(1) Solve
$$\begin{cases} u_t = k_0 u_{xx} + 2\beta u_x & 0<x<L \\ u(0,t) = u(L,t) = 0 & t>0 \\ u(x,0) = f(x) \end{cases}$$

$u = H(t)F(x)$

$$H_t F = k_0 H F_{xx} + 2\beta H F_x$$

$$\frac{1}{k_0}\frac{H_t}{H} = \frac{F_{xx}}{F} + \frac{2\beta}{k_0}\frac{F_x}{F} = -l^2$$

(A) $\frac{1}{k_0}\frac{H_t}{H} = -l^2$ and $\begin{cases} F_{xx} + \frac{2\beta}{k_0}F_x + l^2 F = 0 \\ F(0) = F(L) = 0 \end{cases}$ (B)

solving (B): $F(x) \sim e^{mx} \Rightarrow m^2 + \frac{2\beta}{k_0} m + l^2 = 0$

$$m_{1,2} = -\frac{\beta}{k_0} \pm \frac{1}{2}\sqrt{\frac{4\beta^2}{k_0^2} - 4l^2} = -\frac{\beta}{k_0} \pm \frac{\beta}{k_0}\sqrt{1 - \frac{l^2 k_0^2}{\beta^2}}$$

$$m_{1,2} = -\frac{\beta}{k_0}\left[1 \mp \sqrt{1 - l^2 k_0^2/\beta^2}\right]$$

CASE A: $\beta = 0$ then $F = \sin\frac{n\pi x}{L}$ $n = 1, 2 \ldots$

we get the familiar $u = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} e^{-\frac{n\pi}{L}k_0 t}$

case (B) $\beta \neq 0$:

Write $m_{1,2} = -\frac{\beta}{k_0}\left[1 \mp i\alpha\right]$

$$\alpha = \sqrt{l^2 - \beta^2/k_0^2}$$

for $l^2 > \beta^2/k_0^2$, $\alpha > 0$ and real. In fact, $l^2 < \beta^2/k_0^2$ is not possible

$$F = \left(A\cos\alpha x + B\sin\alpha x\right)e^{-\beta/k_0 x}$$

$$F(0) = 0 = A \text{ and } F(L) = 0 = \sin\alpha L\, e^{-\frac{\beta}{k_0}L} = 0$$

$$\text{or} \quad \alpha_n = \frac{n\pi}{L} \qquad n = 1, 2 \ldots$$

$$F_n = \sin\frac{n\pi}{L}x\, e^{-\frac{\beta}{k_0}x}$$

w/ $\frac{1}{k_0}\frac{H_t}{H} = -l^2 \Rightarrow H_t - k_0 l^2 H = 0$

$$l^2 = \alpha_n + \beta^2/k_0^2$$

$$H_t + k_0\left(\alpha_n^2 + \beta^2/k_0^2\right)H = 0$$

$$H = H_0 e^{-k_0\left(\alpha_n^2 + \beta^2/k_0^2\right)t}$$

$$u = \sum_{n=1}^{\infty} a_n e^{-k_0\left(\alpha_n^2 + \beta^2/k_0^2\right)t} e^{-\beta/k_0 x} \sin\frac{n\pi x}{L}$$

where $a_n = \frac{2}{L}\int_0^L f(x)\, e^{\frac{\beta}{k_0}x}\, dx$ //

(2) a. $w(x,y) = \sum_{m=1}^{\infty} \psi_m(y) \sin\frac{m\pi x}{a}$ solve

$$\Delta w = \sum_{m=1}^{\infty} \psi_{m_{yy}} \sin\frac{m\pi x}{a} - \left(\frac{m\pi}{a}\right)^2 \psi_m \sin\frac{m\pi x}{a} = 0$$

provided $\psi_{m_{yy}} - \left(\frac{m\pi}{a}\right)^2 \psi_m = 0$

so $\psi_m = A_m \cosh\frac{m\pi y}{a} + B_m \sinh\frac{m\pi y}{a}$

Note: $\psi_m(0) = A_m$ $\qquad \psi_m(b) = B_m \sinh\frac{m\pi b}{a} + A_m \cosh\frac{m\pi b}{a} =$

$$w(x,0) = g(x) = \sum_{m=1}^{\infty} \psi_m(0) \sin\frac{m\pi x}{a} = \sum_{m=1}^{\infty} A_m \sin\frac{m\pi x}{a}$$

$$A_m = \frac{2}{a}\int_0^a g(x) \sin\frac{m\pi x}{a}\,dx \text{ , using orthogonality.}$$

$$w(x,b) = h(x) = \sum_{m=1}^{\infty} \psi_m(b) \sin\frac{m\pi x}{a} \text{ ; let } \psi_m(b) \equiv h_m = \frac{2}{a}\int_0^a h(x) \sin\frac{m\pi x}{a}\,dx$$

using orthogonality.

$$B_m = \frac{1}{\sinh\frac{m\pi b}{a}}\left[-A_m \cosh\frac{m\pi b}{a} + h_m\right]$$

~~Switch~~ Switch the roles of x, y to find (V) solution:

$$V(x,y) = \sum_{m=1}^{\infty} \varphi_m(x) \sinh \frac{m\pi y}{b}$$

where $\varphi_m = f_m \cosh \frac{m\pi}{b} x + C_m \sinh \frac{m\pi x}{b}$

$$f(y) = \sum_{m=1}^{\infty} \varphi_m(0) \sinh \frac{m\pi y}{b}$$

$$k(y) = \sum_{m=1}^{\infty} \varphi_m(a) \sinh \frac{m\pi y}{b}$$

$$\text{let } \varphi_m(0) = f_m = \frac{2}{b} \int_0^b f(x) \sinh \frac{m\pi y}{b} dy$$

$$\varphi_m(a) = k_m = \frac{2}{b} \int_0^b k(x) \sinh \frac{m\pi y}{b} dy$$

$$\varphi_m(0) = f_m \quad ; \quad \underbrace{\varphi_m(a)}_{\text{"} k_m} = C_m \sinh \frac{m\pi a}{b} + f_m \cosh \frac{m\pi a}{b}$$

$$\therefore C_m = \frac{k_m - f_m \cosh m\pi a/b}{\sinh \frac{m\pi a}{b}}$$

$$\therefore u = \sum_{m=1}^{\infty} \left[A_m \cosh \frac{m\pi y}{a} + \frac{1}{\sinh \frac{m\pi b}{a}} \left[h_m - A_m \cosh \frac{m\pi b}{a} \right] \sinh \frac{m\pi y}{a} \right] \sin \frac{m\pi x}{a}$$

$$+ \sum_{m=1}^{\infty} \left[f_m \cosh \frac{m\pi x}{b} + \frac{1}{\sinh \frac{m\pi a}{b}} \left[k_m - f_m \cosh \frac{m\pi a}{b} \right] \sinh \frac{m\pi x}{b} \right] \sin \frac{m\pi y}{b}$$

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(3) $\begin{cases} \dfrac{\partial u}{\partial t} - k_0 \dfrac{\partial^2 u}{\partial x^2} = x & 0<x<L \;, t>0 \\ \dfrac{\partial u}{\partial x}(0,t) = \cancel{\dfrac{\partial u}{\partial x}(L,t)} = 0 = U(L,t), & 0<t \\ u(x,0) = f(x) \end{cases}$

Let $u(x,t) = v(x,t) + U(x)$

$\begin{cases} \dfrac{\partial v}{\partial t} - k_0 \dfrac{\partial^2 v}{\partial x^2} = \dfrac{d^2 U}{dx^2} + x \\ \dfrac{\partial}{\partial x}(v+U)\Big|_{x=0} = 0 \qquad \dfrac{\partial}{\partial x}(v+U)\Big|_{x=L} = 0 \end{cases}$

(A) $\begin{cases} \dfrac{\partial v}{\partial t} = k_0 \dfrac{\partial^2 v}{\partial x^2} \\ v_x(0,t) = v_x(L,t) = 0 \\ v(x,0) = f(x) \end{cases}$ (B) $\begin{cases} \dfrac{d^2 U}{dx} + x = 0 \\ U'(0) = U'(L) = 0 \end{cases}$

(B): integrate twice $U_x = -\frac{1}{2}x^2 + c$; $U = \frac{1}{6}x^3 + cx + d$

$\cancel{U'(0) = 0 = -\frac{1}{2}0^2 + c \Rightarrow} \; c = 0$

$\cancel{U'(L) = 0 \text{ if } U_x = -\frac{1}{2}(x-L)^2} \quad d = 0$

$U'(0) = 0 \Rightarrow c = 0 \qquad U(L) = 0 = \frac{1}{6}L^3 + d \Rightarrow d = -\frac{1}{6}L^3$

$\therefore U(x) = \frac{1}{6}(x^3 - L^3)$

Solve (A) $v = H(t)G(x)$

$$\frac{1}{k_0}\frac{H_t}{H} = \frac{G_{xx}}{G} = -k^2$$

$$G_{xx} + k^2 G = 0 \qquad G'(0) = G(L) = 0$$

$$G = A\cos kx + B\sin kx$$

$$G'(0) = -Ak\sin kx + Bk\cos kx \Big|_{x=0} \Rightarrow B = 0$$

$$G(L) = 0 = A\cos kL \Rightarrow k_n = \frac{(n+\frac{1}{2})\pi}{L} \quad n = 0, 1 \ldots$$

$$H_t - k_0 k_n^2 H = 0 \rightarrow H \sim H_0 e^{-k_0 k_n^2 t}$$

$$v = \sum_{n=0}^{\infty} a_n e^{-k_0 k_n^2 t} \cos\frac{(n+\frac{1}{2})\pi x}{L}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{(n+\frac{1}{2})\pi x}{L}\,dx \qquad \cancel{n = 1, 2, \ldots} \quad n = 0, 1, 2 \ldots$$

(4)
$$\begin{cases} \dfrac{\partial u}{\partial t} - k_0 \dfrac{\partial^2 u}{\partial x^2} = g(x,t) \\ u(0,t) = A(t) \quad , \quad u(L,t) = B(t) \\ u(x,0) = 0 \end{cases}$$

let $u = v(x,t) + K(x,t)$

want $\left.\begin{array}{l} K(0,t) = A(t) \\ K(L,t) = B(t) \end{array}\right\} K(x,t) = A(t) + \dfrac{B(t) - A(t)}{L} x$

$$\frac{\partial K}{\partial t} = A' + \frac{B' - A'}{L} x$$

$$\frac{\partial K}{\partial x} = \frac{B-A}{L} \qquad \frac{\partial^2 K}{\partial x^2} = 0$$

$$\therefore \frac{\partial v}{\partial t} + \frac{dA}{dt} + \frac{x}{L}\frac{d}{dt}(B-A) - k_0 \frac{\partial^2 v}{\partial x^2} = g$$

leads to $\dfrac{\partial v}{\partial t} - k_0 \dfrac{\partial^2 v}{\partial x^2} = g - A' - \dfrac{x}{L}\dfrac{d}{dt}(B-A)$

$$v(0,t) = v(L,t) = 0$$

$$v(x,0) = -K(0) = -\left[A(0) + \frac{B(0) - A(0)}{L} x\right] \equiv g(x)$$

$$v(x,0) = g(x)$$

so let $Q(x,t) = q - A'(t) - \frac{x}{L}(B' - A')(t)$

$$\begin{cases} \frac{\partial v}{\partial t} - k_0 \frac{\partial^2 v}{\partial x^2} = Q(x,t) \\ v(0,t) = v(L,t) = 0 \\ v(x,0) = g(x) \end{cases}$$

the associated $\varphi'' + \lambda^2 \varphi = 0$

$$\varphi(0) = \varphi(L) = 0$$

$$\varphi_n = \sin\frac{n\pi x}{L} \qquad \lambda_n = \frac{n\pi}{L} \qquad n = 1, 2 \ldots$$

$$(\ddagger) \quad v(x,t) = \sum_{n=1}^{\infty} \psi_n(t)\varphi_n \quad \& \quad Q(x,t) = \sum_{n=1}^{\infty} a_m(t)\varphi_n(x)$$

$$\varphi_n \frac{d}{dt}\psi_n - k_0 \psi_n \frac{\partial^2 \varphi_n}{\partial x^2} = a_m \varphi_n \qquad n = 1, 2 \ldots$$

$$\frac{d\psi_n}{dt} + k_0 \lambda_n^2 \psi_n = a_n$$

where: $\psi_n = c_n e^{-k_0 \lambda_n^2 t} + \int_0^t a_n(s) e^{-k_0 \lambda_n^2 (t-s)} ds$ in $(\ddagger)$

$$v(x,0) = g(x) = \sum_{n=1}^{\infty} c_n \varphi_n(x)$$

$$c_n = \frac{2}{L}\int_0^L g(x)\varphi_n(x)\,dx \qquad n = 1, 2 \ldots$$

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