Two functions $f(x)$ & $g(x)$, complex (or real) are said to be orthogonal with respect to a weight function $w(x)$ over an interval $I=[a,b]$ if

$$\int_a^b f(x)\,\overline{g(x)}\,w(x)\,dx = 0$$

where $\overline{(\cdot)}$ is complex conjugate

(or $\int_a^b \bar{f}(x) g(x) w(x)\,dx = 0$

The weight function $w(x) \geq 0$ for $x \in (a,b)$

and $\int_a^b w(x)\,dx = A \geq 0$, a number.

ex) Under what conditions are $\varphi_n(x) = e^{inx}$ and $\varphi_m(x) = e^{imx}$ orthogonal wrt $w(x) = 1$ over $I = [-\pi, \pi]$?

$$\int_{-\pi}^{\pi} e^{imx} e^{-inx} \, 1 \, dx$$

$$= \int_{-\pi}^{\pi} e^{i(m-n)x} \, dx$$

$$= \frac{1}{i(m-n)} e^{i(m-n)x} \Big|_{-\pi}^{\pi}$$

$$= \frac{1}{i(m-n)} \left[e^{i(m-n)\pi} - e^{-i(m-n)\pi} \right]$$

$$= \frac{2}{(m-n)} \sin\left[(m-n)\pi\right]$$

Rmk: what values of m & n make this equal to 0?

(i) Case $m \neq n$ let $p = m - n$

$$h(p) = \frac{2}{p} \sin p\pi$$

if p is not an integer ($\neq 0$)

$h(p)$ is generally not equal to 0.

(ii) $p = 0$: for this case, take limit as $p \to 0$, use L'Hopital's Rule:

$$\lim_{p \to 0} \frac{2}{p} \sin p\pi = \lim_{p \to 0} \frac{2\pi \cos p\pi}{1}$$

$$= 2\pi$$

(iii) If $p = \pm 1, \pm 2, \pm 3 \dots$

$p = m - n$

$$\frac{2}{(m-n)} \sin \left[(m-n) \pi \right]$$

$$= \frac{2}{p} \sin p\pi = 0$$

$\therefore$ For $\varphi_n(x) = e^{inx}$, $\varphi_m(x) = e^{imx}$

$$\int_{-\pi}^{\pi} \varphi_n(x) \overline{\varphi_m(x)} \, 1 \, dx = \begin{cases} 0 & m \neq n \\ 2\pi & m = n \end{cases}$$

For $m, n \in \mathbb{Z}$

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STURM-LIOUVILLE PROBLEM

A special family of boundary value problems, they are all linear and are ordinary differential equations of even order.

Rmk: For the special case of order 2 we can always transform the differential equation into Sturm-Liouville form.

Focus on 2nd order Sturm-Liouville Problems (SL) BVP:

Consider some nice function $y(x)$

such that

$$(\dagger) \quad \frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + \left[q(x) + \lambda w(x)\right]y(x) = 0$$

on a bounded interval $x=a$, $x=b$

$b>a$, plus 2 boundary conditions

(to be specified shortly)

See MTH481/581 for more details

with

p, q, w are real functions

let $\mathcal{L} = \frac{d}{dx}\left(p(x)\frac{d}{dx}\right) + q(x)$

then $(\dagger)$

$$(*) \quad \mathcal{L}y + \lambda w y = 0$$

We can take any 2nd order linear BVP ode and transform it into (*):

$$(**) \quad a_0(x)y''(x) + a_1(x)y'(x) + [a_2(x) + \lambda a_3(x)]y(x) = 0$$

($a_0(x) \neq 0$ for $x \in [a,b]$)

(exercise)

if $p(x) \equiv e^{\int \frac{a_1}{a_0}dx}$

$q(x) \equiv \frac{a_2}{a_0} p$

$w(x) \equiv \frac{a_3(x)}{a_0(x)} p$ then (**) can be written as (*) and can thus be of SL form over some interval $[a,b]$

provided $\frac{a_3(x)}{a_0(x)} > 0$ //

SL BVP
$$\begin{cases} \mathcal{L}y + \lambda w(x) y = 0 \\ \alpha_1 y(a) + \alpha_2 y'(a) = 0 \\ \beta_1 y(b) + \beta_2 y'(b) = 0 \end{cases}$$

if $p(x) > 0$, $w(x) > 0$, λ is a number

p, q, w are continuous on $x \in [a,b]$

$\alpha_1, \alpha_2, \beta_1, \beta_2$ are numbers. //

We will now show that solutions for SL BVP have the property

$$\int_a^b w(x)\, \varphi_i(x)\, \varphi_j(x)\, dx = 0$$

where $\varphi_i(x)$ solves $\mathcal{L}\varphi_i + \lambda_i w(x)\varphi_i = 0$

$\varphi_j(x)$ solves $\mathcal{L}\varphi_j + \lambda_j w(x)\varphi_j = 0$

λ_i & λ_j are 2 numbers

plus some Boundary conditions

$\varphi_i(x)$ and $\varphi_j(x)$ are (REAL) orthogonal functions.

We will show that the solutions to the SL problem, with different λ's have the property

$$\int_a^b w(x)\,\varphi_j(x)\varphi_i(x)\,dx = 0$$

i.e. $\varphi_i(x)$ & $\varphi_j(x)$, solutions to SL are orthogonal w.r.t. $w(x)$ on the interval $[a,b]$: to see this:

$$① \quad \frac{d}{dx}\left(p(x)\partial_x\varphi_i\right) + (q+\lambda_i w)\varphi_i = 0$$

$$② \quad \frac{d}{dx}\left(p(x)\partial_x\varphi_j\right) + (q+\lambda_j w)\varphi_j = 0$$

Multiply ① by φ_j and ② by φ_i and subtract:

$$\varphi_j(p\partial_x\varphi_i)_x - \varphi_i(p\partial_x\varphi_j)_x + (\lambda_i - \lambda_j)w\varphi_i\varphi_j = 0$$

or

$$(\lambda_i - \lambda_j)w\varphi_i\varphi_j = \underline{\varphi_i(p\partial_x\varphi_j)_x} - \varphi_j(p\partial_x\varphi_i)_x$$

integrate by parts (IBP):

$$(\lambda_i-\lambda_j)\int_a^b w\varphi_i\varphi_j\,dx = p\partial_x\varphi_j\varphi_i\Big|_a^b - \int_a^b p\varphi_{ix}\varphi_{jx}\,dx$$

$$-p\partial_x\varphi_i\varphi_j\Big|_a^b + \int_a^b p\,\varphi_{ix}\varphi_{jx}\,dx$$

The IBP calculation. Take let $u = \varphi_i$ $dv = (p\partial_x\varphi_j)_x$

then $du = \varphi_{ix}\,dx$ and $v = p\,\partial_x\varphi_j$

so $\int_a^b p\varphi_i(\partial_x\varphi_j)\,dx$

$= p\varphi_i\partial_x\varphi_j\Big|_a^b$

$- \int_a^b p\varphi_{ix}\varphi_{jx}\,dx$

cancel:

$$(\lambda_i-\lambda_j)\int_a^b w(x)\varphi_i(x)\varphi_j(x)\,dx$$

$$= \left(p\,[\partial_x\varphi_j\varphi_i - \partial_x\varphi_i\varphi_j\right|_a^b$$

Rmk: Since $\lambda_i-\lambda_j\neq 0$, then

$$\int_a^b w(x)\varphi_i(x)\varphi_j(x)\,dx = 0$$

provided terms underlined are zero. That is, provided the boundary conditions make the rhs equal to zero.

CASE (A) if $\begin{cases} \varphi_i = 0 & \text{at } x=a \ \& \ x=b \\ \varphi_j = 0 \end{cases}$ DIRICHLET B.C.

$$\Rightarrow \int_a^b w(x)\varphi_i\varphi_j\,dx = 0$$

CASE (B) if $\begin{cases} \varphi_{ix} = 0 & \text{at } x=a,\ x=b \\ \varphi_{jx} = 0 \end{cases}$ NEUMANN B.C.

$$\Rightarrow \int_a^b w(x)\varphi_i\varphi_j\,dx = 0$$

CASE (C) if $\begin{cases} \varphi_i + \gamma\varphi_{ix} = 0 \\ \varphi_j + \gamma\varphi_{jx} = 0 \end{cases}$ ROBIN or MIXED B.C.

$$\Rightarrow \int_a^b w(x)\varphi_i\varphi_j\,dx = 0$$

Rmk: could also have combinations of Dirichlet B.C. at one end & Neumann at the other.

Rmk: * if $p=0$ at $x=a$, $x=b$ then $\int_a^b w\varphi_i\varphi_j\,dx = 0$

* if φ is finite at $x=a,b$ and φ' or $p\varphi'$ tend

to zero at $x=a,b$ then φ_i, φ_j are orthog wrt w as well.

* if $p(a)=p(b) \Rightarrow \int_a^b w\varphi_i\varphi_j\,dx=0$

$$\begin{cases} \varphi_i(a)=\varphi_i(b) \text{ and } \varphi_i'(a)=\varphi_i'(b) \\ \varphi_j(a)=\varphi_j(b) \text{ and } \varphi_j'(a)=\varphi_j'(b) \end{cases}$$

i.e. periodic B.C, i.e.

if $\varphi(x)=\varphi(x+L)$ where $L=b-a$. //

ex) (‡)
$$\begin{cases} y''+\alpha^2 y=0 \\ y(0)=y(l)=0 \end{cases}$$

here $y=y(x)$

λ is a number

or $[p\,y_x]_x+\lambda y=0$

$q=0$, $p=1$, $\lambda=\alpha^2$, $w=1$

we see that (‡) is a S.L. problem.

The solution of the ODE $y''+\alpha^2 y=0$

$$y = A\cos\alpha x + B\sin\alpha x$$

apply B.C. $y = B\sin\frac{n\pi x}{\ell}$

i.e. $\alpha = \frac{n\pi}{\ell}$ $n = 1, 2 \ldots$

B is an arbitrary constant.

i.e. $y(x) = B\,\varphi_n(x)$

$\varphi_n(x) = \sin\frac{n\pi x}{\ell}$ $\lambda_n = \frac{n\pi}{\ell}$

and they are orthogonal for different n

over $x = [0, \ell]$ w.r.t. $w = 1$

i.e.
$$\int_0^\ell \sin\frac{n\pi x}{\ell}\sin\frac{m\pi x}{\ell}\,dx = \begin{cases} 0 & \text{if } n \neq m \\ \ell/2 & \text{if } n = m \end{cases}$$

We say that $\varphi_n(x)$ are eigenfunctions of the SL problem with eigenvalues

$\alpha_n = \lambda_n = \frac{n\pi}{\ell} \quad n = 1, 2 \ldots,$

Because

$$\begin{cases} -\mathcal{L}\varphi_n = \lambda_n \varphi_n \\ \text{with } \varphi_n(0) = \varphi_n(\ell) = 0 \end{cases}$$

$$\mathcal{L} = \frac{d^2}{dx^2}, \quad \lambda_n = \frac{n^2\pi^2}{\ell^2}$$

//

We sometimes find it convenient to normalize the eigenfunctions, then we call them orthonormal:

For the example we just did

$$(\$) \qquad \int_0^\ell \hat{\varphi}_n(x)\, \hat{\varphi}_m(x)\, dx = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{otherwise.} \end{cases}$$

Define $\delta_{nm} \equiv \begin{cases} 1 & n = m \\ 0 & n \neq m \end{cases}$ then $\int_0^\ell \hat{\varphi}_n(x)\hat{\varphi}_m(x)\, dx = \delta_{nm}$.

Let $\hat{\varphi}_j(x) = A\varphi_j(x)$, A is the "normalization factor." Find A s.t. ($) is true:

$$\int_0^\ell \varphi_i(x)\varphi_j(x)\,dx = \frac{\ell}{2}\delta_{ij}$$

or

$$\frac{2}{\ell}\int_0^\ell \varphi_i(x)\varphi_j(x)\,dx = \delta_{ij}$$

$$\therefore \sqrt{\frac{2}{\ell}}\,\varphi_j(x) \equiv \hat{\varphi}_j(x)$$

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EXPANSION OF A FUNCTION IN A SERIES OF ORTHOGONAL FUNCTIONS, on A FIXED INTERVAL $I \equiv [a,b]$

Suppose we know of a family of orthogonal functions $\{\varphi_n(x)\}$ on a fixed interval $[a,b]$. Also, Suppose $f(x)$ has boundary values consistent with $\varphi_n(x)$. Then

$$f(x) = \sum_{n=0}^{\infty} \delta_n \varphi_n(x)$$

where δ_n are coefficients. This is called an eigenfunction expansion of $f(x)$ in terms of eigenfunctions $\{\varphi_n(x)\}$, orthogonal over $[a, b]$ w.r.t $w(x)$, the weight ($w(x) \geq 0$) with

$$\int_a^b w(x)\,dx = C, \text{ a number.}$$

We will also require that

$$\int_a^b |f(x)|^2 dx < \infty$$

i.e. $f(x)$ is an L_2 function.

To find the unknown coefficients A_n:

$$w(x) f(x) = \sum_{n=0}^{\infty} A_n w(x) \phi_n(x)$$

Multiply both sides by $\phi_m(x)$ and integrate from $x = a, b$:

$$\int_a^b w(x) f(x) \phi_m(x)\, dx$$

$$= \int_a^b \sum_{n=0}^{\infty} A_n w(x) \phi_n(x) \phi_m(x)\, dx$$

$$= \sum_{n=0}^{\infty} A_n \int_a^b w(x) \phi_n(x) \phi_m(x)\, dx$$

$$= A_m \int_a^b w(x) \phi_m(x) \phi_m(x)\, dx$$

$$= A_m Q$$

where

$$Q = \int_a^b w(x)\, \varphi_m^2(x)\, dx$$

Solving for

$$A_m = \frac{1}{Q} \int_a^b w(x)\, f(x)\, \varphi_m(x)\, dx$$

$m = 0, 1, 2 \ldots$

There are many families of orthogonal functions

these are associated with SL problems

Family	interval	weight $w(x)$
$\sin\frac{n\pi x}{\ell}, \cos\frac{n\pi x}{\ell}$	$x = [-\frac{\ell}{2}, \frac{\ell}{2}]$	1
Legendre	$-1 \le x \le 1$	1

Chebyshev	$-1 \le x \le 1$	$(1-x^2)^{-1/2}$
Hermite	$-\infty < x < \infty$	e^{-x^2}
Bessel	$0 \le x < \infty$	x

etc

//

There is indeed a connection between SL problem and the eigenvalue problem of certain (finite-dimensional) matrices:

$$\mathcal{L} y = -\lambda w y \quad \text{plus B.C.}$$

or $\left[\frac{1}{w}\mathcal{L} + \lambda\right] y = 0$ or $Ly = \lambda y$

Which looks like

$$(3) \quad A z_m = \lambda_m z \quad \text{where}$$

A is Hermitian : $A^* = A$ (self-adjoint)

A is $n \times n$ matrix. $z_m \in \mathbb{R}^n$, $m = 0, 1 \ldots$ and $\lambda_m \in \mathbb{R}^1$, z_m is the eigenvector and λ_m the associated e'value.

Like the SL, ($\mp$) has real eigenfunctions and real eigenvalues.

Like the SL, ($\mp$) has an infinite, unique eigenvalues that can be arranged as

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots \lambda_n < \cdots$$

where $\lim_{n \to \infty} \lambda_n = \infty$.

So each eigenvalue λ_n has only 1 eigenfunction z_n.

ex) Eigenfunction expansion: let $f(x) = x$

let $I = [0, \ell]$

expand $f(x)$ into a series of eigenfunctions of the SL problem

$$\text{SL} \begin{cases} y'' + k^2 y = 0 \\ y(0) = y(\ell) = 0 \quad \text{B.C.} \end{cases}$$

Rmk: here the B.C. are not consistent with $f(x) = x$

The SL solutions are

$$\varphi_n(x) = \sin\frac{n\pi x}{\ell} \qquad k_n = \frac{n\pi}{\ell}$$

$n = 1, 2, \ldots$

These are orthogonal on $x = [0, \ell]$ wrt $w(x) = 1$

$$f(x) = \sum_{n=0}^{\infty} a_n \varphi_n(x)$$

$$w(x) f(x) \varphi_m(x) = \sum_{n=0}^{\infty} a_n \varphi_n(x) \varphi_m(x) w(x)$$

$w=1$, integrate B.S. from $x=0, l$:

$$\int_0^l f(x) \sin\frac{m\pi x}{l}\, dx = \sum_{n=0}^{\infty} a_n \int_0^l \varphi_m(x) \varphi_n(x)\, dx$$

$$= a_m \int_0^l \varphi_m^2(x)\, dx$$

$$= a_m \int_0^l \sin^2 \frac{m\pi x}{l}\, dx = a_m \frac{l}{2}$$

$$\therefore \quad a_m = \frac{2}{l} \int_0^l f(x)\, \varphi_m(x)\, dx$$

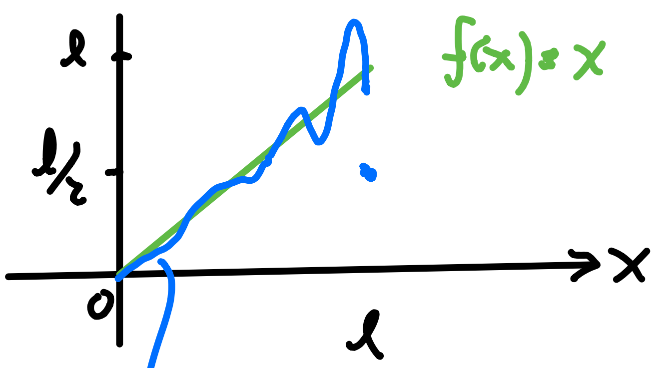
$$a_m = \frac{2}{l}\int_0^l x \sin\frac{m\pi x}{l}\,dx$$

(exercise)

$$a_m = -\frac{2l}{m\pi}\cos m\pi = \frac{2l}{m\pi}(-1)^{m+1}$$

$m = 1, 2, \ldots$

$$\therefore \quad x = \sum_{n=0}^{\infty} a_n \varphi_n(x) = \sum_{n=1}^{\infty} \frac{2l}{\pi}\,\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l}$$



$$\sum_{n=0}^{\infty} a_n \varphi_n(x)$$

Rmk: Since $f(x)$ is not periodic, the eigenfunction expansion will not converge in the uniform norm:

$$\lim_{N\to\infty} \left| f(x) - \sum_{n=0}^{N} a_n \varphi_n(x) \right|$$ we will not get 0.

for $x \in [0, \ell]$

instead

$$\lim_{N\to\infty} \int_0^\ell \left| f(x) - \sum_{n=0}^{N} a_n \varphi_n(x) \right|^2 dx = 0$$ L_2 convergence

If the function $f(x)$ is periodic we would get L_2 and L_∞ convergence.

See notes for tricks involving periodic extensions