HW2 Fourier Series (FS)

(1) Find FS of $f = \sin^2 x \cos x + 3\cos^3 x + 2\sin x \cos x$

using trig identities

$$f = \frac{5}{2}\cos x + \frac{1}{2}\cos 3x + \sin 2x$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2\pi n x}{L} + b_n \sin\frac{2\pi n x}{L}$$

if period $L = 2\pi$, then

$a_0 = 0$, $a_3 = \frac{1}{2}$ $a_1 = \frac{5}{2}$ $b_2 = 1$ //

(2) Complex FS

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x} \quad \text{where } k_n = \frac{2\pi n}{L}$$

for $-\frac{L}{2} < x \leq \frac{L}{2}$ L is the period.

$$c_n = \frac{1}{L}\int_0^L e^{-ik_n x} f(x)\, dx$$

Note, if $f(x)$ is real then $\bar{c}_n = c_{-n}$

We want to show that, if $f \in \mathbb{R} \Rightarrow$

$$(\bigstar) \quad c_n = \begin{cases} a_0/2 & n=0 \\ (a_n - ib_n)/2 & n>0 \\ (a_{-n} + ib_{-n})/2 & n<0 \end{cases}$$

and $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{2n\pi x}{L} + b_n \sin\frac{2n\pi x}{L} \quad (\dagger)$$

To confirm $\bigstar$ for $f(x) \in \mathbb{R}$, Let $c_n = \alpha_n + i\beta_n$

$$\therefore f(x) = \alpha_0 + i\beta_0 + \sum_{n=1}^{\infty} (\alpha_n + i\beta_n) e^{ik_n x}$$

$$+ \sum_{n=-\infty}^{-1} (\alpha_n + i\beta_n) e^{ik_n x} \quad (*)$$

Comparing $(\bigstar)$ and $(\dagger)$ $\quad c_0 = \alpha_0 + i\beta = \alpha_0 = \frac{a_0}{2}$

Rewrite $(*)$ as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (\alpha_n + i\beta_n) e^{ik_n x} + \sum_{n=\infty}^{1} (\alpha_{-n} + i\beta_{-n}) e^{-ik_n x}$$

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (\alpha_n + i\beta_n)(\cos k_n x + i \sin k_n x)$$

$$+ (\alpha_{-n} + i\beta_{-n})(\cos k_n x - i \sin k_n x)$$

clearly, sum is real iff $\alpha_{-n} + i\beta_{-n} = \alpha_n - i\beta_n$

or $\quad \bar{C}_n = C_{-n} \quad \therefore$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos k_n x + b_n \sin k_n x$$

with a_0, a_n, b_n as in $(\bigstar)$. //

Compute Fseries of e^x, using complex FS:

$k_n = \frac{2\pi n}{h}$ where $0 \leq x < h$

$$C_n = \frac{1}{h}\int_0^h e^{-ik_n x} e^x \, dx = \frac{1}{h}\int_0^h e^{x(1-ik_n)x} \, dx$$

$$= \frac{1}{h} \frac{1}{1-ik_n} e^{x(1-ik_n)} \Big|_0^h = \frac{1}{h} \frac{1}{1-ik_n}\left[e^{h(1-ik_n)} - 1\right]$$

$$= \frac{1}{h - i2\pi n}\left[e^h e^{-i2\pi n} - 1\right] =$$

$$c_n = \frac{h + 2\pi i n}{h^2 + 4\pi^2 n^2} \left[e^h - 1\right]$$

$$c_0 = \frac{1}{h}\int_0^h e^h \, dx = \frac{e^h - 1}{h}$$

(3) Parseval's Theorem: Confirm that

$$\frac{1}{L}\int_0^L |f(x)|^2 \, dx = \sum_{n=-\infty}^{\infty} |c_n|^2 = \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 + b_n^2$$

Take $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x}$ $\quad g(x) = \sum_{n=-\infty}^{\infty} d_n e^{ik_n x}$

$$k_n = \frac{2\pi n}{L}$$

$$f g^* = \sum_{n=-\infty}^{\infty} c_n g^* e^{ik_n x}$$ integrating

$$\frac{1}{L}\int_0^L f g^* \, dx = \sum_{n=-\infty}^{\infty} c_n \frac{1}{L}\int_0^L g^*(x) e^{ik_n x} \, dx$$

$$= \sum_{n=-\infty}^{\infty} c_n \frac{1}{L} \int_0^L g e^{-ik_n x} dx = \sum_{n=-\infty}^{\infty} c_n d_n^*$$

if $g = f$

$$\frac{1}{L} \int_0^L |f|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2 = \left(\frac{1}{2} a_0\right)^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

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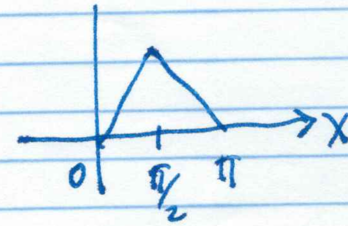
HW2 5/8

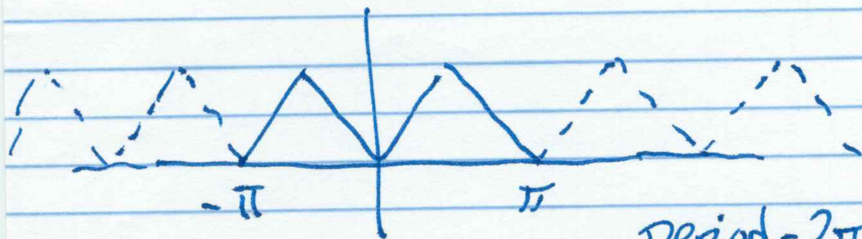
(4) Compute FS of

$$f = \begin{cases} x & 0 \le x < \pi/2 \\ (\pi - x) & \frac{\pi}{2} \le x < \pi \end{cases}$$



by (A) Periodic even extension

$$g = \begin{cases} x & 0 \le x < \pi/2 \\ \pi - x & \pi/2 \le x < \pi \\ -x & -\pi/2 \le x < 0 \\ x + \pi & -\pi \le x < -\pi/2 \end{cases}$$



period = 2π, $l = \pi$

since even $b_n = 0$

$$g(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{\pi} = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \langle g(x) \rangle = \frac{\pi}{4}$$

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(x) \cos nx \, dx = \frac{1}{2\pi} \left\{ \int_{-\pi}^{-\pi/2} + \int_{-\pi/2}^{0} + \int_{0}^{\pi/2} + \int_{\pi/2}^{\pi} \right\}$$

know that $\int x \cos nx \, dx = \dfrac{\cos(nx) + nx \sin(nx)}{n^2}$

HW2 6/8

$$a_n = \frac{1}{2\pi}\left\{ \int_{-\pi}^{-\pi/2} (x+\pi)\varphi_n\,dx + \int_{-\pi/2}^{0} -x\varphi_n + \int_{0}^{\pi/2} x\varphi_n\,dx + \int_{\pi/2}^{\pi} (\pi-x)\varphi_n\,dx \right\}$$

$$\varphi_n = \cos nx$$

Using $z = -x$ $dz = -dx$ on the first 2 integrals

$$a_n = \frac{2}{2\pi}\left\{ \int_{0}^{\pi/2} x\varphi_n\,dx - \int_{\pi/2}^{\pi} x\varphi_n\,dx + \pi\int_{\pi/2}^{\pi}\varphi_n\,dx \right\}$$

$$= \frac{1}{n^2\pi}\left\{ n\pi\sin\frac{n\pi}{2} + 2\cos n\pi/2 - \cos n\pi - 1 - \pi n\sin\frac{n\pi}{2} \right\}$$

$$= \frac{1}{n^2\pi}\left[2\cos\frac{n\pi}{2} - \cos n\pi - 1 \right]$$ n odd are $a_n = 0$
for n even, every other is 0

so let $a_m = \cancel{-\frac{1}{n^2\pi}\frac{1}{m^2}}$ $-\frac{1}{4m^4\pi}$ $m = 1, 3, 5 \ldots$

$$g(x) = \sum_{m=1,3,5\ldots} -\frac{1}{4m^4\pi}\cos 2mx$$

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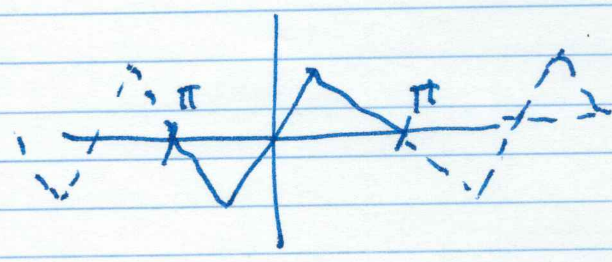
HW2 7/8

by Periodic Odd extension

$$h = \begin{cases} x & 0 \le x < \pi/2 \\ (\pi - x) & \pi/2 \le x < \pi \\ x & -\pi/2 \le x < 0 \\ -(\pi + x) & -\pi < x \le -\frac{\pi}{2} \end{cases}$$



Odd function so $a_0 = a_n = 0$ $\quad n = 1, 2 \ldots$

$$h(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(x) \sin nx \, dx$$

$$= \frac{1}{2\pi} \left\{ -\int_{-\pi}^{-\pi/2} (x + \pi) \sin nx \, dx + \int_{-\pi/2}^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx \right\}$$

$$= \frac{1}{2\pi} \left\{ 2 \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx + \int_{-\pi/2}^{\pi/2} x \sin nx \, dx \right\}$$

$$\int x \sin nx \, dx = \frac{\sin(nx) - xn\cos(nx)}{n^2}$$

HW2 8/8

$$= \frac{1}{2\pi}\left\{ -\frac{2}{n}\pi\cos nx\Big|_{\pi/2}^{\pi} - \frac{2}{n^2}\left(\sin nx - xn\cos nx\right|_{\pi/2}^{\pi} + \frac{\sin nx - xn\cos nx}{n^2}\Big|_{-\pi/2}^{\pi/2}\right\}$$

$$= \frac{1}{2\pi}\left\{ -\frac{2\pi}{n}\cos n\pi + \frac{2\pi}{n}\cos\frac{n\pi}{2} - \frac{2}{n^2}\overset{0}{\cancel{\sin n\pi}} + \frac{2}{n^2}\sin\frac{n\pi}{2} + \frac{2}{n^2}n\pi\cos n\pi - \frac{2}{n^2}\frac{n\pi}{2}\cos\frac{n\pi}{2}\right.$$

$$\left. + \frac{1}{n^2}\sin\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2} - \frac{1}{n^2}\frac{n\pi}{2}\cos\frac{n\pi}{2} + \frac{1}{n^2}\frac{n\pi}{2}\cos\frac{n\pi}{2}\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{4}{n^2}\sin\frac{n\pi}{2}\right\} = \frac{2}{n^2\pi}\sin\frac{n\pi}{2}$$ only odd terms

$$\frac{2}{(2m+1)^2\pi}(-1)^{2m+1} \qquad m = 0, 1, 2\ldots$$

$$\therefore h(x) = \sum_{m=0}^{\infty}\frac{2}{(2m+1)^2\pi}(-1)^{2m+1}\sin(2m+1)x$$

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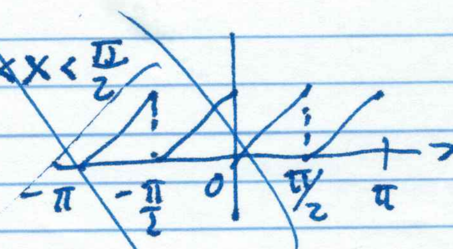
~~Finally look at~~ $f(x) = x \quad 0 < x < \frac{\pi}{2}$



period $= \frac{\pi}{2}$ $\quad l = \frac{\pi}{4}$

$$f = a_0 + \sum_{n=1}^{\infty} a_n\cos\frac{4n\pi x}{\pi} = a_0 + \sum_{n=1}^{\infty} a_n\cos 4nx$$

$$a_0 = \frac{2}{\pi}\left(\frac{\pi}{2}\right)^2\frac{1}{2} = \frac{\pi}{4}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi/2} x\cos 4nx\,dx = \frac{2}{\pi}$$