc}\right) = \frac{1}{2c}\left[H(x+ct) - H(x-ct)\right]$$

Where $H(x)$ is the Heaviside function.

(2) Cauchy Problem on a semi-infinite domain

(a) Show that

$$(\mathcal{F}) \quad \mathcal{F}^{-1}\left(e^{-a|k|}\right) = \frac{a}{\pi} \frac{1}{x^2+a^2} \quad , a>0$$

(b) Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad x \in \mathbb{R} \quad , y>0$

$$u(x,0) = f(x) \qquad x \in \mathbb{R}$$

assuming $u(x,y)$ is bounded.

This is the "Dirichlet" problem on the upper half plane.

(3) $u_t = \nu u_{xx}$ $\quad x>0 \quad t>0$

$u(0,t)=0$

$u(x,0)=\phi(x) \quad x>0$

Heat eq on a semi-infinite domain.

(3a) let
$$\begin{cases} \psi(x) = \phi(x) & \text{if } x>0 \\ \psi(x) = -\phi(-x) & \text{if } x<0 \\ \psi(x=0) = 0 \end{cases}$$

Solve (V)
$$\begin{cases} v_t = \nu v_{xx} & x\in\mathbb{R}, t>0 \\ v(x,0) = \psi(x) & x\in\mathbb{R} \end{cases}$$

(3b) Write solution to (V) as a convolution. Use symmetry, for $y<0$, $y>0$ to split the integral that defines solution of $v(x,t)$ to write the solution of $u(x,t)$

(3c) Find explicit solution if $\cancel{\phi(x) = e^{-x}}$

$$u(x,0) = \phi(x) = 1$$

Use an integral Table.

The trick is to solve an associated problem on an infinite domain, using an appropriate periodic extension of $u(x,y)$, call it $v(x,y)$

(c) Solve $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0 \qquad x \in \mathbb{R} \quad y \in \mathbb{R}$

$$v(x,0) = f(x)$$

$$v(x,y) \text{ bounded} \qquad x \in \mathbb{R},\ y \in \mathbb{R}$$

and then restrict to $0 \le y < \infty$ where $u(x,y) = v(x,y)$

(c1) Use separation of variables and Fourier transforms to the x dependence

$$\mathcal{F}(v(x,y)) = \hat{v}(k,y)$$

Show that boundedness implies that

$$\hat{v}(k,y) = \hat{f}(k)\, e^{-|k|y}$$

where $\hat{f}(k) = \mathcal{F}(f(x))$

(c2) Use the convolution and (‡) to show

$$u(x,y) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\tau)\, d\tau}{(x-\tau)^2 + y^2}$$