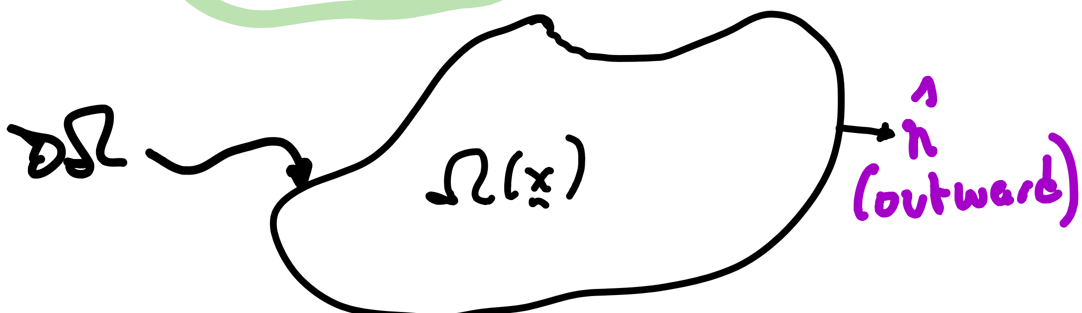
# GREEN'S FUNCTIONS, APPLIED TO NON HOMOGENEOUS ELLIPTIC PROBLEMS

(A) $$\mathcal{L}u = -\nabla\cdot(p\nabla u) + qu = \rho F \text{ in } \Omega$$

$$\left(\alpha u + \beta \frac{\partial u}{\partial n}\right)\bigg|_{\partial\Omega} = B$$

makes this problem non homogeneous

where $\boxed{\dfrac{\partial u}{\partial n} = \hat{n}\cdot\nabla u}$




$\rho F(\underset{\sim}{x})$ is known (inhomogeneous term in the PDE), so is $B$.

Form $\mathcal{L}v = -\nabla\cdot(p\nabla v) + qv = \rho F$ (B)

Multiply (A) by $v$ and (B) by $u$

subtract & integrate:

$$\int_{\Omega} \left(v\mathcal{L}u - u\mathcal{L}v\right) d\underset{\sim}{x}$$

$$= -\int_{\Omega} \nabla\cdot\big(\underbrace{pv\nabla u - pu\nabla v}_{\underset{\sim}{Q}}\big)\, d\underset{\sim}{x} = 0$$

the RHS of (A) & (B) are the same, so zero

Use Divergence Theorem, i.e.

$$\oint_{\Omega} \nabla\cdot\underset{\sim}{Q}\, d\underset{\sim}{x} = \int_{\partial\Omega} \underset{\sim}{Q}\cdot\hat{n}\, dS$$

here $\underset{\sim}{Q} = pv\nabla u - pu\nabla v$

So

$$= -\int_{\Omega} \nabla \cdot (p v \nabla u - p u \nabla v)\, d\underset{\sim}{x}$$

$$= -\int_{\partial\Omega} p(v\nabla u - u\nabla v)\cdot \hat{n}\, dS$$

since $\hat{n}\cdot\nabla w \equiv \frac{\partial w}{\partial n}$, then

$$= \int p\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS$$

we have data for $u, \frac{\partial u}{\partial n}, v, \frac{\partial v}{\partial n}$ at $\partial\Omega$.

$$\therefore \int_{\Omega} (v\mathcal{L}u - u\mathcal{L}v)\, d\underset{\sim}{x} = \int_{\partial\Omega} p\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS \quad (\dagger)$$

Now, suppose $v$ is the solution to

(D) $\mathcal{L} v = \delta(\underset{\sim}{x} - \underset{\sim}{\xi})$

$\delta(\underset{\sim}{x} - \underset{\sim}{\xi})$ is the Dirac Delta function, centered at $\underset{\sim}{\xi}$ inside $\Omega$.

Recall that

$$\begin{cases} \displaystyle\int_{\Omega} \delta(\underset{\sim}{x} - \underset{\sim}{\xi})\, d\underset{\sim}{x} = 1 \\ \displaystyle\int_{\Omega} h(\underset{\sim}{x})\, \delta(\underset{\sim}{x} - \underset{\sim}{\xi})\, d\underset{\sim}{x} = h(\underset{\sim}{\xi}) \end{cases}$$

"sifting property"

Use the sifting property:

(F) $$\int_\Omega u \mathcal{L} v \, d\underset{\sim}{x} = \int_\Omega u \, \delta(\underset{\sim}{x} - \underset{\sim}{\xi}) \, d\underset{\sim}{x} = u(\underset{\sim}{\xi})$$

using (D).

Also,

(G) $$\int_\Omega v \mathcal{L} u \, d\underset{\sim}{x} = \int_\Omega v \rho F \, d\underset{\sim}{x},$$

since $\mathcal{L}u = \rho F$, see (D).

Substitute (F) & (G) into (‡), into

$$\int_\Omega (v \mathcal{L} u - u \mathcal{L} v) \, d\underset{\sim}{x} = \int_{\partial\Omega} p \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) ds \qquad (‡)$$

to get

$$\int_{\Omega} v \rho F \, d\underset{\sim}{x} - u(\underset{\sim}{\xi}) = \int_{\partial\Omega} p \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS.$$

Solving for $u(\underset{\sim}{\xi})$, $\underset{\sim}{\xi} \in \Omega$,

$$u(\xi) = \int_{\Omega} v \rho F \, d\underset{\sim}{x} - \int_{\partial\Omega} p \left( u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \right) dS. \qquad (\star)$$

here $v$ is the solution to $\mathcal{L} v = \delta(\underset{\sim}{x} - \underset{\sim}{\xi})$

so, in principle we know it.

We will call $v$ the Green's function.

We know that $\mathcal{L} v = \delta(\underset{\sim}{x} - \underset{\sim}{\xi})$

But we need B.C. :

use similar boundary conditions as posed by the BVP, i.e. we'll impose

$$\alpha v + \beta \frac{\partial v}{\partial n}\Big|_{\partial\Omega} = 0$$

i.e. the homogeneous version of the original B.C.

$\therefore$ The boundary terms, i.e. the RHS of ☆

$$-\int_{\partial\Omega} p\left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) dS$$

$$= \int -p\,\frac{1}{\alpha}\left[\alpha u\frac{\partial v}{\partial n} - \underbrace{\alpha v}_{-\beta\frac{\partial v}{\partial n}}\frac{\partial u}{\partial n}\right] dS$$

factor $\partial v/\partial n$

$$= -\int p \frac{1}{\alpha}\frac{\partial v}{\partial n}\left(\alpha u + \beta \frac{\partial u}{\partial n}\right) dS$$

$$= -\int p \frac{1}{\alpha} B \frac{\partial v}{\partial n}\, dS = \int p \frac{1}{\beta} B v\, dS$$

In Summary, using (✱):

$$u(\underset{\sim}{\xi}) = \int_{\Omega} v \rho F \, d\underset{\sim}{x} + \int_{\partial\Omega} p \frac{1}{\beta} B v \, dS$$

So $v = G(x, \xi)$ is the Green's Function

it satisfies

$$\mathcal{L} G = \delta(x - \xi)$$

$$\left(\alpha G + \beta \frac{\partial G}{\partial n}\right|_{\partial\Omega} = 0$$

it solves the problem

$$\begin{cases} \mathcal{L}u = \rho F & \text{in } \Omega \\ \left(\alpha u + \beta \dfrac{\partial u}{\partial n}\right)\Big|_{\partial\Omega} = B \end{cases}$$

as

$$(\mathcal{Y}) \quad u(\xi) = \int \rho G F \, d\underset{\sim}{x} - \int \frac{p}{\alpha} B \frac{\partial G}{\partial n} ds + \int p \frac{B}{\beta} G \, ds$$

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Application: in the solution of the forced Helmholtz Eq

$$(\nabla^2 + k^2) u = \rho F$$

with B.C. as above.

In that case $(\mathcal{Y})$ is known as the

Helmholtz Integral solution

This is a very popular formulation in time harmonic problems in acoustics, electromagnetics, etc //

So let's summarize:

# GENERIC RECIPE FOR NON-HOMOGENEOUS LINEAR ODE's:

(A) $Lu(x) = f(x) \quad x \in \Omega = (a,b) \subseteq \mathbb{R}$

(B) $B_k u(x) = h_k(x) \quad x \in \partial\Omega = \{a,b\}$

$k = 1, 2, \ldots, n-1$

$L$ is a linear $n^{\text{th}}$ order differential operator.

(B) are the $n-1$ required boundary conditions involving linear combinations of $u$ and its derivatives, up to order $n-1$, evaluated on the boundary.

We form:

$$(G, Lu) \equiv \int_a^b G(x,y)\, L\, u(y)\, dy$$

Integrate by parts,

$$(*) \quad (G, Lu) = (L^* G, u) + \text{integration boundary terms}$$

$L^*$ is the adjoint to $L$

Rmk: if $L = L^*$ then we say that $L$ is self-adjoint.

If $\quad L^* G(x,y) = \delta(x-y)$

$\qquad\qquad + \text{terms involving b.c.}$

then $(*)$ yields

$$u = -(\text{integration terms}) + \int_a^b G(x,y)\, f(y)\, dy$$

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# GENERAL RECIPE FOR NONHOMOGENEOUS LINEAR PDE

PDE $\quad Lu(P) = f(P) \quad P \in \Omega \subseteq \mathbb{R}^n$

B.C. $\quad Bu_k(p) = h(p) \quad p \in \partial\Omega$

here $P = (x_1, x_2, \dots, x_n) \in \Omega \subseteq \mathbb{R}^n$

and $p$ is a point on the boundary $\partial\Omega$

Let $Q = (x_1, x_2, \dots, x_n)$ another point in $\Omega$.

We have

$$(G, Lu) = \int_{\Omega} G(P,Q)\, Lu(P)\, dP$$

we compute

$$(G, Lu) = (L^*G, u) + \text{integrated terms},$$

by integrating by parts. Then, solve

$$\begin{cases} L^* G(P,Q) = \delta(P,Q), \text{ the dirac delta function} \\ \text{with the homogeneous B.C.} \end{cases}$$

then

$$u(Q) = -\text{integrated terms} + \int G(P,Q) f(P)\, dP$$

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# ADJOINT OPERATORS

They came up in Differential Equations, i.e.

in $(v, Lu) = (L^* v, u)$

+ boundary terms.

but they come up in a variety of other contexts.

ex) Find $L^*$, if

$L = \dfrac{d}{dx}$ and boundary conditions

on $0 \le x \le 1$ $\quad v(0) - 2v(1) = 0$.

Integrate by parts...

$$(u, Lv) = \int_0^1 u \frac{\partial v}{\partial x}\, dx$$

$$= [uv]\Big|_0^1 - \int_0^1 v \frac{du}{dx}\, dx$$

$$= u(1)v(1) - u(0)v(0) - \int_0^1 v \frac{du}{dx}\, dx$$

Apply $v(0) = 2v(1)$:

$$= u(1)\ v(1) - u(0)\, 2v(1) - \int_0^1 v \frac{du}{dx}\, dx$$

$$= v(1)[u(1) - 2u(0)] - \int_0^1 v \frac{du}{dx}\, dx$$

$$= (L^* u, v)$$

$\therefore$ $L^*$ is $-\frac{d}{dx}$ with $u(1) = 2u(0)$ for boundary conditions.

Rmk: $L^* \neq L$, i.e. $L$ is not self adjoint. //

ex) Find $L^*$ for

$$L = e^x \frac{d^2}{dx^2} + e^x \frac{d}{dx} \qquad 0 \le x \le 1$$

and boundary conditions $u'(0) = 0$ and $u(1) = 0$.

$$(v, Lu) = \int_0^1 v \left[ e^x \frac{d^2}{dx^2} + e^x \frac{d}{dx} \right] u \, dx$$

Integrate by parts: to do that expediently

we rewrite

$$(v, Lu) = \int_0 v \, (e^x u')' \, dx$$

$$= \left[ v e^x u' \right]_0^1 - \left[ v' e^x u \right]_0^1 - \int_0^1 u \, (e^x v')' \, dx$$

$$= u'(1) v(1) e^1 + v'(0) u(0) - \int u \left[ e^x v'' + e^x v' \right] dx$$

$= (L^* v, u)$

with $L^* = L$ (selfadjoint)

and $v'(0)=0, v(1)=0$ //

Let's now examine in more detail the following ODE problem:

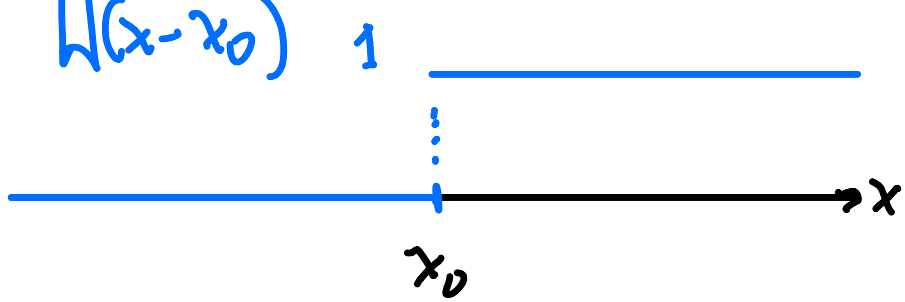
Solve

$$(*) \qquad \frac{d^2K}{dx^2} = \delta(x-\xi)$$

we require 2 boundary conditions, more later on this.

Introduce the Heaviside Function:

$H(x-x_0)$



$$H(x-x_0) = \begin{cases} 0 & \text{for } x < x_0 \\ 1 & \text{for } x \geq x_0 \end{cases}$$

The $H'(x) = \delta(x)$

Integrate (‡)

$$\frac{dK}{dx}(x,\xi) = H(x-\xi) + \alpha(\xi)$$

arbitrary

Integrate again

$$K(x,\xi) = \int_{-\infty}^{\infty} H(x-\xi)\,dx + x\alpha(\xi) + \beta(\xi)$$

arbitrary

$$\therefore (‡) \quad K(x,\xi) = (x-\xi)H(x-\xi) + x\alpha(\xi) + \beta(\xi)$$

K is a continuous, differentiable function.

Let's solve Poisson's Equation in 1D:

$$(\mathcal{F}) \quad \begin{cases} \dfrac{d^2u}{dx^2} = f(x) & 0 \le x \le 1 \\ u(0) = a & u(1) = b \end{cases}$$

We will use $(\mathcal{F})$. We need 2 boundary conditions to find $\alpha, \beta$.

A side Calculation:

Use sifting property of Dirac Delta:

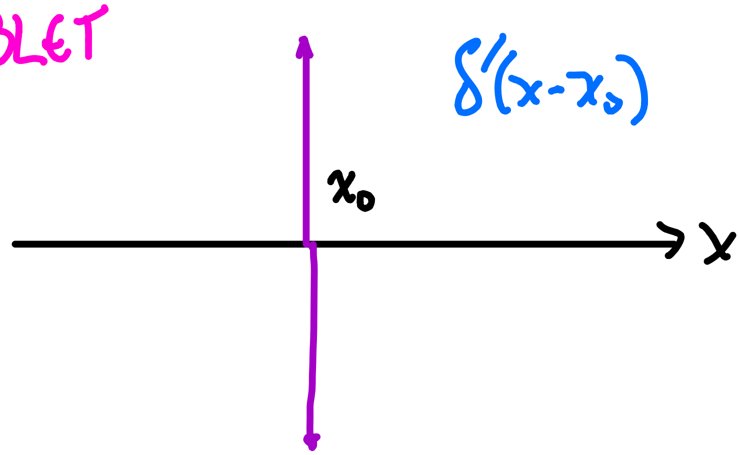
$$g'(0) = \int_{-\infty}^{\infty} g'(x)\,\delta(x)\,dx = -\int_{-\infty}^{\infty} g(x)\,\delta'(x)\,dx$$

by integrating by parts.

Also,

$$g''(0) = -\int_{-\infty}^{\infty} \delta'(x)\, g'(x)\, dx = \int \delta''(x)\, g(x)\, dx$$

DOUBLET




## Another Side Calculation

Suppose $Lu = f$, $(*)$

a linear differential equation. It could be a PDE or ODE, but let's assume an ODE and $u(x)$.

Formally, define $L^{-1}$, is an operator, s.t.

$L^{-1}L = I$, the identity. Hence

$L^{-1}Lu = u(x)$.

Using (*):

$u(x) = L^{-1}Lu = L^{-1}f$

and, from previous discussion

$$L^{-1}u(x) = \int K(x,\xi)\, u(\xi)\, d\xi$$

$$\therefore \quad u(x) = L\int K(x,\xi)\, u(\xi)\, d\xi$$

Since $L$ operates on $x$, we will put it in the integral:

$$u(x) = \int \underbrace{LK(x,\xi)}_{\text{write these}}\, u(\xi)\, d\xi$$

$LK(x,\xi) \equiv g(x,\xi)$

then

$$u(x) = \int g(x,\xi)\, u(\xi)\, d\xi .$$

If true for all $u(x)$ continuous, then

$g(x,\xi) = 0$ when $x \neq \xi$ and

when $x=\xi$ $\int g(x,\xi)u(\xi)d\xi = u(x)$ //

Back to $\begin{cases} u''(x) = f(x) & 0<x<1 \\ u(0)=a \quad u(1)=b \end{cases}$

formally, write (see previous calculation (✱))

$$Ku = -f(x) + b\,\delta'(x-1) - a\,\delta'(x) \quad (\bigstar)$$

i.e. $K = -\dfrac{d^2}{dx^2}$ s.t.

$$Kg = -\delta(x-\xi)$$

it was found that:

$$\therefore\ (\ddagger)\quad K(x,\xi) = (x-\xi)H(x-\xi) + x\alpha(\xi) + \beta(\xi) \quad //$$

Propose a solution to $(\pounds)$ of the form

$$u(x) = u_1(x) + u_2(x).$$

Focus on $u_1$: Compare $(\not{A})$ and $(\ddagger)$ to suggest

Let $g(x,\xi) \equiv (x-\xi)\underline{H(x-\xi)} - \alpha(1-\xi) \qquad (\$)$

for $0 < x, \xi < 1$,

where $\alpha$ is to be determined.

Multiply $(\$)$ by $f(\xi)$ and integrate

$$u_1(x) = \int_0^1 f(\xi)\, g(x,\xi)\, d\xi$$

$\leftarrow$ due to $H(x-\xi)$ in $g$

$$(\exists)\quad u_1(x) = \int_0^x (x-t) f(t)\, dt - x\alpha \int_0^1 (1-t) f(t)\, dt.$$

We can confirm that $u_1$ obeys $Ku_1 = -f$:

Multiply $u_1$ by $K$: $Ku_1 = K\int_0^x (x-t)f(t) - K x \int_0^1 (1-t)f(t)dt$

$$\therefore\ Ku_1 = K\int_0^1 f(t)\,g(x,t)\,dt$$
$$= \int_0^1 f(t)\,K g(x,t)\,dt = -\int_0^1 f(t)\,\delta(x-t)\,dt$$
$$= -f(t).$$

To complete the solution: we require $u_2$ s.t.

$$Ku_2 = b\,\delta'(x-1) - a\,\delta'(x)$$

The idea is to write the $\delta'$ above in terms of $K$: We know that $-\delta(x-t) = Kg$

(i) $\quad -\delta(x-t) = Kg$

and that

③ $g(x,t) = (x-t)\,H(x-t) - x(1-t)$

Take $\frac{d}{dt}$ Ⓐ: $\delta'(x-t) = K\frac{dg}{dt}$

where $\frac{dg}{dt} = -H(x-t) + x$,

obtained by $\frac{d}{dt}$ Ⓑ.

We can thus infer that

$$u_2 = -b\left.\frac{dg}{dt}\right|_{t=1} - a\left.\frac{dg}{dt}\right|_{t=0}$$

where $Ku_2 = b\,\delta'(x-1) - a\,\delta'(x-1)$.

Since $\left.\frac{dg}{dt}\right|_{t=1} = -\underbrace{H(x-1)}_{0 \text{ for } 0 \le x < 1} + x = x \quad \forall x \in [0,1]$

$$\left.\frac{dy}{dt}\right|_{t=0} = -H(x)+x = x-1 \quad \forall x\in[0,1]$$

$\underset{\text{"}-1\text{" for } 0<x\leq 1}{}$

$$\therefore \quad u_2(x) = bx - a(x-1).$$

Finally $\quad u = u_1 + u_2$

$$= \int_0^x (x-t)\, f(t)\, dt - x\int_0^1 (1-t)\, f(t)\, dt$$

$$+ bx - a(x-1)$$

Rmk: How did we know to construct a solution as a linear superposition of two subproblems? experience... it's a trick, but in linear problems you should always assume you can write the solution as a linear superposition of subproblems.

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