HW Waves Solution

(1) $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = u_{xx} + u_{yy}$ $\quad 0 < x < a$, $\quad 0 < y < b$, $\quad t > 0$

$u(0,y,t) = u(a,y,t) = 0$ $\quad u(x,y,0) = f(x)$

$u(x,0,t) = u(x,b,t) = 0$ $\quad \frac{\partial u}{\partial t}(x,y,0) = 0$

$u = H(t)\,\psi(x,y)$

$H_{tt} + \omega^2 H = 0$

$\frac{1}{c^2}\frac{H_{tt}}{H} = \frac{\Delta\psi}{\psi} = -k^2 \Rightarrow \Delta\psi + k^2\psi = 0$

$ck = \omega$

$\Delta\psi + k^2\psi = \psi_{xx} + \psi_{yy} + k^2\psi = 0 \qquad \psi = F(x)G(y)$

$\frac{F_{xx}}{F} + k^2 = -\frac{G_{yy}}{G} = l^2$

$F_{xx} + (k^2 - l^2)F = 0 \qquad G_{yy} + l^2 G = 0$

$F(0) = F(a) = 0 \qquad G(0) = G(b) = 0$

$\psi = \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b} \qquad k^2_{nm} = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$

~~$u = \sum A_n \cos\omega$~~ $\quad u = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(A_{nm}\cos\omega_{nm}t + B_{nm}\sin\omega_{nm}t\right)\psi_{nm}$

$\frac{\partial u}{\partial t} = 0 \therefore B_{nm} = 0$

$$u = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm} \cos \omega_{nm} t \, \psi_{nm}(x,y)$$

$$u(x,y,0) = f(x) = \sum\sum A_{nm} \psi_{nm}(x,y)$$

$$A_{nm} = \frac{4}{ab}\int_0^a dx \int_0^b dy \, f(x,y) \sin \frac{n\pi x}{a} \sin \frac{m\pi y}{b}$$

since $f(x,y) = \sin \frac{\pi x}{a} \sin \frac{\pi y}{b}$

$$u = \sin \frac{\pi x}{a} \sin \frac{\pi y}{b} \cos \omega_{1,1} t$$

$$\omega_{1,1}^2 = c^2 \pi^2 (a^2 + b^2)$$

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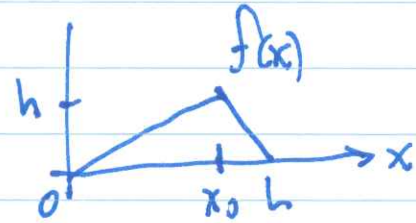
(2) $$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad \begin{matrix} 0<x<L \\ t>0 \end{matrix}$$

B.C. $u(0,t) = u(L,t) = 0$ I.C. $u(x,0) = f(x)$; $u_t(x,0) = g(x)$

$$f(x) = \begin{cases} \dfrac{hx}{x_0} & 0 \le x < x_0 \\ \dfrac{h(L-x)}{L-x_0} & x_0 \le x \le L \end{cases}$$



$u = H(t)\,G(x)$

$$\frac{1}{c^2}\frac{H_{tt}}{H} = \frac{G_{xx}}{G} = -k^2 \qquad \text{let } \omega^2 = k^2 c^2$$

$G_{xx} + k^2 G = 0 \qquad G(0) = G(L) = 0$

$$G = \varphi_n(x) = \sin\frac{n\pi x}{L} \qquad k_n = \frac{n\pi}{L} \quad n = 1, 2 \ldots$$

$H = A_n \cos\omega_n t + B_n \sin\omega_n t$

$$u = \sum_{n=1}^{\infty} \left[A_n \cos\omega_n t + B_n \sin\omega_n t\right]\varphi_n(x)$$

$$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \varphi_n(x)$$

$$A_n = \frac{2}{L}\int_0^L f(x)\,\varphi_n(x)\,dx$$

$$\frac{\partial u}{\partial t} = \sum\left[-\omega_n A_n \sin\omega_n t + \omega_n B_n \cos\omega_n t\right]\varphi_n(x)$$

$$\frac{\partial u}{\partial t}(x,0) = \sum_{n=1}^{\infty} \omega_n B_n \varphi_n(x)$$

$$B_n = \frac{2}{\omega_n L} \int_0^L g(x)\varphi_n(x)\,dx$$

$$\text{if } g(x) = 0 \quad B_n = 0$$

$$A_n = \frac{2}{L} \int_0^{x_0} \frac{hx}{x_0} \sin\frac{n\pi x}{L}\,dx + \frac{2}{L}\int_{x_0}^{L} \frac{h(L-x)}{L-x_0} \sin\frac{n\pi x}{L}\,dx$$

$$A_n = \frac{2h}{x_0(L-x_0)} \frac{L^2}{n^2\pi^2} \sin\frac{n\pi x_0}{L}$$

$$\therefore u(x,t) = \frac{2L^2 h}{\pi^2 x_0(L-x_0)} \sum \frac{1}{n^2} \sin\frac{n\pi x_0}{L} \cos\omega_n t \sin\frac{n\pi x}{L}$$

$$\omega_n = c\frac{n\pi}{L}$$

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(3) Find solution to $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + 3\cos 3t\, e^{-x}$ DE

B.C. $\frac{\partial u}{\partial x}(0,t) = 0 \quad u(L,0) = 0$

I.C. $\frac{\partial u}{\partial t}(x,0) = 0 \quad u(x,0) = \cos\frac{\pi x}{2L}$

let ~~$u(x,t) = V(x) + W(x)$~~

$u(x,t) = V(x,t) + W(x,t)$ where

$$\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}(x,t) = \frac{\partial^2 V}{\partial x^2} + 3\cos 3t\, e^{-x} \quad (1)$$

assume $V(x,t) = A\cos 3t\, e^{-x}$ & subst in (1)

$$-\frac{9}{c^2}Ae^{-x} = Ae^{-x} + 3e^{-x} \Rightarrow A = -\frac{3c^2}{9+c^2}$$

$$\boxed{V(x,t) = -\frac{3c^2}{9+c^2}e^{-x}\cos 3t}$$

Hence $\frac{1}{c^2}\frac{\partial^2 W}{\partial t^2} = \frac{\partial^2 W}{\partial x^2}$

but the B.C. are inhomogeneous unless we choose $W(x,t)$ carefully:

let $W(x,t) = \sum_{n=0}^{\infty} a_n(t)\,\varphi_n(x) + (Cx+D)\cos 3t$

C, D and $a_n(t)$ to be determined.

where φ_n are constructed from the associated SL

$$\text{SL}\quad \begin{cases} \dfrac{\partial^2 \varphi_n}{\partial x^2} = -\lambda_n^2 \varphi_n \\[2ex] \dfrac{\partial \varphi_n}{\partial x}(0) = \varphi_n(L) = 0 \end{cases}$$

solving the SL $\quad \varphi_n = A_n \cos \lambda_n x + B_n \sin \lambda_n x$

$\dfrac{\partial \varphi_n}{\partial x}(0) = 0 \Rightarrow B_n = 0$

$\varphi_n(L) = 0 \Rightarrow \boxed{\lambda_n = \dfrac{(n+\frac{1}{2})\pi}{L} \quad n = 0, 1, 2 \ldots}$

$$\boxed{\varphi_n = \cos \lambda_n x \quad n = 0, 1 \ldots}$$

Substituting $u(x,t)$ in B.C.

$$\left.\frac{\partial V}{\partial x}\right|_{x=0} + \left.\frac{\partial W}{\partial x}\right|_{x=0} = 0$$

$$-A\cos 3t + C\cos 3t \Rightarrow \boxed{C = A = \frac{-3c^2}{9+c^2}}$$

$$u(L,t) = 0 = (CL+D)\cos 3t + Ae^{-L}\cos 3t = 0$$

$$+AL + D + Ae^{-L} = 0$$

$$D = -AL - Ae^{-L} = -A(1+e^{-L})$$

$$\boxed{D = \frac{3c^2}{9+c^2}(1+e^{-L})}$$

So now the B.C. on W are homogeneous and automatically satisfied.

The I.C. $\frac{\partial u}{\partial t}(x,0) = 0$ and $u(x,0) = \frac{\cos \pi x}{2L}$

$$\frac{\partial W}{\partial t}(x,0) = 0 = \sum_{n=0}^{\infty} \frac{d}{dt} a_n(0)\, \varphi_n(x) \Rightarrow \boxed{\begin{array}{c} a_n'(0) = 0 \\ n = 0, 1, \ldots \end{array}}$$

We expand $Cx + D \equiv g(x) = \sum_{n=0}^{\infty} d_n \varphi_n(x)$

$$d_n = \frac{2}{L}\int_0^L (Cx+D)\,\varphi_n(x)\,dx \quad n = 0, 1, \ldots$$

$$\text{or} \quad \begin{cases} d_n = \frac{2}{L} C \int_0^L x\,\varphi_n(x)\,dx & n = 1, 2, \ldots \\ d_0 = D \end{cases}$$

so $w(x,t) = \sum_{n=0}^{\infty} a_n(t)\,\varphi_n(x) + \sum_{n=0}^{\infty} d_n\,\varphi_n(x)\cos 3t$

the only thing to pin down is $a_n(t)$.

Substitute $w(x,t)$ into $\frac{1}{c^2}\frac{\partial^2 w}{\partial t^2} = \frac{\partial^2 w}{\partial x^2}$

$\cancel{\sum \frac{1}{c^2}\frac{d^2 a_n}{dt} - \frac{9}{c^2}\sum d_n \varphi_n}$ let $\tilde{d}_n(t) = d_n \cos 3t$

$$\sum_{n=0}^{\infty} \frac{1}{c^2}\frac{d^2 a_n}{dt^2}\varphi_n - \frac{9}{c^2}\sum_{n=0}^{\infty}\tilde{d}_n\varphi_n = \frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty} a_n(t)\varphi_n + \frac{\partial^2}{\partial x^2}\sum_{n=0}^{\infty}\tilde{d}_n\varphi_n$$

but RHS $\frac{\partial^2 \varphi_n}{\partial x^2} = -\lambda_n^2\varphi_n$ so

$$\sum_{n=0}^{\infty}\varphi_n\left[\frac{1}{c^2}\frac{d^2 a_n}{dt^2} - \frac{9}{c^2}\tilde{d}_n + \lambda_n^2 a_n + \lambda_n^2\tilde{d}_n\right] = 0$$

using orthogonality

$$\frac{d^2}{dt^2}a_n + \omega_n^2 a_n = \cancel{(\omega_n^2 - 9)}\, d_n\cos 3t$$
$$= (-\omega_n^2 + 9)\, d_n \cos 3t$$

we already found that $\frac{da_n(0)}{dt} = 0$

we need another initial condition

$$u(x,0) = \cos\frac{\pi x}{2L} = v\Big|_{t=0} + w\Big|_{t=0}$$

$$\cos\frac{\pi x}{2L} = \frac{-3c^2}{9+c^2} e^{-x} + \sum_{n=0}^{\infty} a_n(0)\varphi_n(x) + \sum d_n \varphi_n(x)$$

$$\sum_{n=0}^{\infty} a_n(0)\varphi_n(x) = -\sum_{n=0}^{\infty} d_n \varphi_n(x) + \cos\frac{\pi x}{2L} + \frac{3c^2}{9+c^2} e^{-x}$$

$$\therefore\ a_n(0) = -d_n + \underbrace{\frac{2}{L}\int_0^L \cos\frac{\pi x}{2L}\varphi_n(x)\,dx}_{n=0 \text{ only}} + \frac{2}{L}\frac{3c^2}{9+c^2}\int_0^L e^{-x}\varphi_n\,dx$$

$$\therefore\ a_0(0) = -D + 1 + \frac{6c^2}{(9+c^2)L}\int_0^L e^{-x}\,dx \qquad n=0$$

$$= -D + 1 - \frac{6c^2}{L(9+c^2)}\left(e^{-L} - 1\right)$$

$$a_n(0) = -d_n + \frac{6c^2}{L(9+c^2)}\int_0^L e^{-x}\varphi_n(x)\,dx \qquad n = 1, 2 \ldots$$

Hence we can solve

$$\left\{ \begin{array}{l} \dfrac{d^2 a_n}{dt^2} + \omega_n^2 a_n = (-\omega_n^2 + 9)\, d_n \cos 3t \\ \text{with } \dfrac{da_n}{dt}(0) = 0 \\ \text{and } a_n(0) \text{ as per above.} \end{array} \right.$$

the general solution $a_n(t) = a_n^H(t) + a_n^P(t)$

$$a_n^P = K_1 \cos 3t + K_2 \sin 3t$$

$$-9K_1 \cos 3t - 9K_2 \sin 3t + \omega_n^2 K_1 \cos 3t + \omega_n^2 K_2 \sin 3t$$
$$= (-\omega_n^2 + 9)\, d_n \cos 3t$$

let $K_2 = 0$

$$(-9 + \omega_n^2) K_1 = (-\omega_n^2 + 9)\, d_n$$

$$\Rightarrow K_1 = 1$$

$$\therefore \; a_n^P = \cos 3t$$

$$\begin{cases} \dfrac{d^2 a_n^H}{dt^2} + \omega_n^2 a_n^H = 0 \\ \dfrac{d a_n^H}{dt}(0) = 0 \\ a_n(0) = \text{----} \end{cases}$$

$$a_n^H = K_3 \cos\omega_n t + K_4 \sin\omega_n t$$

$$\frac{d a_n^H}{dt}(0) = 0 = K_4$$

$$\therefore \quad a_n^H = K_3 \cos\omega_n t$$

$$a_n^H(0) = K_3 = -d_n + \frac{6c^2}{L(9+c^2)} \int_0^L e^{x} \varphi_n(x)\, dx$$

$$n = 1, 2 \ldots$$

$$a_0^H(0) = -D + 1 - \frac{6c^2}{L(9+c^2)} \left(e^{-L} - 1\right)$$

So we have a fully explicit solution. //