Math 482/582 –Midterm

Name: ______________________________

Score

1.	**(20)**	
2.	**(20)**	
3.	**(30)**	
4.	**(30)**	
Total		

1. (20 %) Let $f(x) = \exp(-x^2)$, for $0 \leq x < 1$, a portion of a Gaussian. Propose periodic extensions that lead to a Fourier series, valid in $0 \leq x < 1$ that have the following characteristics.

   - (A) with period 1, sine and cosine series.
   - (B) cosine series only, with period 2.
   - (C) sine terms only, with period 2.

   You do not need to compute a series, simply make drawings for each case. Be sure to label the period in each of the four figures.

   Answer:

   (A) $f = g$, for $0 \leq x < 1$ where $g = \exp(-x^2)$ and $f(x+1) = f(x)$, and all x.
   (B) $f = g$, for $-1 \leq x < 1$ where $g = \exp(-x^2)$ and $f(x+2) = f(x)$, and all x.
   (C) A possible but not unique example: $f = g$ for $0 \leq x < 1$ and $f = -g$ for $-1 \leq x < 0$ and $g = \exp(-x^2)$ and $f(x+2) = f(x)$, and all x.

2. (20 %) Solve for $u(x,t)$, obeying

$$u_t + 3u_x = 0, \quad t > 0, x \in \mathbb{R}, u(x,0) = e^{|x|}.$$

$dx/dt = 3$, hence $x = 3t + \xi$.

$du/dt = 0$ hence $u(x,t) = u(\xi, 0)$.

Solving for $\xi = x - 3t$.

$$u(x,t) = \exp[|x - 3t|], \quad t \geq 0, x \in \mathbb{R}.$$

3. (30 %) The function $u(x,t)$ obeys the following problem:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + qu, \quad t > 0, \quad 0 < x < L, \\ u(0,t) &= u_x(L,t) = 0, \\ u(x,0) &= f(x), \quad f(x) \text{ is a bounded continuous function.} \end{aligned}$$

Here q is a constant. Solve for $u(x,t)$ and find for what values of q the solution is stable, in the sense, the $\lim_{t\to\infty} u(x,t)$ is bounded.

Answer:

Assume separation of variables $u(x,t) = H(t)G(x)$. Then

$$\frac{H_t}{H} - \frac{G_{xx}}{G} - q = 0.$$

This leads to

$$H_t + \omega^2 H = 0,$$

and

$$G_{xx} + k^2 G = 0,$$

where $k^2 = \omega^2 + q$. The solution of the G equation with $G(0) = G'(L) = 0$ is

$$G_n(x) = A_n \phi_n(x), \quad n = 0, 1, 2, ...$$

where $\phi_n(x) = \sin[k_n x]$, where

$$k_n = (n + 1/2)\pi/L, \quad n = 0, 1, 2, ...$$

This means that $\omega^2 = [(n + 1/2)\pi/L]^2 + q$.
The solution of the H equation is

$$H(t) = C_n \exp(-\omega_n^2 t).$$

The solution

$$u(x,t) = \sum_{n=1}^{\infty} A_n \exp(-\omega_n^2 t)\phi_n(x).$$

where

$$\omega_n^2 = ([(n + 1/2)\pi/L]^2 + q),$$

and $k_n = (n + 1/2)\pi/L$. Using orthogonality and $u(x,0) = g(x)$, we find that

$$A_n = \frac{2}{L}\int_0^L g(x)\sin[k_n x]dx, \quad n = 1, 2, 3...$$

For solutions to remain constant in time or decay, we require $\omega_n^2 \geq 0$, hence $[(n + 1/2)\pi/L]^2 - q \geq 0$, so $(n = 0)$, $q \leq \pi^2/4L^2$.

4. (30 %) Solve for $u(x,t)$, obeying

$$\begin{aligned}
\frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u}{\partial x^2} + qu, \quad t > 0, \quad 0 < x < L, \\
u(0,t) &= u(L,t) = 0, \qquad t \geq 0 \\
u(x,0) &= 0, \\
u_t(x,0) &= g(x), \qquad 0 < x < L.
\end{aligned}$$

Extra Credit: Find the condition required on q for the solution to remain bounded.

Answer:

Assume separation of variables $u(x,t) = H(t)G(x)$. Then

$$\frac{H_{tt}}{H} - \frac{G_{xx}}{G} - q = 0.$$

This leads to

$$H_{tt} + \omega^2 H = 0,$$

and

$$G_{xx} + k^2 G = 0,$$

where $k^2 = \omega^2 + q$. The solution of the G equation with $G(0) = G(L) = 0$ is

$$G_n(x) = A_n \phi_n(x), \quad n = 1, 2, ...$$

where $\phi_n(x) = \sin[k_n x]$, where

$$k_n = n\pi/L, \quad n = 1, 2, ...$$

This means that $\omega^2 = [n\pi/L]^2 - q$.
The solution of the H equation is of the form

$$H(t) = C_n \cos(\omega_n t) + D_n \sin(\omega_n t).$$

Since $u(x,0) = 0$, then $C_n = 0$. The solution

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(\omega_n t) \phi_n(x).$$

where

$$\omega_n = \sqrt{([n\pi/L]^2 - q)},$$

and $k_n = n\pi/L$. Using orthogonality and $u_t(x,0) = g(x)$, we find that

$$A_n = \frac{2}{\omega_n L} \int_0^L g(x) \sin[k_n x] dx, \quad n = 1, 2, 3...$$

For solutions that oscillate or decay, we require $q \leq (\pi/L)^2$, *i.e.*, we require that ω_n remain real, for all n.