One-Way Wave Equation

(see Logan's Book)

This is a first-order (in space and time) PDE. We focus on its solution.

Let c constant

$$CP \begin{cases} \text{PDE} & u_t + c(x,t,u)\, u_x = 0 \quad t>0 \\ & \qquad x \in \Omega \text{ (possibly } \mathbb{R}^1) \\ \text{I.C.} & u(x,0) = f(x) \quad , x \in \Omega \end{cases}$$

The simplest version:

Consider $x \in \mathbb{R}^1$ and c, a given constant.

We can confirm that

$$u(x,t) = f(x-ct)$$

is a solution to (CP), since

$$u_t = -c f'(x-ct)$$

$$u_x = f'(x-ct)$$

//

ex) $f(x) = A\cos(kx)$ $\quad$ A is constant

($) $\quad u(x,t) = A\cos[k(x-ct)]$

note that $kc \equiv \omega$

we call ω the frequency

k the wave number

c the phase speed

Note also that $\lambda \equiv \frac{2\pi}{k}$

λ is the wavelength

($) Describes a wave of constant shape, that propagates with constant speed c.

If $\begin{cases} c>0 & \text{wave propagates to the right,} \\ c<0 & \text{wave propagates to the left.} \end{cases}$

A useful fact:

$$\frac{du(x,t)}{dt} = \frac{\partial u}{\partial x}(x(t),t)\frac{dx}{dt} + \frac{\partial u}{\partial t}$$

by PDE: $c = \frac{dx}{dt}$ $\therefore$

$$\frac{du}{dt} = u_x c + u_t = 0$$

by PDE.

Consider

$$\text{C.P}\begin{cases} u_t + c(x,t)u_x = 0 & x \in \mathbb{R}^1 \quad t>0 \\ u(x,0) = \phi(x) \end{cases}$$

$$\frac{du}{dt} = u_x c + u_t = 0$$

$$\therefore \frac{dx}{dt} = c(x,t)$$

This is the characteristic equation

which generates a characteristic curve $\mathcal{C}$ (in space-time). Along $\mathcal{C}$:

$$(\star) \qquad \frac{du}{dt} = u_x \frac{dx}{dt} + u_t = u_x c(x,t) + u_t = 0$$

ex) CP

$$\begin{cases} u_t + 2t u_x = 0 & x \in \mathbb{R}^1, t > 0 \\ u(x,0) = e^{-x^2} & x \in \mathbb{R}^1 \end{cases}$$

Here $c(x,t) = 2t$

using $(\star)$

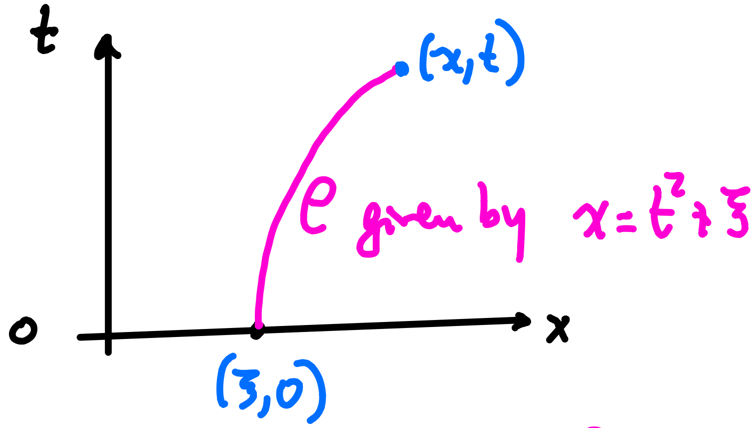
$$\frac{du}{dt} = u_x 2t + u_t = 0$$

$$(\ast) \qquad \frac{dx}{dt} = 2t \qquad t \geq 0$$

yields the characteristic $\mathcal{C}$:

Integrating $(\ast)$:

$x = t^2 + k \qquad k$ a constant //



Since u is constant along $\mathcal{C}$ curve then

$$u(x,t) = \exp(-\xi^2) = \exp(-(x-t)^2)$$

for $t \geq 0$ and $x \in \mathbb{R}^1$

Wave speeds up, at rate $2t$, but retains its shape.

Nonlinear Waves

$$CP \begin{cases} u_t + c(u,x,t)\,u_x = 0 & x \in \mathbb{R}^1 \quad t > 0 \\ u(x,0) = \phi(x) & x \in \mathbb{R}^1 \end{cases}$$

Here c depends on u itself $\therefore$ nonlinear

In what follows, assume that $c(u,x,t) > 0$

$$\begin{cases} \dfrac{du}{dt} = 0 & \text{again, but} \\ \dfrac{dx}{dt} = c(u,x,t) \quad (*) & \text{for } \mathcal{C} \end{cases}$$

i.e. $$\frac{du}{dt} = \frac{\partial u}{\partial x} c(u(x,t),x,t) + \frac{\partial u}{\partial t} = 0$$

u is constant along characteristics and these are, since $dx/dt = c$, note that

$$\frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d}{dt} c\big(u(x,t),x,t\big) = \frac{\partial c}{\partial u}\frac{du}{dt} = 0$$

So along characteristics the acceleration depends on the gradient of the speed c wrt u itself.

To find $\mathcal{C}$ through (x,t) we note that

$$\frac{dx}{dt} = c\big(u(\xi,0),x,t\big)$$

$$\frac{dx}{dt} = c\left(u(x,t), x, t\right) = c\left(\phi(\xi), x, t\right).$$

Applying $(\bigstar)$ at $(\xi, 0)$, after integrating we obtain

$$x = c\left(\phi(\xi)\right)t + \xi \qquad (\dagger)$$

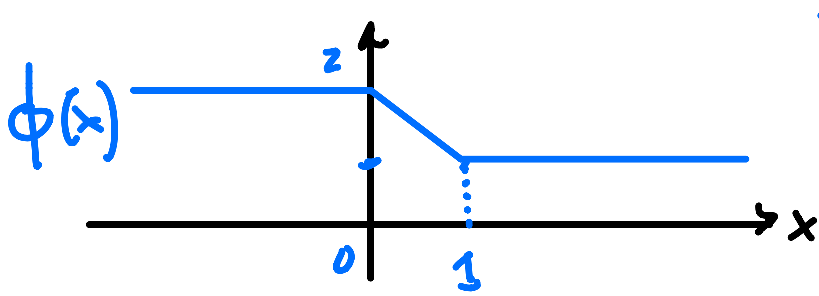
which defines $\xi = \xi(x,t)$ implicitly.

So $u(x,t) = \phi(\xi)$

where ξ is given by the solution to $(\dagger)$

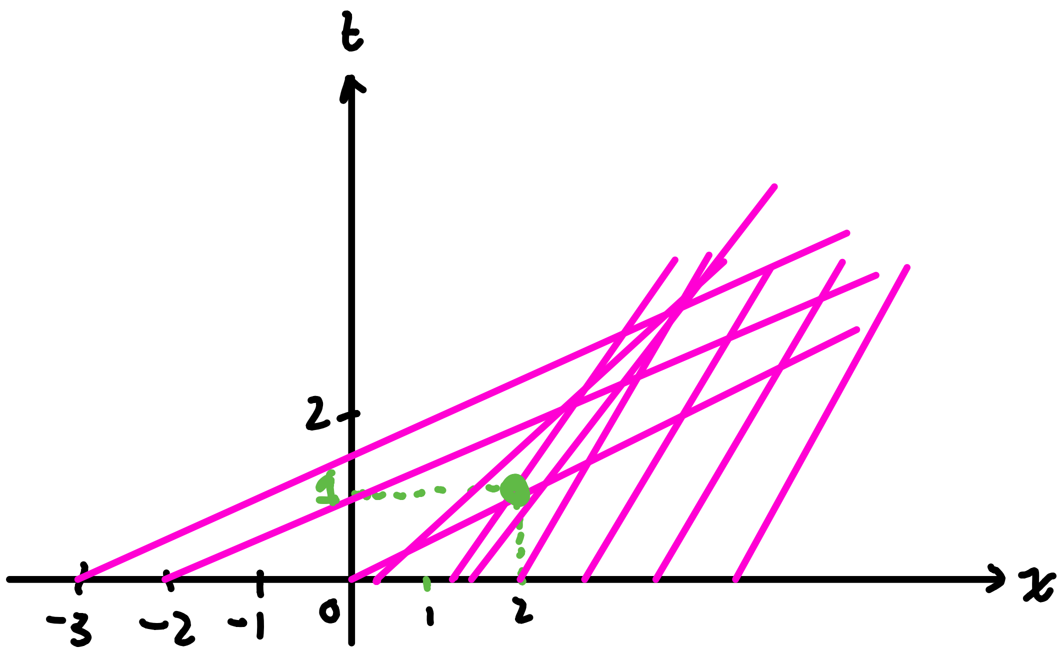
Inviscid Burger's Equation

ex) CP $\begin{cases} u_t + u u_x = 0 & x \in \mathbb{R}^1 \quad t > 0 \\ \phi(x) = u(x,0) = \begin{cases} 2 & x < 0 \\ 2-x & x \in [0,1] \\ 1 & x > 1 \end{cases} \end{cases}$

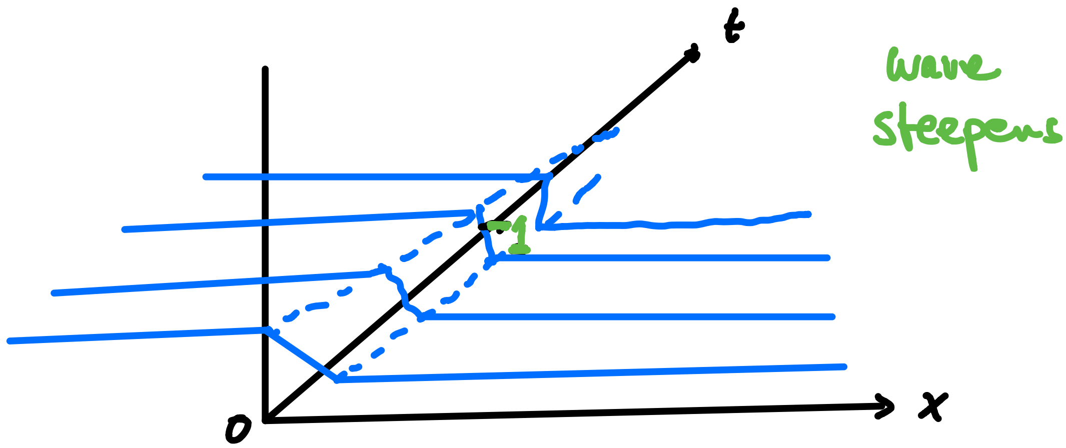


Since $c(u) = u$ the characteristics are straight lines emanating from $(\xi, 0)$ with speed $c(\phi(\xi)) = \phi(\xi)$

Since $c > 0$, the wave travels to the right, but the $\xi < 0$ travel at speed 2 and $\xi > 1$ travel at rate 1



As we shall see, the characteristics will not cross in this example for $0 < t < 1$. We will say that $t_0 = 1$ is the shock time, in our example. See below:



At $t=1$ wave becomes multi-valued.
This is called a shock

The solution of the CP is no longer a traditional function, after the shock, we call it a weak solution.

To find the solution

for $t<1$ $\quad u(x,t)=2$, for $x<2t$

$\quad u(x,t)=1$, for $x>t+1$

for $2t<x<t+1$

$$x = c(\phi(\xi))t+\xi$$

or $\quad \xi = (2-\xi)t+\xi$

we can solve for $\xi = \dfrac{x-2t}{1-t}$

$\therefore \quad u(x,t) = \phi(\xi)$ yields

$$u(x,t) = \frac{2-x}{1-t} \qquad 2t < x < t+1 \quad t<1$$

So the general solution to CP:

$$\begin{cases} u(x,t) = 2, & \text{for } x < 2t \\ u(x,t) = 1, & \text{for } x > t+1 \\ u(x,t) = \dfrac{2-x}{1-t} & 2t < x < t+1 \quad t<1 \end{cases}$$

How do we find the shock time?

For the case with $c'(u) > 0$

$$\begin{cases} u_t + c(u)u_x = 0 & t>0 \\ u(t,0) = \phi(x) & x \in \mathbb{R}^1 \end{cases}$$

The shock (or blow up) time t_0 is found as follows:

if $c'(u) > 0$ with $\phi(x) \geq 0$ $\phi'(x) < 0$
for sufficiently long time t, the solution becomes multivalued; i.e. u_x is unbounded.

To find u_x differentiate

$$x = c(\phi(\xi))t + \xi \quad \text{wrt } x:$$

$$1 = c'(\phi(\xi))\phi'(\xi)t + \xi_x$$

Solving for

$$\xi_x = \frac{1}{1 + c'(\phi)\phi'(\xi)t}$$

then for $u = \phi(\xi)$

$$u_x = \frac{\phi'(\xi)}{1 + c'(\phi)\phi' t}$$

So when the denominator $\to 0$ we get

$$\left\{ t_0 = \min_{\xi} \frac{-1}{\phi'(\xi)\, c'(\phi(\xi))} \right.$$

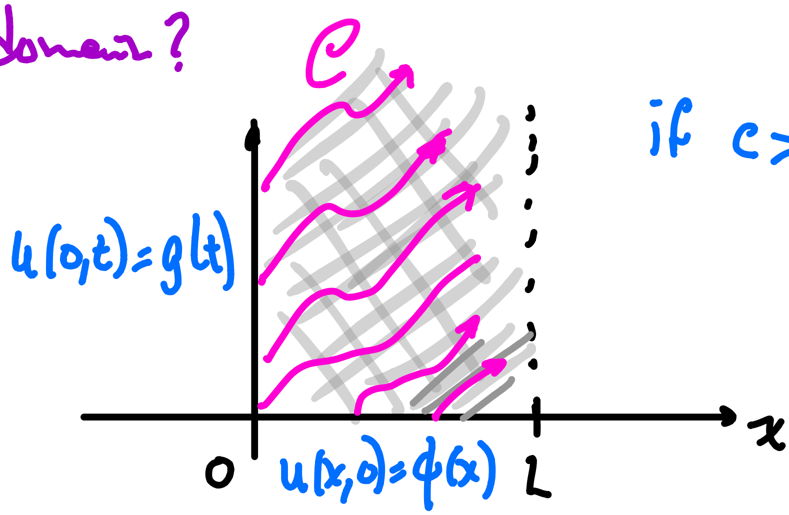
which will be meaningful, so long as $t_0 > 0$ (the initial time).

In the example $c(u) = u$, $\phi(\xi) = 2 - \xi$

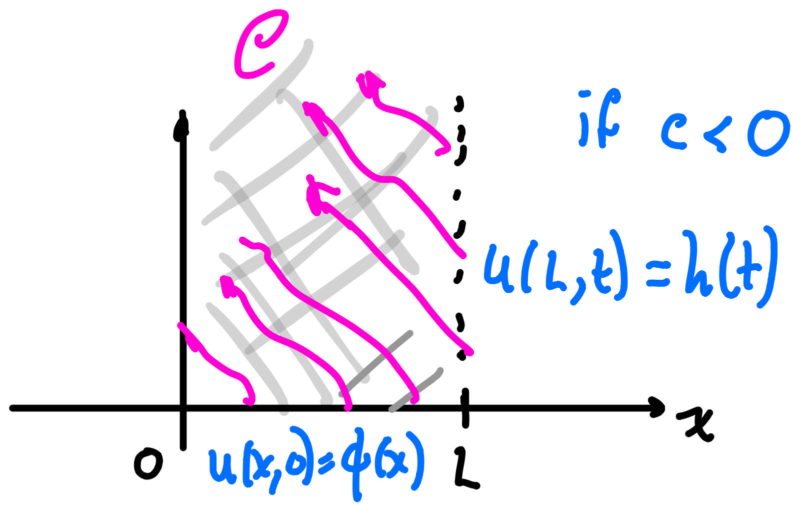
$$\phi'(\xi)\, c'(\overset{=u}{\phi}) = -1\,(1) = -1$$

$\therefore$ $t_0 = 1$ the shock time. //

Thus for $x \in \mathbb{R}^1$, the CP. What happens when we have a finite domain, or semi-infinite domain?



if $c > 0$



Forced one-way wave equation

$$a(x,t,u)\,u_x + b(x,t,u)\,u_t = f(x,t,u)$$

S'pse $a, b, f \in C^1(D)$

D is a space-time domain.

WLOG consider

$$\text{CP} \begin{cases} a(x,t,u)\,u_x + u_t = f(x,t,u) & \begin{cases} t>0 \\ x\in\mathbb{R}^1 \end{cases} \\ u(x,0) = g(x) \end{cases}$$

As before

(A) $$\frac{dx}{dt} = a(x,t,u)$$

(B) $$\frac{du}{dt} = a u_x + u_t = f(x,t,u)$$

Note: before, $f=0$.

Set $x = \xi$ at $t=0$ $\therefore$

$$u(\xi,0) = g(\xi).$$

(A) and (B) is a system of differential equations with a solution that depends on 2 arbitrary constants.

Along characteristics

$$x = F(t, c_1, c_2)$$

$$u = G(t, c_1, c_2)$$

and the constants can be evaluated

by constraining solution to obey I.C.

$$c_1 = c_1(\xi) \qquad c_2 = c_2(\xi)$$

then

$$(\ast) \begin{cases} x = F(t;\, c_1(\xi), c_2(\xi)) \\ u = G(t;\, c_1(\xi), c_2(\xi)) \end{cases}$$

and (C) $\xi = \xi(x,t)$ via solution of (A).

Then, (C) is substituted into $(\ast)$

obtaining final solution. //

ex) Let's try this method

on the following example.

This is a nonlinear wave with a

"reaction" term proportional to u:

$$\begin{cases} u_t + u u_x + u = 0 & x \in \mathbb{R}^1 \quad t > 0 \\ u(x,0) = -\frac{x}{2} & x \in \mathbb{R}^1 \end{cases}$$

Along $\mathcal{C}$:

(£) $\frac{dx}{dt} = u$

$(\$)$ $\frac{du}{dt} = -u$ or $\frac{du}{u} = -dt$

Solving $(\$)$:

$u = c_1 e^{-t}$. Since $u(x,0) = -x/2$

$u(x,0) = c_1$ and $u(\xi,0) = -\xi/2$. So, at $t=0$

$u(\xi,0) = c_1 = -\xi/2$ $\therefore$

Solving (£): $x = \underbrace{+c_1 e^{-t}}_{u} + c_2 = -\frac{\xi}{2} e^{-t} + c_2$

Now, we find c_2:

$x(0) = \xi$ $\therefore$ $-\frac{\xi}{2} + c_2 = \xi \Rightarrow c_2 = 3\xi/2$

$$x = -\frac{\xi}{2}e^{-t} + \frac{3\xi}{2} = \frac{\xi}{2}\left(3 - e^{-t}\right)$$

Next, solve for $\xi = \xi(x)$:

$\xi = 2x/(3 - e^{-t})$. Since $u = -\frac{\xi}{2}e^{-t}$

$\therefore\ u(x,t) = \dfrac{xe^{-t}}{3-e^{-t}} = \dfrac{x}{3e^{t}-1}$. Blow up t_0 occurs when $3e^{t_0} - 1 = 0$ or $t_0 = \log(1/3)$ //

exercise: try the above method to solve

$$2xuu_x + 2tuu_t = u^2 - x^2 - t^2$$
$$u(x,0) = g(x)$$

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