$$\star \begin{cases} \dfrac{\partial u}{\partial t} - \nu \dfrac{\partial^2 u}{\partial x^2} = x \\ \dfrac{\partial u}{\partial x}(0,t) = 0 \qquad \dfrac{\partial u}{\partial x}(L,t) = -L^2/2\nu \\ u(x,0) = \cos 3\dfrac{\pi x}{L} - \dfrac{L^2 x}{2\nu} \end{cases}$$

let $u(x,t) = U(x) + v(x,t)$

$-\nu U_{xx} = x \quad \therefore U_x = -\frac{1}{2} x^2/\nu$, satisfies B.C.

$U = -\frac{1}{6} x^3/\nu + U_0$

replace U into $u(x,t)$ and into ($\star$)

$$\begin{cases} \dfrac{\partial v}{\partial t} - \nu \dfrac{\partial^2 v}{\partial x^2} = 0 \\ \dfrac{\partial v}{\partial x}(0,t) = \dfrac{\partial v}{\partial x}(L,t) = 0 \\ v(x,0) = \cos 3\dfrac{\pi x}{L} \end{cases}$$

$v = \sum_{n=0}^{\infty} T_n(t)\, \varphi_n(x)$, where $\varphi_n'' + k_n^2 \varphi_n = 0$; $\varphi_x(0) = \varphi_x(L) = 0$

$\varphi_n(x) = \cos \dfrac{n\pi x}{L} \qquad n = 1, 2, \ldots$ ($n=0$ is also valid, but see later)

$$v(x,t) = \sum_{n=0}^{\infty} a_n e^{-\nu k_n^2 t} \varphi_n(x)$$

$$v(x,0) = \cos\frac{3\pi x}{L} = \sum_{n=0}^{\infty} a_n e^{-\nu k_n^2 t}\Big|_{t=0} \varphi_n(x)$$

or $$\cos\frac{3\pi x}{L} = \sum_{n=0}^{\infty} a_n \varphi_n(x)$$

using orthogonality $\quad a(n=3) = 1 \qquad a_n = 0 \;\; n \neq 3$

$$\therefore\; v(x,t) = \cos\frac{3\pi x}{L} e^{-\nu \frac{9\pi^2}{L^2} t}$$

$$u(x,t) = U(x) + v(x,t) = -\frac{1}{6} x^3/\nu + u_0 + \cos\frac{3\pi x}{L} e^{-\nu\frac{9\pi^2}{L^2} t}$$

but $\cancel{u_0 = 0}$ since $\frac{1}{L}\int_0^L u\, dx = 0$

$$\frac{1}{L}\int_0^L u\, dx = -\frac{1}{6L\nu}\int_0^L x^3\, dx + u_0 \frac{1}{L}\int_0^L dx = 0$$

$$\therefore\; u_0 = \frac{L^4}{24 L\nu} = \frac{L^3}{24\nu}$$

$$\therefore\; u(x,t) = \frac{L^3}{24\nu} - \frac{1}{6}\frac{x^3}{\nu} + \cos\frac{3\pi x}{L} e^{-\nu\frac{9\pi^2}{L^2} t}$$

substitute $u(x,t)$ into DE

$$\frac{\partial v}{\partial t} - \nu \frac{\partial^2 v}{\partial x^2} = 0$$

subst $u(x,t)$ into B.C. + I.C.

$$\frac{\partial v}{\partial x}(0,t) = \frac{\partial v(L,t)}{\partial x} = 0$$

$$v(x,0) = f(x) - U(x)$$

$$v(x,t) = \sum_{n=1}^{\infty} c_n e^{-\nu k_n^2 t} \phi_n(x) + c_0$$

$$\phi_n = \cos\frac{n\pi x}{L} \qquad n = 0, 1, 2 \ldots$$

but $c_0 = 0$ since $\int_0^L u\,dx = 0$

$$\therefore v(x,t) = \sum_{n=1}^{\infty} c_n e^{-\nu k_n^2 t} \phi_n(x)$$

$$v(x,0) = \sum_{n=1}^{\infty} c_n \phi_n(x) = f(x) - U(x)$$

$$v(x,0) = c_1\varphi_1(x) + c_2\varphi_2(x) + (1+c_3)\varphi_3(x) + \sum_{n=1}^{\infty} c_n\varphi_n(x)$$

$$= \cos\frac{3\pi x}{L} + \frac{(x-L)^3}{6\nu} - \frac{L^2 x}{2\nu}$$

Using orthogonality:

$$c_n = \frac{2}{L}\int_0^L \frac{(x-L)^3}{6\nu}\cos\frac{n\pi x}{L}\,dx$$

$$-\frac{2}{L}\int_0^L \frac{L^2 x}{2\nu}\cos\frac{n\pi x}{L}\,dx$$

$$c_n = -\frac{L^3}{\nu n^4\pi^4}\left[\pi^2 n^2(-1)^n + 2(-1)^n - 2\right]$$

$$\therefore\ u(x,t) = \frac{(x-L)^3}{6\nu} - \frac{L^2 x}{2\nu} + e^{-\frac{\nu 9\pi^2}{L^2}t}\cos\frac{3\pi x}{L}$$

$$-\frac{L^3}{\nu\pi^4}\sum_{n=1}^{\infty}\left[(\pi^2 n^2+2)(-1)^n - 2\right]e^{-\frac{\nu n^2\pi^2}{L^2}t}\cos\frac{n\pi x}{L}$$

//

(2) Solve

$$D_t^2 \qquad \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + g(x,t)$$

B.C. $u(0,t) = u(L,t) = 0$

I.C. $u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0$

Let $g(x,t) = \sum_{n=1}^{\infty} c_n(t)\,\varphi_n(x)$.

where $\varphi_n'' + k_n^2 \varphi_n = 0$

$\varphi_n(0) = \varphi_n(L) = 0$

or $\varphi_n = \sin\frac{n\pi x}{L}$ $\qquad k_n = \frac{n\pi}{L} \qquad n=1,2\ldots$

$$c_n(t) = \frac{2}{L}\int_0^L g(x,t)\,\varphi_n(x)\,dx$$

$$c_n(t) = \frac{2}{rL}\int_0^L Q(x)\,\varphi_n(x)\,dx \qquad 0 \le t < r$$

so

$$c_n(t) = 0 \qquad t \geq r$$

$n=1$: $C_n(t) = \begin{cases} \frac{1}{r} & 0 \le t \le r \\ 0 & t \ge r \end{cases}$

$$u(x,t) = \sum_{n=1}^{\infty} U_n(t)\,\varphi_n(x)$$, substitute in DE

$$\sum_{n=1}^{\infty} \frac{1}{c^2}\frac{d^2U_n}{dt^2}\varphi_n - \sum_{n=1}^{\infty}\frac{\partial^2\varphi_n}{\partial x^2}U_n = \sum_{n=1}^{\infty} C_n(t)\varphi_n$$

Using orthogonality and $\frac{\partial^2\varphi}{\partial x^2} = -k_n^2\varphi_n$

$$\frac{1}{c^2}\frac{d^2U_n}{dt^2} + k_n^2 U_n = C_n(t)$$

$$\frac{d^2U_n}{dt^2} + \omega_n^2 U_n = c^2 C_n(t)$$

$$\omega_n = k_n c$$

$U_n(t) = U_n^H(t) + U_n^P(t)$ homog + particular

but $U_n^H = 0$ since $u(x,0) = \frac{\partial u}{\partial t}(x,0) = 0$

hence $U_n(t) = U_n^p(t)$

assume $U_n^p = At^0$.

then $\omega_n^2 A = c^2 C_n$

$$A = \frac{C_n}{k_n}$$

but $C_n = G(t) = \begin{cases} \frac{1}{r} & 0 \le t \le r \\ 0 & t > r \end{cases}$

Look at $t < r$:

$$A = \frac{1}{k_1 r}, \text{ for } 0 \le t \le r$$

$$A = \frac{L}{\pi r} \quad \text{for } 0 \le t < r$$

$$u(x,t) = \frac{L}{\pi r} \sin \frac{\pi x}{L} \quad //$$

For $t \ge r$:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \qquad u(0,t) = u(L,t) = 0$$

$$u(x,r) = \frac{L}{\pi r} \sin \frac{\pi x}{L} \qquad \frac{\partial u}{\partial t}(x,r) = 0$$

for $t \geq r$

Reset time:

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \omega_n (t-r) + B_n \sin \omega_n (t-r) \right] \varphi_n(x)$$

$$u(x,r) = \frac{L}{\pi r} \sin \frac{\pi x}{L} \qquad \frac{\partial u}{\partial t}(x,r) = 0$$

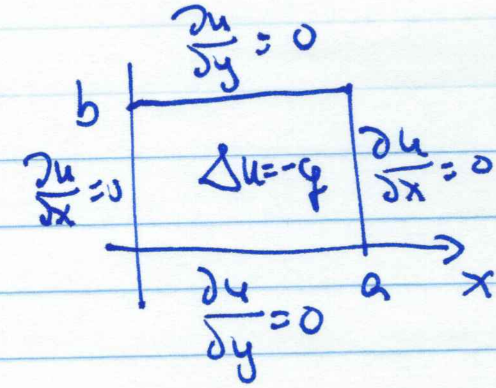
$$\therefore \; B_n = 0 \quad \text{and} \quad A_n = A_1 = \frac{L}{\pi r}$$

$$u(x,t) = \frac{L}{\pi r} \cos \frac{c\pi}{L}(t-r) \sin \frac{\pi x}{L} \quad , \quad t \geq r$$

(3) $$\Delta u = -q(x,y)$$

$$\frac{\partial u}{\partial x}(0,y) = \frac{\partial u}{\partial x}(a,y) = 0$$

$$\frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,b) = 0$$



$$q(x,y) = \sum_n \sum_m d_{nm} \varphi_{nm}(x,y)$$

$$u(x,y) = \sum_n \sum_m c_{nm} \varphi_{nm}(x,y)$$

where $\begin{cases} \Delta \varphi_{nm} + \lambda^2 \varphi_{nm} = 0 \\ \dfrac{\partial \varphi}{\partial n} = 0 \end{cases}$

$$\varphi_{nm} = A_n(x)\, B_m(y)$$

$$\frac{A_{xx}}{A} + \frac{B_{yy}}{B} = -\lambda^2 \Rightarrow \begin{cases} \cancel{A_{xx} + \lambda A = B_{yy}}\ 0 \\ \cancel{B_{yy} + \gamma B =} \end{cases}$$

$\cancel{A_{xx} + \lambda A = 0}$

$$\frac{A_{xx}}{A} + \lambda^2 = -\frac{B_{yy}}{B} = \gamma^2$$

$$\therefore \quad B_{yy} + \gamma^2 B = 0$$

$$A_{xx} + (\lambda^2 - \gamma^2) A = 0$$

$B = \alpha \cos \gamma y + \beta \sin \gamma y \qquad B' = -\alpha\gamma \sin \gamma y + \beta\gamma \cos \gamma y$

$B'(0) = B'(b) = 0 \qquad \Rightarrow \beta = 0$

$\sin \gamma b = 0 \quad \gamma = \frac{m\pi}{b} \quad m = 1, 2 \ldots$

$$B_m = \cos \frac{m\pi y}{b}$$

similarly $\qquad \lambda_{nm}^2 = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$

$A_n = \cos \frac{n\pi x}{a} \qquad n = 1, 2 \ldots$

$$u(x,y) = \sum_{n,m=1}^{\infty} C_{nm} \cos \frac{m\pi y}{b} \cos \frac{n\pi x}{a}$$

Subst into DE

$$\Delta \sum_{nm=1}^{\infty} c_{nm} \varphi_{nm} = -\sum_{nm=1}^{\infty} d_{nm} \varphi_{nm}$$

$$= -\sum_{nm=1}^{\infty} c_{nm} \lambda_{nm}^2 \varphi_{nm} = -\sum_{nm=1}^{\infty} d_{nm} \varphi_{nm}$$

$$\therefore \quad c_{nm} = \frac{d_{nm}}{\lambda_{nm}^2} \qquad n,m = 1,2\ldots$$

we note that $\int_0^a dx \int_0^b dy\, g(x,y) = 0$

otherwise we would have an $n=m=0$ contribution

which we could not resolve.

(4) Find bounded solution to

$$\begin{cases} \frac{\partial^2 u}{\partial x \partial y} + \Delta u = 0 & x \in \mathbb{R} \\ & 0 < y < \infty \\ u(x,0) = f(x) \end{cases}$$

FT

$$\begin{cases} u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{u}(k,y)\, e^{-ikx}\, dk \\ \hat{u}(k,y) = \int_{-\infty}^{\infty} u(x,y)\, e^{ikx}\, dx \end{cases}$$

$$\frac{\partial}{\partial y}\int_{-\infty}^{\infty} \frac{\partial u}{\partial x} e^{ikx}\, dx + \frac{\partial^2}{\partial y^2}\int_{-\infty}^{\infty} u e^{ikx}\, dx - k^2 \int_{-\infty}^{\infty} u e^{ikx}\, dx = 0$$

$$\begin{cases} ik\frac{\partial \hat{U}}{\partial y} + \frac{\partial^2 \hat{U}}{\partial y^2} - k^2 \hat{U} = 0 & -\infty < k < \infty \\ \hat{f}(k,0) = \hat{F}(k) \end{cases}$$

$$\hat{U} = \alpha e^{+\frac{k}{2}[\sqrt{3}+i]y} + \beta e^{-\frac{k}{2}[\sqrt{3}+i]y}$$

$-\infty < k < \infty$

For bounded solution

$$\hat{U} = \hat{F}(k)\, e^{\frac{ik}{2}y}\, e^{-\frac{\sqrt{3}}{2}|k|y}$$

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(k)\, e^{\frac{ik}{2}y}\, e^{-\frac{\sqrt{3}}{2}|k|y}\, e^{-ikx}\, dk$$

but $u(x,y)$ is real $\Rightarrow$ $\hat{U}(-k,y) = \overline{\hat{U}(k,y)}$

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{0} F(k)\, e^{\frac{iky}{2}}\, e^{-\frac{\sqrt{3}}{2}|k|y}\, e^{-ikx}\, dk$$

$$+\frac{1}{2\pi}\int_{0}^{\infty} \quad '' \qquad\qquad = I + II$$

~~let $z=-y$ $dz=-dy$~~ We transform I by letting $k=-k'$

$$u(x,y) = \frac{1}{2\pi}\int_{\infty}^{0} F(-k')\, e^{-\frac{ik'y}{2}}\, e^{-\frac{\sqrt{3}}{2}|k'|y}\, e^{ik'x}\, dk'$$

For a bounded solution

$$\hat{U} = \hat{F}(k)\, e^{\frac{iky}{2} - \frac{\sqrt{3}}{2}|k|y}$$

$$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(k)\, e^{\frac{iky}{2} - \frac{\sqrt{3}}{2}|k|y}\, e^{-ikx}\, dk$$

$$\hat{W}(k) = e^{\frac{iky}{2} - \frac{\sqrt{3}}{2}|k|y}$$

$$W(x,y) = \mathcal{F}^{-1}\left(\hat{W}(k)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\frac{ik}{2}y - \frac{\sqrt{3}}{2}|k|y}\, e^{-ikx}\, dk$$

$$\cancel{u(x,y) = \int_{-\infty}^{\infty} f(\xi)\, W(x-\xi, y)\, dx}$$

$$u(x,y) = \int_{-\infty}^{\infty} f(\xi)\, W(x-\xi, y)\, d\xi \quad \left\}\begin{array}{l} x \in \mathbb{R}^1 \\ y \geq 0 \end{array}\right.$$

we need to use 2 FT identities

$$\mathcal{F}^{-1}\left(e^{ikb}\hat{F}(k)\right) = f(x-b)$$

and $\mathcal{F}^{-1}\left(e^{-|k|y}\right) = \dfrac{1}{\pi}\,\dfrac{y}{x^2+y^2}$

here $b = \frac{1}{2}$ and $y \to \sqrt{3}y/2$

so $\mathcal{F}^{-1}(w(k,y)) = \dfrac{1}{\pi}\,\dfrac{\frac{\sqrt{3}y}{2}}{\frac{3}{2}y^2 + (x-\frac{1}{2}y)^2}$

$$\therefore\ u(x,y) = \frac{\sqrt{3}}{2\pi}\int_{-\infty}^{\infty} f(\xi)\, y \Big/ \left[\frac{3}{2}y^2 + \left(x - \frac{1}{2}y - \xi\right)^2\right] d\xi$$

//

$y \geq 0$

$x \in \mathbb{R}^1$

(5) Heat flow in a disk of radius R when the circumference of disk is insulated and initial temperature is

$$f(r) = \begin{cases} 1 & 0 \le r < R/2 \\ 0 & R/2 < r < R \end{cases}$$

$u = u(r,t)$

$$\frac{\partial u}{\partial t} = \nu \frac{1}{r} \partial (r u_r) = u_{rr} + \frac{1}{r} u_r$$

$u(r,t)$ bounded

$$\frac{\partial u}{\partial r}(R,t) = 0$$

$u = G(r)T(t)$

$$G T_t = \nu \left(\frac{1}{r} G_r + G_{rr} \right) T$$

$$\frac{T_t}{\nu T} = \frac{1}{G} \left(\frac{1}{r} G_r + G_{rr} \right) = -k^2$$

$$T = T(0) e^{-k^2 \nu t}$$

$$G_{rr} + \frac{1}{r} G_r + k^2 G = 0$$

$$r^2 G_{rr} + r G_r + r^2 k^2 G = 0$$

$$G = A J_0(kr) + B Y_0(kr)$$

G is bounded hence $B = 0$

$$G = A J_0(kr)$$

$$\frac{\partial G}{\partial r}(R) = 0 \quad \cdot \quad J_0'(kR) = 0$$

$$\therefore k_n = \frac{\alpha_n}{R} \quad n = 1, 2 \ldots$$

are roots of $J_0'(k_n r)$

$$\therefore \quad u(x,t) = \sum_{n=1}^{\infty} T_n e^{-k_n^2 \nu t} J_0(k_n r)$$

to find T_n use orthogonality:

$$u(r,0) = f(r) = \sum_{n=1}^{\infty} T_n J_0(k_n r)$$

$$T_n = \frac{\int_0^R r J_0(k_n r) f(r)\, dr}{\int_0^R r J_0^2(k_n r)\, dr}$$

$n = 1, 2 \ldots$

$$T_n = \frac{1}{Q_n} \int_0^{R/2} r J_0(k_n r)\, dr = \frac{1}{Q_n} r J_1(k_n r) \Big|_0^{R/2}$$

$$T_n = \frac{1}{Q_n} \frac{R}{2} J_1(k_n R/2)$$

$$Q_n = \cancel{\frac{1}{2} J_1^2(R)} \frac{R^2}{2} J_1^2(k_n R)$$

$$T_n = \frac{1}{R} J_1(k_n R/2) \big/ J_1^2(k_n R)$$

//