HW Heat Equation & Laplace's Equation

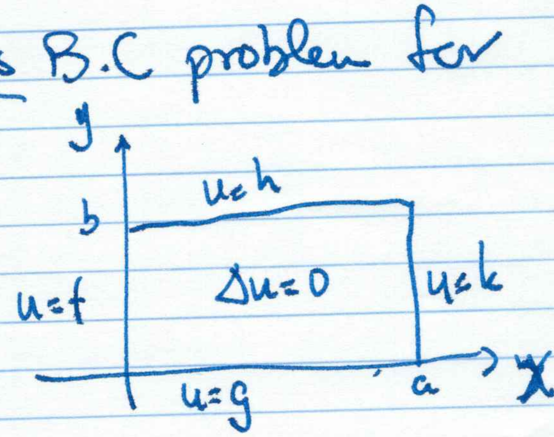
(1) Suppose a uniform plate with heat diffusivity k_0 and convection constant 2β, solve

$$\begin{cases} \dfrac{\partial u}{\partial t} = k_0 \dfrac{\partial^2 u}{\partial x^2} + 2\beta \dfrac{\partial u}{\partial x} & 0<x<L \\ & t>0 \\ u(0,t) = u(L,t) = 0 & t>0 \\ u(x,0) = f(x) & 0<x<L \end{cases}$$

In what ways is the solution, with $\beta = 0$ and $\beta \neq 0$ different? consider $\beta \in \mathbb{R}^1$

(2) Solving the non-homogeneous B.C problem for

$$\begin{cases} \Delta u = 0 & \begin{cases} 0<x<a \\ 0<y<b \end{cases} \\ u(x,0)=g(x) & 0<x<a \\ u(x,b)=h(x) & 0<x<a \\ u(0,y)=f(y) & 0<y<b \\ u(a,y)=k(y) & 0<y<b \end{cases}$$



Strategy: let $u(x,y) = W(x,y) + V(x,y)$, where

$$(W) \begin{cases} \Delta W = 0 \\ W(x,0) = g(x) \\ W(x,b) = h(x) \\ W(0,y) = 0 \\ W(a,y) = 0 \end{cases} \quad \text{and} \quad (V) \begin{cases} \Delta V = 0 \\ V(x,0) = 0 \\ V(x,b) = 0 \\ V(0,y) = f(y) \\ V(a,y) = k(y) \end{cases}$$

solve each of these seperately & add them. So, it suffices to show how to solve the (W) problem.

(a) Show that $W(x,y) = \sum_{m=1}^{\infty} \psi_m(y) \sin\frac{m\pi x}{a}$, solves (W) under certain conditions.

(b) argue why $g(x) = \sum_{m=1}^{\infty} \psi_m(0) \sin\frac{m\pi x}{a}$, $h(x) = \sum_{m=1}^{\infty} \psi_m(b) \sin\frac{m\pi x}{a}$

(c) hence $\psi_m(0) = g_m$ and $\psi_m(b) = h_m$ $\quad m=1,2\ldots$

where g_m and h_m are constants. How do we find g_m and h_m (what's the formula?)

(d) Solve the ODE for $\Psi_m(y)$ $m=1,2\ldots$

(e) Finally, show that

$$w(x,y)=\sum_{m=1}^{\infty} g_m \frac{\sinh m\pi(b-y)}{\sinh \frac{m\pi b}{a}} \sin\frac{m\pi x}{a}$$

$$+\sum_{m=1}^{\infty} h_m \frac{\sinh\frac{m\pi y}{a}}{\sinh\frac{m\pi b}{a}} \sin\frac{m\pi x}{a}$$

(f) To find v, follow same procedure, with the obvious modifications, i.e. you want the Sturm Liouville to be in y. (work this one out, if you'd like).

(3) (Variation of parameters and inhomogeneous heat equation)

a) Find solution to $\frac{\partial u}{\partial t} - k_0\frac{\partial^2 u}{\partial x^2} = x$ $0<x<L$, $t>0$

B.C. $\frac{\partial u}{\partial x}(0,t)=\frac{\partial u}{\partial x}(L,t)=0$ $0<t$

I.C. $u(x,0)=f(x)$ $0<x<L$

(b) Find $\lim_{t\to\infty} u(x,t)$. This is the "asymptotic or long-time" heat distribution on this rod, with constant heat diffusion $k_0>0$.

(4) Solve $\frac{\partial u}{\partial t} - k_0 \frac{\partial^2 u}{\partial x^2} = q(x,t)$ $\quad 0<x<L \,, t>0$

B.C. $u(0,t) = A(t)$ $\quad u(L,t) = B(t)$ $\quad t>0$

I.C. $u(x,0) = 0$ $\quad 0<x<L$

(3) Solve $\frac{\partial u}{\partial t} - k_0 \frac{\partial^2 u}{\partial x^2} = x$ $\quad 0<x<L, t>0$

$\frac{\partial u}{\partial x}(0,t) = 0$ $\quad u(L,t) = 0$

$u(x,0) = f(x)$