HW1/8

HW1 Sturm Liouville HW Solutions

(1) $$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} + \frac{\sin(m+n)x}{m+n}\right]_{x=-\pi}^{\pi}$$

since $m-n$ and $m+n$ are ~~integers~~ natural when m, n integers, then the integral is zero unless $m=n$. When $m=n$

$$\int_{-\pi}^{\pi} \cos^2 mx \, dx = \frac{2}{2}\pi = \pi$$

$\therefore$ $\cos mx, \cos nx$ are orthogonal unless $m=n$.

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{x=-\pi}^{\pi}$$

same argument is to be made for this case. ~~cos~~ $\sin mx$ and $\sin nx$ are orthogonal unless $m=n$, when $\int_{-\pi}^{\pi} \sin^2 mx \, dx = \frac{2\pi}{2} = \pi$.

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = \frac{1}{2}\left[\frac{\cos(m+n)x}{m+n} - \frac{\cos(m-n)x}{m-n}\right]_{x=-\pi}^{\pi}$$

$$= \frac{1}{2}\left[\frac{\cos(m+n)\pi}{m+n} - \frac{\cos(m-n)\pi}{m-n}\right] - \frac{1}{2}\left[\frac{\cos(m+n)(-\pi)}{m+n} - \frac{\cos(m-n)(-\pi)}{m-n}\right]$$

$= 0$. //

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12) Show that $\varphi'' + \frac{\gamma}{x}\varphi' + \alpha\varphi = 0$

can be converted into a SL problem

let $p = e^{\int \frac{\gamma}{x}dx} = e^{\gamma \ln x} = e^{\ln x^{\gamma}} = x^{\gamma}$

$q = 0 \quad w = p$ and $\lambda = \alpha$

$$\boxed{\frac{d}{dx}\left[x^{\gamma}\varphi'\right] + \alpha x^{\gamma}\varphi = 0}$$

for $x \neq 0$, $\gamma \geq 0$

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[illegible]

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for $x \neq 0$, $\gamma \geq 0$

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3) Find the eigenfunctions & eigenvalues of the SL

$$\begin{cases} y'' + \lambda^2 y = 0 & 0 < x < l \\ y'(0) = y(l) = 0 \end{cases}$$

(a) $y = A\cos\lambda x + B\sin\lambda x$

(b) $y' = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$

$y'(0) = 0 \Rightarrow B = 0$

$y = A\cos\lambda x \qquad y(l) = 0 \qquad \cos\lambda l = 0$

$$\begin{cases} \lambda_n = (n+\frac{1}{2})\pi / l & n = 0, 1, 2 \ldots \\ y_n = \cos\frac{(n+\frac{1}{2})\pi}{l} x \end{cases}$$

4) Find eigenfunctions/eigenvalues of SL

$$\begin{cases} y'' + \lambda^2 y = 0 & 0 < x < 1 \\ y(0) + y'(0) = 0 & y(1) = 0 \end{cases}$$

Using (a) & (b) above:

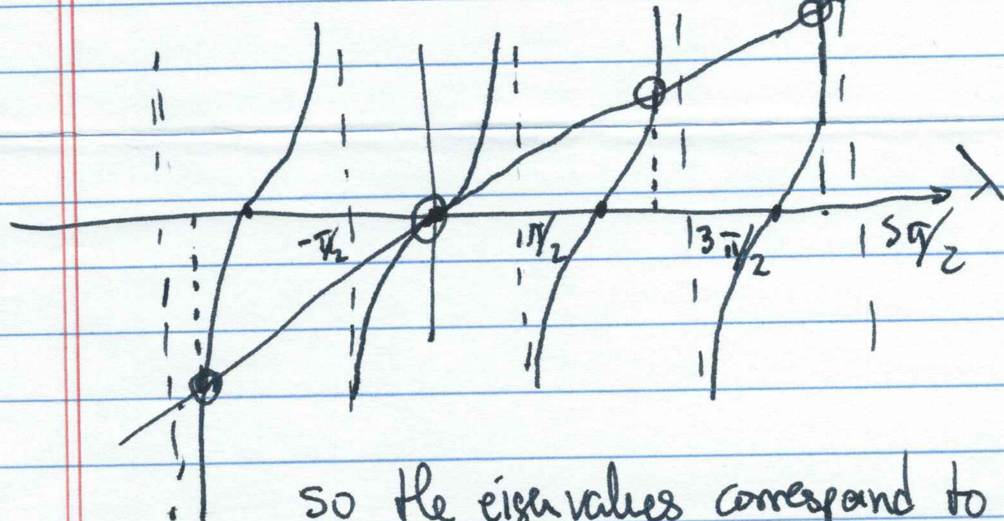
$A\cos\lambda 0 + \cancel{B\sin\lambda 0} - \cancel{A\lambda\sin\lambda 0} + B\lambda\cos\lambda 0 = 0$

or $A + B\lambda = 0 \Rightarrow A = -B\lambda$

$y(1) = -B\lambda\cos\lambda + B\sin\lambda = 0$

hence $\sin\lambda = \lambda\cos\lambda$

or $\tan\lambda = \lambda$



so the eigenvalues correspond to ~~intersection~~ values of λ where $\tan\lambda$ and λ intersect.

$$\cdots\cdots \lambda_{-2} < \lambda_{-1} < \lambda_0 = 0\ ,\ \lambda_1 > \lambda_0\ ,\ \lambda_2 > \lambda_1 \cdots$$

the eigenfunctions are

$$\varphi_n = -\lambda_n \cos\lambda_n x + \sin\lambda_n x$$
$$n = 0, \pm 1, \pm 2 \ldots$$

//

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(5) (Hermite Polynomials)

(a) Show that $y'' - 2xy' + 2ny = 0$ (‡)

n is a non-negative integer or 0 : $n = 0, 1, 2 \ldots$

is an SL equation. Hint: multiply equation by $W(x) \equiv \exp(-x^2)$

$$e^{-x^2} y'' - 2x e^{-x^2} y' + 2n e^{-x^2} y = 0$$

$$\frac{d}{dx}\left[e^{-x^2} y' \right] + 2n e^{-x^2} y = 0$$

which is of the form

$$\frac{d}{dx}\left[p \frac{dy}{dx} \right] + w \lambda y = 0 \quad , \text{with } w > 0$$

(b) Show that (‡) has Hermite function solutions

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \qquad n = 0, 1, 2 \ldots$$

Use these facts:
$$\begin{cases} \frac{d}{dx}\left[e^{-x^2} H_n \right] = -g H_{n+1} \\ \frac{d}{dx} H_n = 2n H_{n-1} \end{cases}$$

(both of these are easy to prove)

$$H_n'' - 2x H_n' + 2n H_n = 0$$

$$(2n H_{n-1})' - 2x H_n' + 2n H_n = 0 \qquad \text{mult by } g = e^{-x^2}$$

$$g 2n H_{n-1}' - 2x g H_n' + 2n g H_n = 0$$

$$2n g H_{n-1}' - 4n x g H_{n-1} - 2n (g H_{n-1})' = 0$$

$$\cancel{2ng H_{n-1}'} - 4nxg H_{n-1} - 2ng' H_{n-1} - \cancel{2ng H_{n-1}'} = 0$$

$$-4nxg H_{n-1} - 2ng' H_{n-1} = 0$$

$$(2xg + g') H_{n-1} = 0 \quad \text{because}$$

$$g = e^{-x^2} \qquad g' = -2xg \qquad //$$

(c) Compute H_0, H_1, H_2, H_3, H_4

$$H_0 = 1 \qquad H_1 = 2x \quad H_2 = 4x^2 - 2$$

$$H_3 = 8x^3 - 12x \quad H_4 = 16x^4 - 48x^2 + 12$$

We note that they are all polynomials in x, because the quantity e^{x^2} cancels the e^{-x^2} in the derivatives
We also note that n even give even polynomial and n odd give odd polynomials
Finally the highest degree of polynomial is n.

(d) Show that

$$\int_{-\infty}^{\infty} H_n(x)\, H_m(x)\, w(x)\, dx = 0 \quad \text{if } n \neq m$$

(Note: $\int_{-\infty}^{\infty} e^{-x^2} H_n^2(x)\, dx = 2^n n! \sqrt{\pi}$, from Table of integrals)

$$H_n'' - 2x H_n' + 2n H_n = 0 \quad \text{or}$$

$$e^{x^2} \frac{d}{dx}\left[e^{-x^2} H_n'\right] + 2n H_n = 0 \qquad \times H_m$$

& subtract

$$e^{x^2} \frac{d}{dx}\left[e^{-x^2} H_m'\right] + 2m H_m = 0 \qquad \times H_n$$

$$2(m-n)\, e^{-x^2} H_n H_m = H_m \frac{d}{dx}\left[e^{-x^2} H_n'\right] - H_n \frac{d}{dx}\left[e^{-x^2} H_m'\right]$$

note: $H_i \frac{d}{dx}\left[e^{-x^2} H_j'\right] = \frac{d}{dx}\left[H_i e^{-x^2} H_j'\right] - e^{-x^2} H_i' H_j'$

$\therefore$ substituting & integrating from $-\infty$ to ∞:

$$2(m-n) \int_{-\infty}^{\infty} e^{-x^2} H_n H_m\, dx = \int_{-\infty}^{\infty} \frac{d}{dx}\left[e^{-x^2}\left(H_m H_n' - H_n H_m'\right)\right] dx = 0$$

$$\therefore \int_{-\infty}^{\infty} e^{-x^2} H_n H_m\, dx = 0, \text{ unless } n = m \qquad //$$

(e) if $f(x)$ is defined for $x \in (-\infty, \infty)$

$$f(x) = \sum_{n=0}^{\infty} c_n H_n$$

$$w(x) f(x) H_m(x) = \sum_{n=0}^{\infty} c_n w(x) H_n(x) H_m(x)$$

and integrate:

$$\int_{-\infty}^{\infty} w(x) f(x) H_m(x)\, dx = \sum_{n=0}^{\infty} c_n \int_{-\infty}^{\infty} w(x) H_n(x) H_m(x)\, dx$$

$$= c_m Q$$

$$\therefore \quad c_m = \frac{1}{Q} \int_{-\infty}^{\infty} w(x) f(x) H_m(x)\, dx \qquad m = 0, 1, 2 \ldots$$

$$Q = \int_{-\infty}^{\infty} w(x) H_m^2(x)\, dx = 2^m m! \sqrt{\pi}$$

(from table of integrals)