HW1 Sturm Liouville Problems

1) Use a table of integrals to show that

(a) $\varphi_n = \cos nx$ $\qquad \varphi_m = \cos mx$

(b) $\varphi_n = \sin nx$ $\qquad \varphi_m = \sin mx$

(c) $\varphi_n = \sin nx$ $\qquad \varphi_m = \cos nx$

are orthogonal on some interval $[-l, l]$ with respect to $w(x) = 1$. What is that l? What conditions need to be satisfied for m & n?

2) Consider $\varphi'' + \frac{\gamma}{x}\varphi' + \alpha\varphi = 0$

α, γ are constants. Can this ODE be converted to a Sturm Liouville problem? If so, show how.

3) Find the eigenfunctions & eigenvalues of the SL (Sturm-Liouville) problem

$$\begin{cases} y'' + \lambda^2 y = 0 & 0 < x < l \\ y'(0) = y(l) = 0 \end{cases}$$

4) Find the eigenvalues & eigenfunctions of the SL

$$\begin{cases} y'' + \lambda^2 y = 0 & 0 < x < 1 \\ y(0) + y'(0) = 0 \;\; \& \;\; y(1) = 0 \end{cases}$$

5) (Hermite Polynomials)

(a) Show that $y'' - 2xy' + 2ny = 0$ (#)

$n = 0, 1, 2 \ldots$

is a SL equation. Hint: multiply (#) by $w = e^{-x^2}$

(b) Show that (#) has solutions

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2} \quad n = 0, 1, 2 \ldots$$

(Hermite Polynomials)

Use these facts:
$$\begin{cases} \frac{d}{dx}\left[e^{-x^2} H_n\right] = -g H_{n+1} \\ \frac{d}{dx} H_n = 2n H_{n-1} \end{cases}$$
(both easy to show)

(c) Compute H_0, H_1, H_2, H_3, H_4. Why are they polynomials? What other 3 properties do you notice for these polynomials?

(d) Prove that H_0 and H_1 are orthogonal. Show that H_1 & H_2 are orthogonal.

(e) If $f(x)$ is integrable on $(-\infty, \infty)$, write the formula that finds c_m is

$$f(x) = \sum_{m=0}^{\infty} c_m H_m(x)$$