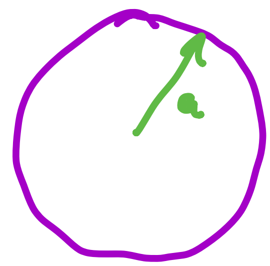
Solve $\Delta u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$ PDE

$0<r<a$
$0<\theta<2\pi n$
$t>0$



B.C. $\begin{cases} u(a,\theta,t)=0 \\ u(0,0,t) \text{ bounded} \\ u(r,\theta,t) = u(r,\theta+n2\pi,t) \quad n\in\mathbb{Z} \\ \text{(periodic)} \end{cases}$

I.C. $u(r,\theta,0) = f(r)$, $u_t(r,\theta,0) = g(r)$

$\Delta u = \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial r^2}$, since $u = u(r,t)$

let $u(r,t) = R(r)W(t)$

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{\partial R}{\partial r} + k^2 R = 0$$

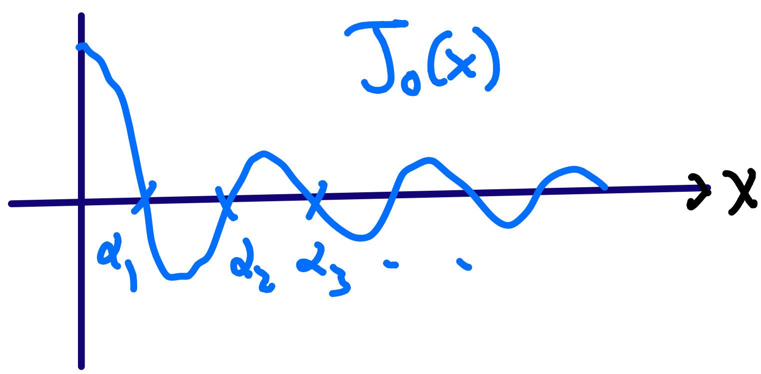
$$R(r) = A J_0(kr) + B Y_0(kr)$$

Since $u(r,\theta,t)$ is bounded everywhere, including at $r=0$, then $B=0$

$$R(r) = A J_0(kr)$$

at $r=a$

$$R(a) = A J_0(ka) = 0$$



$$k_n = \alpha_n / a \qquad n = 1, 2 \ldots$$

The PDE: $\quad \Delta R W = \frac{1}{c^2} R W''$

So $W'' + \omega_n^2 W = 0 \quad (*)$

where $\omega_n = c^2 k_n^2$

The solution to $(*)$ is in terms of $\{\cos \omega_n t, \sin \omega_n t\}$. Let's use the complex representation:

$$W_n = D_n e^{i\omega_n t},$$

D_n is complex.

$$u(r,\theta,t) = \sum_{n=1}^{\infty} D_n J_0(\lambda_n r) e^{i\omega_n t} \qquad (\ddagger)$$

$$D_n \in \mathbb{C}$$

$$u(r,\theta,0) = \sum_{n=1}^{\infty} D_n J_0(\lambda_n r)$$

Here we use

$$\int_0^a r J_0(\alpha_n r) J_0(\alpha_m r)\, dr = \frac{J_1^2(\alpha_n)}{2} \delta_{nm}$$

$$\mathrm{Re}(D_n) = \frac{2}{a^2 J_1^2(\alpha_n)} \int_0^a r f(r) J_0(\alpha_n r/a)\, dr$$

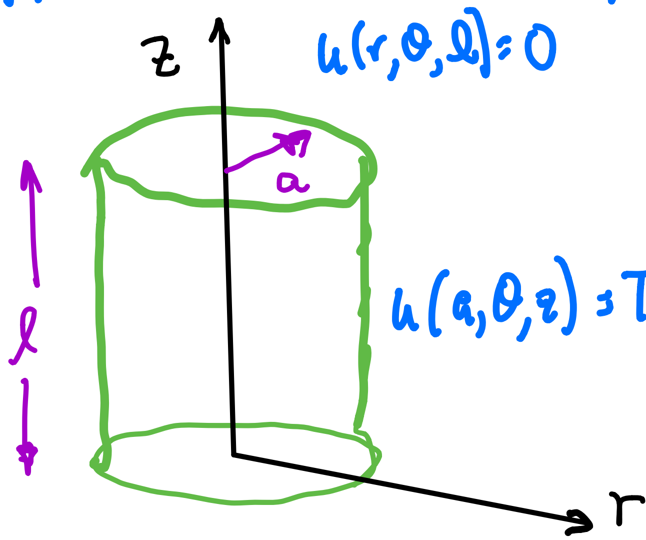
Differentiate $(\ddagger)$ wrt t:

$$u_t(r,\theta,t) = \sum_{n=1}^{\infty} i\omega_n D_n J_0\left(\frac{\alpha_n r}{a}\right) e^{i\omega_n t}$$

$u_t(r,\theta,0) = g(r)$

$$\therefore\ i\omega_n D_n = \frac{2}{a\alpha_n J_1^2(\alpha_n)} \int_0^a r\, g(r)\, J_0(\alpha_n r/a)\, dr$$

ex) Find the solution $u(r,\theta,z)$, $\Delta u = 0$



$u(r,\theta,\ell) = 0$

$u(a,\theta,z) = T_0 z(\ell - z)$

$u(r,\theta,0) = 0$

Since no θ dependence $u = u(r,z)$

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial z^2} = 0 \quad \text{PDE}$$

$u(r,0) = u(r,l) = 0 \qquad 0 \le r \le a$

$u(a,z) = T_0 z(l-z) \qquad 0 \le z \le l$

u is bounded inside the cylinder

$u = R(r)Z(z)$, subs into PDE:

$$Z R_{rr} + \frac{1}{r} Z R_r = - Z_{zz} R$$

divide by ZR:

$$\frac{R_{rr}}{R} + \frac{1}{r}\frac{R_r}{R} = -\frac{Z_{zz}}{Z} = +\lambda^2$$

$$-Z_{zz} = +\lambda^2 Z$$

or $\quad Z_{zz} + \lambda^2 Z = 0$

$$r^2 R_{rr} + r R_r - \lambda^2 r^2 R = 0$$

The "R" equation has a solution in terms

$$R(r) = c_1 I_0(\lambda r) + c_2 K_0(\lambda r)$$

I_0 and K_0 are a type of Bessel functions known as Modified Bessel functions.

$$\begin{cases} Z'' + \lambda^2 Z = 0 \\ Z(0) = 0 \quad Z(l) = 0 \end{cases}$$

$$Z_n = \sin \frac{n\pi z}{l} \qquad n = 1, 2, \ldots$$

$$\lambda_n = \left(\frac{n\pi}{l}\right) \qquad n = 1, 2, \ldots$$

$\therefore \; R(r) = c_1 I_0\left(\frac{n\pi r}{l}\right) + c_2 K_0\left(\frac{n\pi r}{l}\right)$

K_0 is unbounded at $r=0$ $\therefore \; c_2 = 0$

$$u(r,z) = \sum_{n=1}^{\infty} c_n I_0\left(\frac{n\pi r}{l}\right) \sin\frac{n\pi z}{l}$$

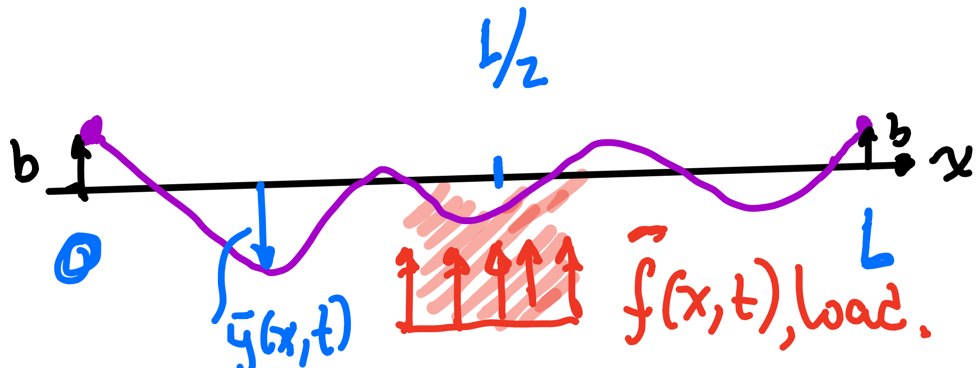
$$u(a,z) = \sum_{n=1}^{\infty} c_n I_0\left(\frac{n\pi a}{l}\right) \sin\frac{n\pi z}{l}$$

$$= T_0 z(l-z)$$

We use orthogonality of $\sin\frac{n\pi z}{l}$

$$C_n = \frac{T_0}{I_0\left(\frac{n\pi a}{l}\right)} \frac{2}{l} \int_0^l z(l-z) \sin \frac{n\pi z}{l} \, dz$$

Ex)



Beam, loaded in the middle and hinged at the 2 end points

The beam is of length L.

$\bar{y} = \bar{y}(x,t)$ is the displacement

$$\rho A \frac{\partial^2 \bar{y}}{\partial t^2} = \bar{f}(x,t) - \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 \bar{y}}{\partial x^2}\right) \quad \text{(PDE)}$$

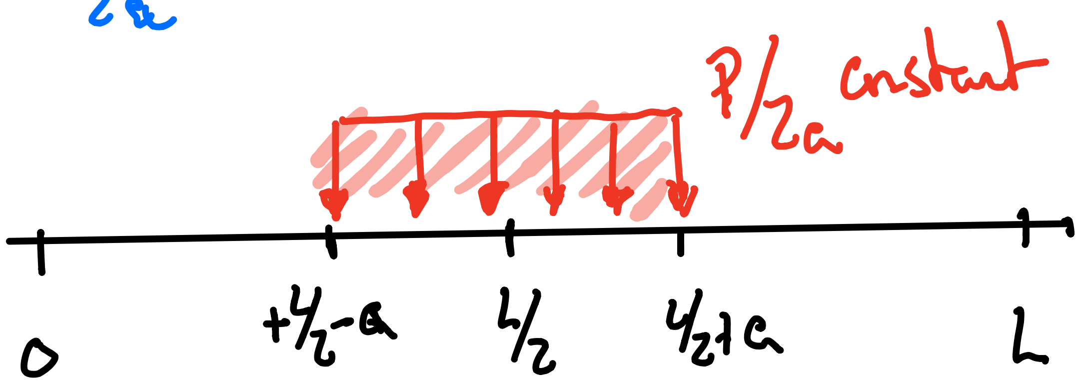
here $\begin{cases} A(x) \text{ is beam cross section} \\ \rho(x) \text{ the density of beam} \\ E(x) \text{ is the Young's modulus} \\ I(x) \text{ is the inertia.} \end{cases}$

We'll assume A, ρ, E, I are constant.

LOAD:

$$\bar{f}(x,t) = \sin\omega t \begin{cases} \frac{P}{2a} & \frac{L}{2} - a < x < \frac{L}{2} + a \\ 0 & \text{otherwise} \end{cases}$$

$\frac{P}{2a}$ is constant



We require 4 B.C. (in space)

$\bar{y}(0,t) = b \qquad \frac{d^2\bar{y}}{dx^2}(0,t) = 0$

$\bar{y}(L,t) = b \qquad \frac{d^2\bar{y}}{dx^2}(L,t) = 0$

let $\bar{y}(x,t) = \sin\omega t \, y(x)$

substitute into the PDE

$$\underbrace{-\rho A \omega^2 \sin\omega t \, y(x)}_{\rho A \frac{\partial^2 \bar{y}}{\partial t^2}}$$

$$= f(x)\sin\omega t - \frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 y}{\partial x^2}\right)\sin\omega t$$

where $f(x) = \begin{cases} \frac{P}{2a} & \frac{L}{2} - a < x < \frac{L}{2} + a \\ 0 \end{cases}$

factor out $\sin \omega t$:

$$\frac{\partial^2}{dx^2}\left(EI y''\right) - \rho A y = f(x) \quad (\dagger)$$

$$y = y(x)$$

let $\beta^4 = \frac{\rho A}{EI}$ $\quad F = \frac{f(x)}{EI}$

$$y^{IV} - \beta^4 y = F \quad \text{is } (\dagger)$$

$$y(0) = y(L) = b$$

$$y''(0) = y''(L) = 0$$

let $y = y_I + y_{II}$

$$\begin{cases} y_I^{IV} - \beta^4 y_I = 0 \\ y_I(0) = y_I(L) = b \\ y_I''(0) = y_I''(L) = 0 \end{cases}$$

$$\begin{cases} y_{II}^{IV} - \beta^4 y_{II} = F(x) \\ y_{II}(0) = y_{II}(L) = 0 \\ y_{II}''(0) = y_{II}''(L) = 0 \end{cases}$$

① Find eigenfunctions:

$$\begin{cases} u^{iv} - \beta^4 u = 0 & \text{D.E.} \\ u(0) = u(L) = u''(0) = u''(L) = 0 \end{cases}$$

the solution is found by assuming $u = e^{\alpha x}$ in D.E.

we get $\alpha^4 - \beta^4 = 0$. There are 2 real roots & 2 complex conjugate roots:

$$u = c_1 \sin\beta x + c_2 \cos\beta x + c_3 \sinh\beta x + c_4 \cosh\beta x$$

Apply B.C.

$u(0) = 0 = c_2 + c_4$ $\quad u''(0) = -c_2 + c_4 = 0 \Rightarrow c_2 = c_4 = 0$

$u(L) = 0 = c_1 \sin\beta L + c_3 \sinh\beta L$

$u''(L) = 0 = -c_1 \beta^2 \sin\beta L + c_3 \beta^2 \sinh\beta L$

or $\sin\beta L = 0$ $\therefore \beta_n = \frac{n\pi}{L}$ $\quad n = 1, 2 \ldots$

$\therefore u_n = \sin\frac{n\pi x}{L}$ $\quad n = 1, 2 \ldots$

② Expand $F(x)$ in a series in u_n wrt $w(x) = 1$:

$$\frac{P}{2aEI} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

$$b_n = \frac{2}{L}\,\frac{P}{2aEI}\int_{L/2-a}^{L/2+a} \sin\frac{n\pi x}{L}\,dx$$

$$\therefore\ b_n = -\frac{2}{L}\,\frac{P}{2aEI}\left(\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\Bigg|_{L/2-a}^{L/2+a}$$

But

$$\cos\left(\frac{n\pi}{2}+\frac{n\pi a}{L}\right)-\cos\left(\frac{n\pi}{2}-\frac{n\pi a}{L}\right) = -2\sin\frac{n\pi}{2}\sin\frac{n\pi a}{L}$$

$$\therefore\ b_n = \frac{2}{aEI}\,\frac{1}{n\pi}\sin\frac{n\pi}{2}\sin\frac{n\pi a}{L} \qquad n = \text{even}.$$

③ Substitute series of $F(x)$ into ODE:

$$y_{II}^{IV} - \beta^4 y_{II} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

$$\text{let } y_{II} = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} a_n \sin\beta_n x$$

$$y_{II}^{IV} = \sum_{n=1}^{\infty} a_n \beta_n^4 \sin\beta_n x$$

$$\therefore\ \sum_{n=1}^{\infty}\left[a_n\beta_n^4\sin\beta_n x - \beta^4 a_n \sin\beta_n x\right] = \sum_{n=1}^{\infty} b_n \sin\beta_n x$$

$$\therefore \sum_{n=1}^{\infty}\left[a_n\left(\beta_n^4-\beta^4\right)-b_n\right]\sin\beta_n x = 0$$

$$\therefore a_n = \frac{b_n}{\beta_n^4-\beta^4}$$

$$y_{II} = \sum_{n=1}^{\infty}\frac{b_n}{\beta_n^4-\beta^4}\sin\frac{n\pi x}{L}$$

$$\text{with } b_n = \frac{2P}{aEI}\frac{1}{n\pi}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{L}$$

$$\text{or } b_n = \frac{2PL}{EI}\sin\frac{n\pi}{2}\frac{\sin\theta}{\theta}, \quad \theta = \frac{n\pi a}{L}$$

$$\lim_{\theta\to 0} b_n = \frac{2PL}{EI}\sin\frac{n\pi}{2}$$

④ Solve y_I

$$(*)\begin{cases} y_I^{IV} - \beta^4 y_I = 0 \\ y_I(0) = y_I(L) = b \\ y_I''(0) = y_I''(L) = 0 \end{cases}$$

We show that the solution is $y_I = b$:

let $y_I = b + v(x)$

y_I requires v & derivatives are 0 at boundaries.

substitute in $(*)$:

$$v^{IV} - b\beta^4 - \beta^4 v = 0$$

$$(\ddagger)\begin{cases} v^{IV} - \beta^4 v = b\beta^4 \quad \text{(D.E.)} \\ v(0) = v(L) = v''(L) = v''(0) = 0 \end{cases}$$

by inspection,

$$v = \alpha_1 x^4 + \alpha_2 x^3 + \alpha_3 x^2 + \alpha_4 x + \alpha_5$$
$$v' = 4\alpha_1 x^3 + 3\alpha_2 x^2 + 2\alpha_3 x + \alpha_4$$
$$v'' = 12\alpha_1 x^2 + 6\alpha_2 x + 2\alpha_3$$

$$v''' = 24\alpha_1 x + 6\alpha_2$$

$$v'''' = 24\alpha_1$$

The D.E.

$$24\alpha_1 - \beta^4(\alpha_1 x^4 + \alpha_2 x^3 + \alpha_3 x^2 + \alpha_4 x + \alpha_5) = b\beta^4$$

$\therefore$ let $24\alpha_1 - \beta^4 \alpha_5 = b\beta^4$

$$v''(L) = 0 = 12\alpha_1 L^2 + 6L\alpha_2 + 2\alpha_3 = 0$$

$$v(L) = 0 = \alpha_1 L^4 + \alpha_2 L^3 + \alpha_3 L^2 + \alpha_4 L + \alpha_5 = 0$$

let $\alpha_3 = 0$, $\alpha_4 = 0$

$$\alpha_2 = -2\alpha_1 L$$

$$\alpha_5 = \frac{24\alpha_1 - b\beta^4}{\beta^4}$$

$$\alpha_1 L^4 - 2\alpha_1 L^4 - \frac{24\alpha_1 - b\beta^4}{\beta^4} = 0$$

$$\therefore \alpha_1\left(-L^4 - \frac{24}{\beta^4}\right) = b$$

$$\alpha_1\left(\beta^4L^4 - 24\right) = -\beta^4 b$$

$$\alpha_1 = \frac{\beta^4 b}{24-\beta^4L^4}$$

$$\alpha_5 = \frac{\beta^2 b}{24-\beta^4L^4} - b$$

$$\alpha_2 = \frac{-2L\beta^4 b}{24-\beta^4L^4}$$

$$\therefore\ v(x) = \frac{\beta^4 b}{24-\beta^4L^4}x^4 - \frac{2L\beta^4 b}{24-\beta^4L^4}x^3 - \frac{\beta^2 b}{24-\beta^4L^4} - b$$

$$y_I = b + v(x)$$

$$y_I = \frac{\beta^4 b}{24-\beta^4L^4}x^4 - \frac{2L\beta^4 b}{24-\beta^4L^4}x^3 - \frac{\beta^2 b}{24-\beta^4L^4} \quad //$$