THE WAVE EQUATION (HYPERBOLIC PROBLEM)

Consider the following model for waves on a string of finite length:

PDE $\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$ $\quad 0 < x < \ell$, $t > 0$

c is a constant.

u represents the displacement of the string.

B.C. $u(0,t) = 0$, $u(\ell,t) = 0$ Dirichlet B.C.

(clamped string at $x=0$, $x=\ell$)

I.C.
$$\begin{cases} u(x,0) = g(x) & \text{(initial displacement)} \\ & 0 < x < \ell \\ \frac{\partial u}{\partial t}(x,0) = f(x) & \text{(initial velocity)} \end{cases}$$

Later on we'll use the following specific I.C.:

$$(\dagger) \qquad g(x) = \begin{cases} \frac{2b}{\ell} x & 0 \le x \le \ell/2 \\ \frac{2b}{\ell}(\ell - x) & \frac{\ell}{2} < x \le \ell \end{cases}$$

and

$$f(x) = 0$$

i.e., we specify an initial displacement and a zero initial velocity.

We will try SEPARATION OF VARIABLES:

Assume $u = H(t)\,F(x)$ and

substitute into PDE and B.C., we obtain from

PDE: $$\frac{1}{c^2}\frac{H_{tt}}{H} = \frac{F_{xx}}{F} = -k^2$$

let $\omega^2 \equiv c^2 k^2$ where

ω the angular frequency $= 2\pi\nu$

ν is the frequency,

k is the wavenumber $k = \frac{2\pi}{\lambda}$

λ is the wavelength

$\omega = 2\pi\nu$

$$c = \frac{\omega}{k} = \lambda\nu$$

$k = \frac{2\pi}{\lambda}$

c is the phase speed.

Separation of variables applied to B.C.:

B.C. $F(x=0)\,G(t) = 0 \Rightarrow F(0) = 0$

$F(x=\ell)\,G(t) = 0 \Rightarrow F(\ell) = 0$

So the 2 problems to solve are:

$$H_{tt} + \omega^2 H = 0$$ (A) an initial value problem

$$F_{xx} + k^2 F = 0$$ (B) a boundary value problem

Solving (B):

$$F = A\cos kx + B\sin kx$$

$$F(0) = F(l) = 0 \quad \therefore$$

$$F = B_n \sin\frac{n\pi x}{l} = B_n \varphi_n(x)$$

$$k_n = \frac{n\pi}{l} \quad n = 1, 2 \ldots$$

$$\therefore \quad \omega_n = c\, k_n \quad n = 1, 2 \ldots$$

Solving (A): $H_{tt} + \omega_n^2 H = 0$

$$H_n(t) = C_n \cos\omega_n t + D_n \sin\omega_n t$$

The full solution is thus:

$$\therefore u(x,t) = \sum_{n=1}^{\infty} \left[C_n \cos\omega_n t + D_n \sin\omega_n t\right] \varphi_n(x).$$

Apply I.C. to find C_n and D_n. We'll need

$$u_t(x,t) = \sum_{n=1}^{\infty} \left[-\omega_n C_n \sin \omega_n t + \omega_n D_n \cos \omega_n t \right] \varphi_n(x).$$

At $t=0$,

$$u(x,0) = g(x) = \sum_{n=1}^{\infty} C_n \varphi_n(x)$$

$$u_t(x,0) = f(x) = \sum_{n=1}^{\infty} \omega_n D_n \varphi_n(x).$$

We use orthogonality of $\{\varphi_n(x)\}$:

$$C_n = \frac{2}{\ell} \int_0^{\ell} g(x) \varphi_n(x) \, dx$$

$$D_n = \frac{2}{\ell \omega_n} \int_0^{\ell} f(x) \varphi_n(x) \, dx$$

In our specific problem: since $f(x) = 0$,

$\therefore D_n = 0.$

Using the specific $g(x)$ given by $(\ddagger)$, we get

$$C_n = \frac{2}{\ell}\int_0^\ell g(x)\,\varphi_n(x)\,dx = \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2} \qquad n=1,2,\dots$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{8b}{n^2\pi^2}\sin\frac{n\pi}{2}\cos\omega_n t\,\sin k_n x$$

WAVES IN MORE THAN 1 SPACE DIMENSION:

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \nabla^2 u \qquad \text{Wave Equation}$$

In 2D (x,y) $\quad \nabla^2 = \nabla\cdot\nabla = \partial_{xx} + \partial_{yy}$

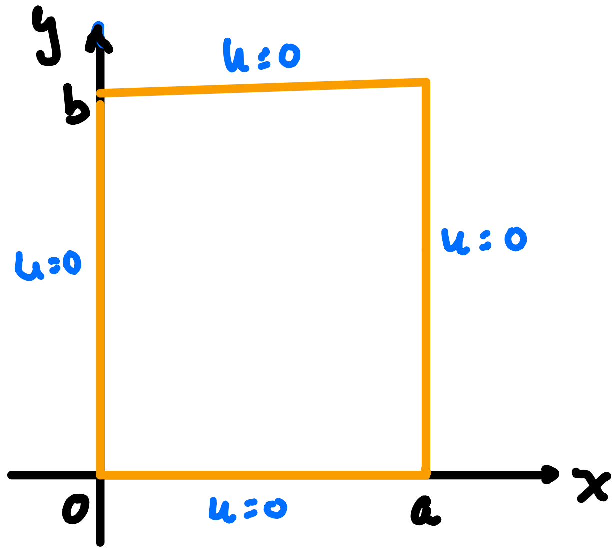
In 3D (x,y,z) $\quad \nabla^2 = \nabla\cdot\nabla = \partial_{xx} + \partial_{yy} + \partial_{zz}$

WAVES ON A CLAMPED MEMBRANE

PDE $\quad \frac{1}{c^2}u_{tt} = u_{xx} + u_{yy}, \qquad \begin{cases} 0<x<a \\ 0<y<b \\ t>0 \end{cases}$

B.C. $\begin{cases} u(0,y,t) = 0 \\ u(a,y,t) = 0 \\ u(x,0,t) = 0 \\ u(x,b,t) = 0 \end{cases}$ Clamped (Dirichlet B.C.)

$$\text{I.C.}\begin{cases} u(x,y,0) = f(x,y) \\ u_t(x,y,0) = g(x,y) \end{cases}$$



Proceed with separation of Variables

$$u(x,y,t) = H(t)\,G(x,y)$$

Substitute in PDE + B.C.

$$\frac{1}{c^2} H_{tt} G = H(G_{xx} + G_{yy})$$

$$\frac{1}{c^2}\frac{H_{tt}}{H} = \frac{G_{xx}+G_{yy}}{G} = -k^2$$

let $c = \omega/k$ (★)

or
$$\begin{cases} H_{tt} + \omega^2 H = 0 & \text{(A)} \\ G_{yy} + G_{xx} + k^2 G = 0 & \text{(B)} \end{cases}$$

Solution to the H equation (A):

$$H(t) = A\cos\omega t + B\sin\omega t$$

The (B) equation is also solved by separation of variables:

$$G = M(x)\,N(y)\text{, substitute in (B):}$$

$$M N_{yy} + M_{xx} N + k^2 M N = 0$$

$$\frac{N_{yy}}{N} + \frac{M_{xx}}{M} + k^2 = 0$$

$$\frac{N_{yy}}{N} = -k^2 - \frac{M_{xx}}{M}$$ (separated solution)

$$\frac{N_{yy}}{N} = -l^2 = -k^2 - \frac{M_{xx}}{M}$$

(C) $$N_{yy} + l^2 N = 0$$

$$-\frac{M_{xx}}{M} - k^2 + l^2 = 0$$

or $$M_{xx} + (k^2 - l^2)M = 0$$

let $$\lambda^2 = k^2 - l^2 \quad (\neq)$$

(D) $$M_{xx} + \lambda^2 M = 0$$

SL in the x & y equations, (C) and (D)

$$G(x,y) \propto \left\{ \begin{matrix} \sin \lambda x \\ \cos \lambda x \end{matrix} \right\} \left\{ \begin{matrix} \sin l y \\ \cos l y \end{matrix} \right\}$$

Assembling H & G solutions

we get u:

$u(x,y,t)$ will be a linear combination of sin & cos in x, y, t. However, we apply the B.C. to find the specific e'functions & e'values.

For this specific example have clamped B.C. at the edges:

$$\begin{cases} M_{xx} + k^2 M = 0 \\ M(0) = M(a) = 0 \quad \text{Dirichlet} \end{cases}$$

$$\begin{cases} N_{yy} + l^2 N = 0 \\ N(0) = N(b) = 0 \quad \text{Dirichlet} \end{cases}$$

$$M_n = \sin \frac{n\pi x}{a} \qquad n = 1, 2, \ldots$$

$$N_m = \sin \frac{m\pi y}{b} \qquad m = 1, 2, \ldots$$

$$k_n^2 = \frac{n^2\pi^2}{a^2} \qquad n = 1, 2 \ldots$$

$$l_m^2 = \frac{m^2\pi^2}{b^2} \qquad m = 1, 2 \ldots$$

by $(*)$ $\quad k^2 = k^2 - l^2$

$$\therefore \ k^2 = k^2 + l^2 = \frac{n^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}$$

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[A_{nm}\cos\omega_{nm}t + B_{nm}\sin\omega_{nm}t\right] \cdot \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$$

To find A_{nm} & B_{nm} use orthogonality:

Use I.C.

$$f(x,y) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{nm}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}$$

$$\therefore \quad A_{nm} = \frac{4}{ab}\int_0^a dx \int_0^b dy\, f(x,y) \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}$$

use $u_t(x,y,0)$ to find B_{nm}. For this we need

$$u_t(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(-\omega_{nm} A_{nm} \sin\omega_{nm} t + \omega_{nm} B_{nm} \cos\omega_{nm} t\right)$$

$$\cdot \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}$$

$u_t(x,y,0) = g(x,y)$

Using orthogonality:

$$B_{nm} = \frac{1}{\omega_{nm}} \frac{4}{ab}\int_0^a dx \int_0^b dy\, g(x,y) \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}$$

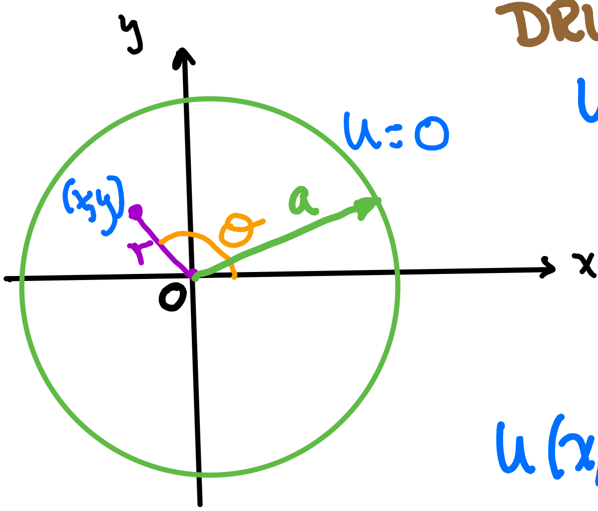
$$\omega_{nm} = c\, k_{nm} = c\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$$

$n = 1, 2 \ldots$

$m = 1, 2 \ldots$

//

WAVE EQUATION ON A CLAMPED CIRCULAR DRUMHEAD



u is periodic in θ and bounded everywhere, including at the origin

$$u(x,y,t) = u(r,\theta,t)$$

$$x = r\cos\theta \qquad \tan\theta = y/x \qquad r = \sqrt{x^2+y^2}$$
$$y = r\sin\theta$$

In this specific instance we'll assume that $u(r,\theta,0)$ is not zero, but $u_t(r,\theta,0)=0$ i.e. zero initial velocity:

In Polar Coordinates, a separation of variables is possible:

PDE $$\Delta u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \qquad \begin{cases} t>0 \\ 0\le r<a \\ 0\le\theta<2\pi \end{cases}$$

where $$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$

B.C. $\begin{cases} u(a,\theta,t) = 0 & \text{Dirichlet B.C.} \\ u \text{ is bounded at } r=0 \\ u(r,\theta,t) = u(r,\theta+2n\pi,t) & \text{Periodic} \end{cases}$

I.C. $\begin{cases} u(r,\theta,0) = f(r) \\ u_t(r,\theta,0) = 0 \end{cases}$

For this specific problem there is no θ-dependency:

$$\therefore \quad u(r,\theta,t) = u(r,t)$$

$$\Delta u = \frac{1}{c^2} u_{tt} \quad \text{becomes}$$

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{c^2} u_{tt}$$

Seperation of variables:

$$u(r,t) = H(t)R(r)$$

Substitute into PDE + B.C.

$$H R_{rr} + \frac{1}{r} R_r H = \frac{1}{c^2} H_{tt} R$$

$$\frac{R_{rr} + \frac{1}{r} R_r}{R} = \frac{1}{c^2} \frac{H_{tt}}{H} = -k^2$$

$$c = \frac{\omega}{k}$$

$$\begin{cases} H_{tt} + \omega^2 H = 0 \quad (A) \therefore \\ H(t) = A\cos\omega t + B\sin\omega t \\ r^2 R_{rr} + r R_r + k^2 r^2 R = 0 \quad (B) \end{cases}$$

The solution to (B):

$$R = C J_0(kr) + D Y_0(kr)$$

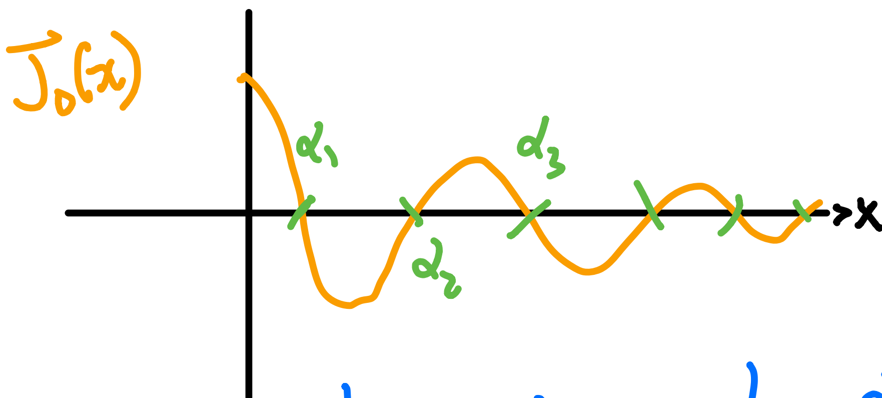
but u is bounded (at origin)

$\therefore \quad D = 0.$

$$R = C\,J_0(kr)$$

$R(a) = 0$ Results from application of B.C. on u at $r = a$.

$$J_0(ka) = 0$$



$$k_n a = \alpha_n \quad \therefore \; k_n = \frac{\alpha_n}{a}$$

α_n are the roots of $J(x)$

$$u(r,t) = \sum_{n=1}^{\infty} \left(\underbrace{A_n \cos \omega_n t + B_n \sin \omega_n t} \right) J_0(k_n r)$$

Comes from the H equation solution.

$u_t(r,0) = 0$ by I.C.

$\therefore B_n = 0 \Rightarrow$

$$u(r,t) = \sum_{n=1}^{\infty} A_n \cos\omega_n t \, J_0(k_n r)$$

Apply the other I.C. to determine the A_n's:

$$u(r,0) = f(r) = \sum_{n=1}^{\infty} A_n J_0(k_n r)$$

The $J_0(k_n r)$ are orthogonal wrt r:

$$\int_0^a r f(r) J_0(k_m r)\, dr$$

$$= \sum_{n=1}^{\infty} A_n \underbrace{\int_0^a J_0(k_m r) J_0(k_n r)\, r\, dr}_{Z_n}$$

Solving for A_n (using orthogonality):

$$A_n = \frac{1}{Z_n} \int_0^a r f(r) J_0(k_n r)\, dr, \quad n = 1, 2, \ldots$$

$$Z_n = \frac{a^2 J_1^2(\alpha_n)}{2}$$ This is a known outcome:

$$\int_0^a r J_0^2(k_n r)\, dr = \frac{J_1^2(\alpha_n) a^2}{2} \quad n = 1, 2, \ldots$$

(see Bessel function identities in a table of functions)

TIME HARMONIC SOLUTION TO THE WAVE EQUATION

Recall $$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \Delta u \quad (*)$$

with $\underset{\sim}{r} = (x, y, z)$ position and time t

can be seperated

$$u(\underset{\sim}{r}, t) = H(t) F(\underset{\sim}{r})$$

substitute into (*):

$$\frac{1}{c^2} \frac{H_{tt}}{H} = \frac{\Delta F}{F} = -k^2$$

let $\omega^2 = k^2 c^2$

$$\begin{cases} H_{tt} + \omega^2 H = 0 \\ \Delta F + k^2 F = 0 \end{cases}$$

the $H(t)$ equation involves $\cos\omega t, \sin\omega t$

"time harmonic"

we note that $\frac{\partial}{\partial t} \cos\omega t = -\omega \sin\omega t$

$$\frac{\partial^2}{\partial t^2} = -\omega^2 \cos\omega t$$

$\therefore$ if we assume $u(\underset{\sim}{r}, t) = g(\underset{\sim}{r}) \cos\omega t$

and substitute this into (*)

we get

$$-\frac{\omega^2}{c^2} g(\underset{\sim}{r}) = \Delta g(\underset{\sim}{r})$$

(after dividing both sides by $\cos \omega t$).

$$(\Delta + k^2) g(\underset{\sim}{r}) = 0$$

Helmholtz Equation

Similarly, if $u(\underset{\sim}{r}, t) = \sin \omega t \, f(\underset{\sim}{r})$, say,

then $$(\Delta + k^2) f(\underset{\sim}{r}) = 0$$

So k^2 is the eigenvalue of the operator Δ

and g or f are the eigenfunctions

WLOG let $g(\underset{\sim}{r}) = f(\underset{\sim}{r}) = f$.

Coming back to the time harmonic solution

$u(\underset{\sim}{r},t)$ of $(*)$

$$(\maltese) \qquad u(\underset{\sim}{r},t) = \sum_n \begin{pmatrix} a_n \cos\omega_n t \\ b_n \sin\omega_n t \end{pmatrix} \varphi_n(\underset{\sim}{r})$$

Where the eigenfunctions $\varphi_n(\underset{\sim}{r})$ are

given by the

$$\left(\Delta + k_n^2\right)\varphi_n(\underset{\sim}{r}) = 0$$

with boundary conditions on $\varphi_n(r)$

We can write $(\maltese)$ more compactly as:

$$(\dagger) \quad u(\underset{\sim}{r},t) = \sum_{n=-\infty}^{\infty} \alpha_n e^{i\omega_n t} \varphi_n(\underset{\sim}{r})$$

here α_n are complex.

Rmk: if $u(\underset{\sim}{r},t)$ is real then $\bar{\alpha}_n = \alpha_{-n}$.

If we want to find the part of the solution associated with $\cos \omega_n t$, then we take the real part of $(\ddagger)$. Similarly, the imaginary part yields the part of the solution proportional to $\sin \omega_n t$.

TIME HARMONIC SOLUTION TO WAVE EQUATION:

Consider $$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \Delta u \qquad (\ddagger)$$

with suitable boundary conditions on $u(\underset{\sim}{r},t)$.

$\underset{\sim}{r} = (x,y,z)$, $-\infty < t < \infty$.

Assume $u(\underset{\sim}{r},t) = e^{i\omega t} F(\underset{\sim}{r})$

we need $\frac{\partial^2 u}{\partial t^2} = -\omega^2 u$ (1)

substitute (1)

$$-\frac{\omega^2}{c^2} u = \Delta u$$

or

$$-\frac{\omega^2}{c^2} e^{i\omega t} F(\underset{\sim}{r}) = e^{i\omega t} \Delta F(\underset{\sim}{r})$$

$\therefore$ for every ω we need to solve the

Helmholtz Equation

$$(HE) \quad (\Delta + k^2) F(\underset{\sim}{r}) = 0$$

where $c = \frac{\omega}{k}$

$\overrightarrow{\text{To solve}}$ (HE) for the eigenvalues of Δ, i.e. k_n^2 and the eigenfunctions $F(\underset{\sim}{r}) = \phi_n(\underset{\sim}{r})$

we need to specify B.C.

Once we obtain the solution to HE then the solution associated with $\sin \omega_n t$ is

$$\mathrm{Im}\left(u(\underline{r},t)\right)$$

to get the part of the solution associated with $\cos \omega t$, take

$$\mathrm{Re}\left(u(\underline{r},t)\right)$$

//

FORCED (NON-HOMOGENEOUS) WAVE EQUATION

For illustration, we show the 2D-space case:

PDE $\quad u_{tt} = c^2 \Delta u + g(x,y,t) \quad \begin{cases} 0<x<a \\ 0<y<b \\ t>0 \end{cases}$

here $\quad \Delta = \partial_{xx} + \partial_{yy}$

B.C. $\quad \begin{cases} u(x,0,t) = \dfrac{\partial u}{\partial y}(x,b,t) = 0 \\ u(0,y,t) = \dfrac{\partial u}{\partial x}(a,y,t) = 0 \end{cases}$

$$\text{I.C.}\begin{cases} u(x,y,0) = f(x,y) \\ \frac{\partial u}{\partial t}(x,y,0) = g(x,y) \end{cases}$$

Assume

$$u(x,y,t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} B_{nm}(t)\,\varphi_{nm}(x,y)$$

with

$$B_{nm}(t) = \frac{4}{ab}\int_0^b\int_0^a u(x,y,t)\,\varphi_{nm}(x,y)\,dx\,dy$$

This is an eigenfunction expansion $\{\varphi_{nm}(x,y)\}$. These eigenfunctions satisfy the HE:

$$(HE)\left[\begin{aligned} &(\Delta + k_{nm}^2)\,\varphi_{nm}(x,y) = 0 \\ &\varphi_{nm}(x,0) = \frac{\partial}{\partial y}\varphi_{nm}(x,b) = 0 \\ &\varphi_{nm}(0,y) = \frac{\partial}{\partial x}\varphi_{nm}(a,y) = 0 \end{aligned}\right.$$

(1) Solve the HE problem:

To find $\varphi_{nm}(x,y)$ assume a seperable solution to HE:

$$\varphi_{nm}(x,y) = H_n(x)\, R_m(y)$$

$$H_{xx} R + H R_{yy} + k^2 HR = 0$$

or
$$\frac{H_{xx}}{H} + \frac{R_{yy}}{R} + k^2 = 0$$

$$\therefore \quad \frac{H_{xx}}{H} = -\frac{R_{yy}}{R} + k^2 = -l^2$$

$$\text{A} \quad \begin{cases} H_{xx} + l^2 H = 0 \\ H(0) = H_x(a) = 0 \end{cases}$$

$$\text{B} \quad \begin{cases} R_{yy} + (k^2 + l^2) R = 0 \\ R(0) = R_y(b) = 0 \end{cases}$$

Solving A $\quad H = A\cos lx + B\sin lx$

$W(0) = 0 = A \quad \therefore$

$$W = B \sin lx$$

$$W' = lB \cos lx$$

$$W'(a) = 0 = lB \cos la$$

$$\therefore \quad l_n = \frac{(n+\frac{1}{2})\pi}{a} \qquad n = 0, 1 \ldots$$

$$W_n = \sin l_n x \quad .$$

By the same arguments we find B:

$$R_m = \sin \frac{(m+\frac{1}{2})\pi y}{b} \qquad m = 0, 1 \ldots$$

The eigenfunctions of HE are then

$$\varphi_{nm}(x) = \sin \frac{(n+\frac{1}{2})\pi x}{a} \sin \frac{(m+\frac{1}{2})\pi y}{b} .$$

(2) Perform an eigenfunction expansion of $q(x,y,t)$:

$$Q_{nm}(t) = \frac{4}{ab}\int_0^b dy \int_0^a dx\, q(x,y,t)\,\varphi_{nm}(x,y)$$

(3) Substitute $u(x,y,t)$ and $q(x,y,t)$ in terms of $\varphi_{nm}(x,y)$ into PDE:

$$u_{tt} = c^2\left(u_{xx} + u_{yy}\right) + q(x,y,t)$$

$$\sum_n \sum_m \frac{d^2 B_{nm}}{dt^2}\varphi_{nm} = c^2 \sum_n \sum_m \left(\frac{d^2\varphi_{nm}}{dx^2} + \frac{d^2\varphi_{nm}}{dy^2}\right) B_{nm}$$

$$+ \sum_n \sum_m Q_{nm}\varphi_{nm}$$

$$\sum_n \sum_m \left[\frac{d^2 B_{nm}}{dt} \varphi_{nm} - c^2 \frac{d^2 \varphi_{nm}}{dx^2} B_{nm} - c^2 \frac{d^2 \varphi_{nm}}{dy^2} B_{nm} \right]$$

$$= \sum_n \sum_m Q_{nm} \varphi_{nm}$$

or

$$\sum_n \sum_m \left[\frac{d^2 B_{nm}}{dt} + c^2 k_{nm}^2 B_{nm} \right] \varphi_{nm} = \sum_n \sum_m Q_{nm} \varphi_{nm}$$

$(-\Delta \varphi = +k^2 \varphi$ via HE$)$

$$\therefore \quad (\text{¥}) \quad \frac{d^2 B_{nm}}{dt^2} + c^2 k_{nm}^2 B_{nm} = Q_{nm}(t) \qquad \begin{matrix} n = 0, 1, \ldots \\ m = 0, 1, \ldots \end{matrix}$$

To solve (¥) we use the I.C. we project the I.C. onto the space spanned by $\varphi_{nm}(x,y)$

$$(\mathcal{J}')\quad \begin{cases} B_{nm}(0) = f_{nm} & n = 0, 1 \ldots \\ \dfrac{d}{dt} B_{nm}(0) = g_{nm} & m = 0, 1 \ldots \end{cases}$$

$$\begin{cases} f_{nm} = \dfrac{4}{ab} \displaystyle\int_0^b dy \int_0^a dx \, f(x,y) \, \varphi_{nm}(x,y) \\ g_{nm} = \dfrac{4}{ab} \displaystyle\int_0^b dy \int_0^a dx \, g(x,y) \, \varphi_{nm}(x,y) \end{cases}$$

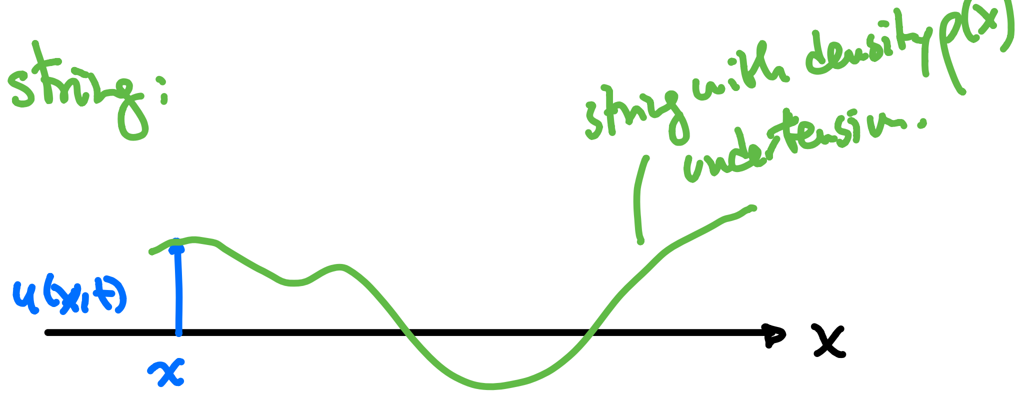
To solve $(\mathcal{Y})$ with $(\mathcal{J}')$ we use variation of parameters, on this inhomogeneous linear constant coefficient 2nd order ODE

THE FORCED WAVE EQUATION & RESONANCE & GREEN'S FUNTION

We first derive the equation for the dynamics of a vibrating string: it obeys the wave equation

What follows generalizes to 2 & 3 space dimensions...

This string:



$u(x,t)$ is the displacement of the string

$[u(x,t)] = $ length (L)

x is the position along string and

t is time.

$[x] = L$ (length)

$[t] = T$ (time)

Let $\theta(x,t)$ angle between the horizontal and the string

$[\theta] = 1$

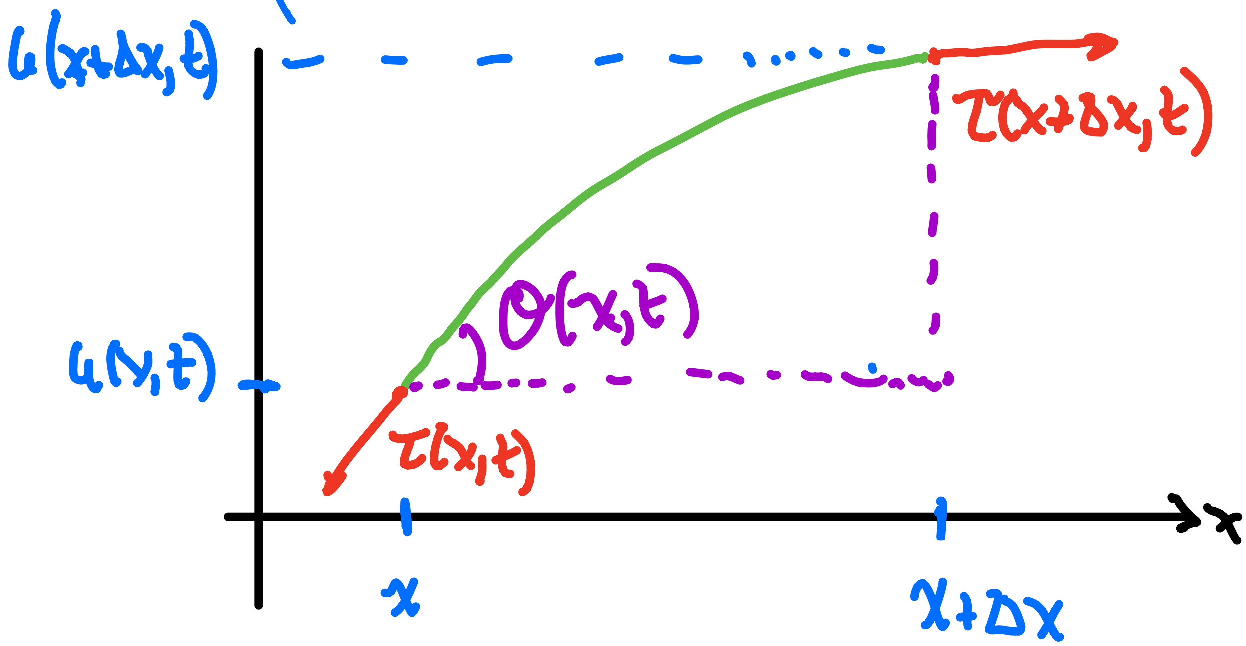
let $\tau(x,t)$ tension

$$[\tau(x,t)] = MLT^{-2}$$

(force)

let $\rho(x)$ string density

$$[\rho] = ML^{-1}$$



The total mass

$$\approx \rho(x)\sqrt{\Delta x^2 + \Delta u^2}$$

Newton's Second Law $(\cancel{A})$
(in the vertical direction:)

$$\rho(x)\sqrt{\Delta x^2+\Delta u^2}\,\frac{\partial^2 u}{\partial t^2} =$$

$$\tau(x+\Delta x,t)\sin\left[\theta(x+\Delta x,t)\right]$$

$$-\tau(x,t)\sin\left[\theta(x,t)\right] + F(x,t)\Delta x$$

$F(x,t)$ represents a (vertical) external force per unit length (if present)

$F(x,t)$ combines the applied and

restoring forces. An example of an applied force is that the string is being plucked or pulled vertically. An example of a restoring force may be friction, or a magnetic field applied to a magnetizable string.

Continuing:

Divide (★) by Δx and $\Delta x \to 0$:

($):

$$\rho(x)\sqrt{1+\left(\frac{du}{dx}\right)^2}\,dx\,\frac{\partial^2 u}{\partial t^2} = dx\left[\frac{\partial}{\partial x}\left\{\tau(x,t)\sin\theta(x,t)\right\} + F(x,t)\right]$$

Since

$$\tau(x+\Delta x,t)\sin\left[\theta(x+\Delta x,t)\right]$$

$$\approx \tau(x,t)\sin\left[\theta(x,t)\right] + \frac{\partial \tau}{\partial x}\sin\left[\theta(x+\Delta x,t)\right]\Delta x$$

$$+\mathcal{O}(\Delta x^2)$$

$$\simeq \tau(x,t)\sin[\theta(x,t)] + \frac{\partial \tau}{\partial x}\Big[\sin(\theta(x,t))$$

$$+ \Delta x \cos\theta(x,t) + \mathcal{O}(\Delta x^2)\Big]\Delta x$$

$$+ \mathcal{O}(\Delta x^2)$$

$$\simeq \underline{\tau(x,t)\sin[\theta(x,t)]} + \frac{\partial \tau}{\partial x}(x,t)\sin\theta(x,t)\,\Delta x$$

$$+ \mathcal{O}(\Delta x^2)$$

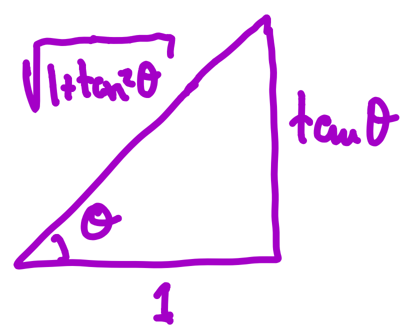
The terms underlined cancel, we omit $\mathcal{O}(\Delta x^2)$ terms, and take $\Delta x \to 0$:

Then (‡) yields:

$$\rho(x)\sqrt{1+\left(\frac{du}{dx}\right)^2}\,\frac{\partial^2 u}{\partial t^2} = \frac{\partial \tau}{\partial x}\sin\theta + F \qquad (‡)$$

approximately.

Note also $\tan\theta = \frac{\partial u}{\partial x}$ and



$$\sin\theta = \frac{1}{\gamma}\frac{\partial u}{\partial x} \; ; \; \cos\theta = \frac{1}{\gamma}$$

where $\gamma = \sqrt{1 + \left(\frac{\partial u}{\partial x}\right)^2}$

$\theta = \arctan\left[\frac{\partial u}{\partial x}\right]$ and $\frac{\partial \theta}{\partial x} = \frac{1}{\gamma^2}\frac{\partial^2 u}{\partial x^2}$

$\therefore$

Assume $|\theta(x,t)| \ll 1$ (small displacements)

then $\begin{cases} \sin\theta \approx \frac{\partial u}{\partial x} \\ \frac{\partial \theta}{\partial x} \approx \frac{\partial^2 u}{\partial x^2} \end{cases}$ turns (#) into

$$(¥) \quad \rho(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial \tau}{\partial x}\frac{\partial u}{\partial x} + \tau(x,t)\frac{\partial^2 u}{\partial x^2} + F(x,t)$$

The balance of forces in the horizontal direction is

$$\tau(x+\Delta x,t)\cos\left[\theta(x+\Delta x,t)\right] - \tau(x,t)\cos\left[\theta(x,t)\right] = 0.$$

Divide by Δx and $\Delta x \to 0$

Which allows us to conclude that

$$\frac{\partial}{\partial x}\left[\tau \cos\left(\theta(x,t)\right)\right] = 0$$

$$\approx \frac{\partial}{\partial x}\tau(x,t) = 0$$

(since $|\theta| \ll 1 \therefore \cos\theta \approx 1$)

which can be integrated to suggest that

$\therefore \quad \tau(x,t) = \tau(t)$, only a function of t.

Replace this in $(\neq)$

$$\rho(x)\frac{\partial^2 u}{\partial t^2} = \tau(t)\frac{\partial^2 u}{\partial x^2} + F(x,t)$$

$$(\neq) \quad \frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + f(x,t)$$

"WAVE EQUATION"

$$\begin{cases} c^2 = \dfrac{\tau}{\rho} & \text{wave speed} \\ f = F/\rho & \text{acceleration per unit length.} \end{cases}$$

GREEN'S FUNCTIONS & RESONANCE

Green's functions arise in the solution of non-homogeneous ODE's and PDE's.

Consider, for specificity, with $u(x,t)$ and time-harmonic forcing, i.e.

$$F(x,t) = \rho(x)\, g(x)\, \mathrm{Re}\left(e^{-i\omega t}\right) - V(x)\, u$$

in the following PDE:

$$(*)\quad \rho \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[\tau \frac{\partial u}{\partial x}\right] + \mathrm{Re}\left(e^{i\omega t}\right) g\, \rho - V(x)\, u$$

Here $V(x)\, u(x,t)$ is a conservative force, i.e. derived from the gradient of a field potential:

$$V(x) \equiv -\nabla_u \phi(u)$$

here, $\phi(u)$ is the potential.

Assume $u = \mathrm{Re}\left(\mu(x) e^{-i\omega t}\right)$, a time harmonic solution. Then $\frac{\partial^2 u}{\partial t^2} = -\omega^2 u(x) e^{-i\omega t}$ and (*) is

$$(**)\quad -\frac{d}{dx}\left[\tau \frac{\partial u}{\partial x}\right] + V(x) u(x) - \omega^2 \rho(x) u(x) = \rho(x) g(x).$$

Let $\mathcal{L} = -\frac{d}{dx}\left[\tau \frac{d}{dx}\right] + V(x)$

Then (**) can be written as a S.L problem with forcing:

$$(NH)\quad \mathcal{L}u - \omega^2 \rho u = \rho g$$

To solve, we need 2 boundary conditions on $u(x)$. For specificity, assume that

$$u(0) = u(L) = 0.$$

Also, for simplicity assume $V(x) = 0$.

We invoke an associated SL problem:

$$\begin{cases} \mathcal{L}u_n = \omega_n^2 \rho u_n \\ u_n(0) = u_n(L) = 0 \end{cases}$$

We get $u(x) = \sum_{n=1}^{\infty} c_n \varphi_n(x)$ $(\ddagger)$

where $\varphi_n(x) = \sin\frac{n\pi x}{L}$

Substitute $(\ddagger)$ into (NH),

$$\mathcal{L}u - \omega^2 \rho u = \rho g,$$

to get $$\sum_{n=1}^{\infty} c_n (\omega_n^2 - \omega^2) \rho \varphi_n(x) = \rho g$$

Using orthogonality:

$$c_n = \frac{1}{\omega_n^2 - \omega^2} \int_0^L \rho(y) g(y) \varphi_n(y) dy$$

We'll write this as

$$c_n = \frac{1}{\omega_n^2 (1 - \omega^2/\omega_n^2)} \langle g | \varphi_n \rangle$$

Where $\langle g | \varphi_n \rangle = \int_0^L \rho(y)\, g(y)\, \varphi_n(y)\, dy$

$$\therefore \quad u(x) = \sum_{n=1}^{\infty} \frac{1}{\omega_n^2 \left(1 - \omega^2/\omega_n^2\right)} \langle g | \varphi_n \rangle \varphi_n(x)$$

We'll write this as

$$u(x) = \int_0^L G(x,y)\, \rho(y)\, g(y)\, dy$$

$G(x,y)$ is the Green's function

$$G(x,y) = \sum_{n=1}^{\infty} \frac{\varphi_n(x)\, \varphi_n(y)}{\omega_n^2 \left(1 - \omega^2/\omega_n^2\right)}$$

The Green's Function has several important properties:

1. $G(x,y) = G(y,x)$, Symmetric.

2. It depends on the forcing frequency ω

3. $G(x,y)$ satisfies the boundary conditions at $x=0$ & $x=L$.

4. $G(x,y)$ becomes unbounded when $\omega_n = \omega$.

$\therefore$ $u(x,t)$ becomes unbounded. In other words, whenever the driving frequency ω matches any of the "natural frequencies" ω_n $|u(x,t)|$ becomes unbounded. We call this Resonance. //

Green's functions appear in a all sorts of linear ODE's & PDE's in the solution to the non-homogeneous part.

Intuition using the finite dimensional matrix problem:

We want to solve

$$Ax = b, \text{ then } x = A^{-1}b$$

for simplicity, take $A \in \mathbb{R}^{n \times n}$

$x, b \in \mathbb{R}^n$

Assume $A = A^{+}$ (Hermitian)

and assume you know the e'values & e'vectors

$$A\phi_k = \lambda_k \phi_k$$

$(\lambda_k \neq 0)$

Let $x = A^{-1} b$

written as

$$x = G b$$

G is a matrix.

If $A = \Phi \Lambda \Phi^{+}$

Φ is a matrix with ϕ_k as columns and

$$\Lambda = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{bmatrix}$$

then $A^{-1} = \Phi \Lambda^{-1} \Phi^{+}$

$$A^{-1} = \sum_k \frac{\phi_k \phi_k^{+}}{\lambda_k} = G$$