One-Way Wave Equation

(see Logan's Book)

This is a first-order (in space and time) PDE. We focus on its solution.

\[ \text{PDE} \quad U_t + c(x, t, u) U_x = 0 \quad t > 0 \]

\[ \text{I.C.} \quad U(x, 0) = f(x) \quad x \in \mathbb{R} \text{ (possibly } \mathbb{R}^1) \]

The simplest version:

Consider \( x \in \mathbb{R}^1 \) and \( c \), a given constant.

We can confirm that

\[ u(x, t) = f(x - ct) \]

is a solution to (CP), since

\[ U_t = -c f'(x - ct) \]

\[ U_x = f'(x - ct) \]
\[ f(x) = A \cos(kx) \quad A \text{ is constant} \]
\[ u(x, t) = A \cos \left[ k(x-ct) \right] \]

note that \( kc = wo \)

We call \( \omega \) the frequency
\( k \) the wave number
\( c \) the phase speed

Note also that \( \lambda = \frac{2\pi}{k} \)
\( \lambda \) is the wavelength

(\$) Describes a wave of constant shape, that propagates with constant speed \( c \).

If \( c > 0 \) wave propagates to the right;
If \( c < 0 \) wave propagates to the left.

A useful fact:
\[
\frac{du(x,t)}{dt} = \frac{\partial u(x(t),t)}{\partial x} \frac{dx}{dt} + \frac{\partial u}{\partial t}
\]

by PDE: \( c = \frac{dx}{dt} \) \therefore

\[
\frac{du}{dt} = U_x c + U_t = 0
\]

by PDE.

Consider

\[ CP \begin{cases} Ut + C(x,t) Ux = 0 & x \in \mathbb{R}^1 \ t > 0 \\ u(x,0) = \phi(x) \end{cases} \]

\[
\frac{du}{dt} = U_x c + U_t = 0
\]

\[ \therefore \frac{dx}{dt} = C(x,t) \]

This is the characteristic equation.
which generates a characteristic curve \( C \) (in space-time). Along \( C \):

\[
\frac{du}{dt} = u_x \frac{dx}{dt} + u_t = u_x(x,t) + u_t = 0
\]

Example (Ex)

\[
\begin{cases}
    u_t + 2t u_x = 0 & x \in \mathbb{R}^1, t > 0 \\
    u(x,0) = e^{-x^2} & x \in \mathbb{R}^1
\end{cases}
\]

Here \( C(x,t) = 2t \)

Using (1):

\[
\frac{du}{dt} = u_x 2t + u_t = 0
\]

(2)

\[
\frac{dx}{dt} = 2t \quad t > 0
\]

yields the characteristic \( C \):

Integrating (2):

\[
x = t^2 + k \quad k \text{ a constant}
\]
Nonlinear Waves

Consider the following system:

\[
\begin{cases}
U_t + C(U, x, t) U_x = 0 & x \in \mathbb{R}, \ t > 0 \\
U(x, 0) = \phi(x) & x \in \mathbb{R}
\end{cases}
\]

Here, \(c\) depends on \(u\) itself, i.e., nonlinear.

In what follows, assume that \(c(u, x, t) > 0\).
\[
\begin{align*}
\frac{du}{dt} &= 0 \text{ again, but} \\
\frac{dx}{dt} &= c(u_x,x,t) \quad (\star) \quad \text{for } E \\
\text{i.e. } \frac{du}{dt} &= \frac{\partial u}{\partial x} c(u(x,t),x,t) + \frac{\partial u}{\partial t} = 0 \\
\end{align*}
\]

\(u\) is constant along characteristics ad these are, since \(\frac{dx}{dt} = c\), note that
\[
\frac{d^2x}{dt^2} = \frac{d}{dt} \left( \frac{dx}{dt} \right) = \frac{d}{dt} c(u(x,t),x,t) = \frac{\partial c}{\partial u} \frac{du}{dt}=0
\]

So along characteristics the acceleration depends on the gradient of the speed \(c\) wrt \(u\) itself.

To find \(E\) through \((x,t)\) we note that
\[
\frac{dx}{dt} = c(u(x,0),x,t)
\]
\[ \frac{dx}{dt} = c(u(x,t), x,t) = c(\phi(t), x,t). \]

Applying (A) at \((5,0)\), after integrating we obtain

\[ x = c(\phi(t)) t + 5 \quad (†) \]

which defines \(5 = \xi(x,t)\) implicitly.

So \( u(x,t) = \phi(\xi) \)

where \(\xi\) is given by the solution to (†)

**Inviscid Burger's Equation**

\[ \begin{align*}
  \phi(x) & \quad \text{given by the solution to (†)} \\
  \phi(x) = u(x,0) = \begin{cases} 
  2 & x < 0 \\
  2-x & x \in [0,1] \\
  1 & x > 1 
\end{cases} 
\end{align*} \]
Since \( c(u) = u \), the characteristics are straight lines emanating from \((3,0)\) with speed \( c(\phi(t)) = \phi(t) \).

Since \( c > 0 \), the wave travels to the right, but the \( \psi < 0 \) travel at speed 2 and \( \psi > 1 \) travel at rate 1.

As we shall see, the characteristics will not cross in this example for \( 0 < t < 1 \).

We will say that to = 1 is the shock time, in our example. See below:
At \( t = 1 \) wave becomes multi-valued. This is called a shock.

The solution of the CP is no longer a traditional function, after the shock, we call it a weak solution.

To find the solution:

- For \( t < 1 \) \( u(x_1, t) = 2 \), for \( x < 2t \)
  \[ u(x_1, t) = 1 \], for \( x > t + 1 \)

- For \( 2t < x < t + 1 \)
  \[ x = c \phi(\delta) t + \frac{3}{2} \]

or \[ \delta = (2 - \frac{3}{2}) t + \frac{5}{2} \]
we can solve for \[ s = \frac{x-2t}{1-t} \]

\[ u(x,t) = \phi(s) \]

\[ u(x,t) = \frac{2-x}{1-t}, \quad 2t < x < t+1, \quad t < 1 \]

So the general solution to CP:

\[
\begin{cases}
    u(x,t) = 2, \text{ for } x < 2t \\
    u(x,t) = 1, \text{ for } x > t + 1 \\
    u(x,t) = \frac{2-x}{1-t}, \quad 2t < x < t+1, \quad t < 1
\end{cases}
\]
if $c'(u) > 0$ with $\phi(x) > 0, \phi'(x) < 0$

for sufficiently long time $t$, the solution becomes multi-valued, i.e. $u_x$ is unbounded.

To find $u_x$ differentiate

$$x = c(\phi(t)) t + \xi \text{ wrt } x,$$

$$1 = c'(\phi(t)) \phi'(t) t + 5_x$$

Solving for

$$5_x = \frac{1}{1 + c'(\phi) \phi' t}$$

Then for $u = \phi(t)$

$$u_x = \frac{\phi'(t)}{1 + c'(\phi) \phi' t}$$

So when the denominator $\to 0$ we get
\[
\begin{align*}
    t_0 &= \min \left\{ \frac{-1}{\phi'(\tau) c'(\phi(\tau))} \right\} \\
    \text{which will be meaningful, so long as } t_0 > 0 \text{ (the initial time)}.
\end{align*}
\]

In the example \( \phi(u) = u \), \( \phi'(3) = 2 - \frac{x}{3} \)

\[
\phi'(3) c'(\phi) = -1(1) = -1
\]

\[
\therefore \quad t_0 = 1 \quad \text{the shock time.}
\]

Thus for \( x \in \mathbb{R}^1 \), the CP. What happens when we have a finite domain, or semi-finite domain?

If \( c > 0 \)

\[
0 < u(x,t) = \phi(x)
\]

\[
1 > u(x,t) = \phi(x)
\]
Forced one-way wave equation

\[ a(x,t,u) u_x + b(x,t,u) U_t = f(x,t,u) \]

Space \( a, b, f \in C^1(D) \)

\( D \) is a space-time domain.

WLOG consider

\[ \begin{cases} 
    a(x,t,u) u_x + U_t = f(x,t,u) & t > 0 \\
    u(x,0) = g(x) 
\end{cases} \]

CP
As before

1. \[ \frac{dx}{dt} = a(x,t,u) \]

2. \[ \frac{du}{dt} = a u_x + u_t = f(x,t,u) \]

Rule: Before, \( f = 0 \).

Set \( x = \xi \) at \( t = 0 \), i.e.

\[ u(\xi,0) = g(\xi) \]

(1) and (2) is a system of differential equations with a solution that depends on 2 arbitrary constants.

Along characteristics

\[ x = F(t,c_1,c_2) \]

\[ u = G(t,c_1,c_2) \]

and the constants can be evaluated
by constraining solution to obey I.C.

\[ C_1 = c_1(3) \quad C_2 = c_2(3) \]

Then

\[ \begin{cases} x &= F(t; c_1(3), c_2(3)) \\ u &= G(t; c_1(3), c_2(3)) \end{cases} \]

And \( \mathcal{O} \equiv \mathcal{O}(x, t) \) via solution of \( \Box \).

Then, \( \mathcal{O} \) is substituted into \( \Box \)

obtaining final solution.

\( \square \)

ex)

Let’s try this method

on the following example.

This is a nonlinear wave with a "reaction" term proportional to \( u \)
\[
\begin{cases}
    u_t + u u_x + u = 0 & x \in \mathbb{R}^1, t > 0 \\
    u(x, 0) = -\frac{x}{2} & x \in \mathbb{R}^1
\end{cases}
\]

Along \( C \):
\[
\begin{align*}
    (\text{a}) & \quad \frac{dx}{dt} = u \\
    (\text{b}) & \quad \frac{du}{dt} = -u \quad \text{or} \quad \frac{du}{u} = -dt
\end{align*}
\]

**Solving (b):**

\[
u = c_1 e^{-t}. \quad \text{Since} \quad u(x, 0) = -\frac{x}{2}
\]

\( u(x, 0) = c_1 \) and \( u(3, 0) = -\frac{3}{2} \). So, at \( t = 0 \)

\[
u(3, 0) = c_1 = -\frac{3}{2} \implies c_1 = -\frac{3}{2}
\]

**Solving (a):**

\[
x = t e^{-t} + c_2 = -\frac{3}{2} e^{-t} + c_2
\]

Now, we find \( c_2 \):

\[
x(0) = 5 \implies -\frac{3}{2} + c_2 = 5 \implies c_2 = \frac{33}{2}
\]
\[ x = -\frac{3}{2}e^{-t} + \frac{3 \varepsilon}{2} = \frac{3}{2}(3 - e^{-t}) \]

Next, solve for \( \varepsilon = \varepsilon(x) \):

\[ \varepsilon = \frac{2x}{(3e^{-t})}. \] Since \( u = \frac{3}{2}e^{-t} \)

\[ \therefore u(x,t) = \frac{xe^{-t}}{3e^{-t}} = \frac{x}{3e^{t} - 1}. \] Blow up at

occurs when \( 3e^{t} - 1 = 0 \) or \( t_{0} = \log \left( \frac{1}{3} \right) \)

Exercise: try the above method to solve

\[ 2xu_{xx} + 2tu_{x} = u_{t} - x^{2}t^{2} \]

\[ u(x,0) = g(x) \]