SOLVING INHOMOGENEOUS EQUATIONS FOR HEAT FLOW

Special Case

PDE $\quad \dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial x^2} + q(x) \qquad t>0,\ 0<x<L$

B.C. $\quad u_x(0,t) = A \qquad u(L,t) = B \qquad t>0$

I.C. $\quad u(x,0) = f(x) \qquad 0<x<L$

Rmk: what's special is that $q(x,t) = q(x)$ only (i.e. only spatially-dependent), and further, the B.C. are not time dependent.

Rmk: The above problem is linear in $u(x,t)$ so we'll use linear superposition.

We know that the solution to $u_t = \nu u_{xx}$ has exponentially decaying solutions. So in IBVP

let $\quad u(x,t) = v(x,t) + U(x)$

Substituting into IBVP:

PDE $\quad \dfrac{\partial v}{\partial t} - \nu \dfrac{\partial^2 v}{\partial x^2} = \nu \dfrac{\partial^2 U}{\partial x^2} + g(x)$

B.C. $\quad v_x(0,t) + U'(0) = A$

$\qquad v(L,t) + U(L) = B$

$\therefore \quad$ Take $\quad \text{II} \begin{cases} \nu \dfrac{\partial^2 U}{\partial x^2} = -g(x) \\ U'(0) = A \\ U(L) = B \end{cases}$

we're left with

$$\text{II} \begin{cases} \dfrac{\partial v}{\partial t} = \nu \dfrac{\partial^2 v}{\partial x^2} \\ v_x(0,t) = 0 \\ v(L,t) = 0 \end{cases} \qquad v(x,0) = f(x)$$

Solving II: integrate twice

$$U(x) = A(x-L) + B + \frac{1}{\nu}\int_x^L ds \int_0^s q(r)\,dr$$

(exercise)

System III (see previous notes for the solution)

then reassemble

$$u(x,t) = V(x,t) + U(x)$$

Rmk:
$$\lim_{t\to\infty} u(x,t) = \lim_{t\to\infty} v(x,t) + \lim_{t\to\infty} U(x) = U(x)$$

Rmk: $U(x)$ is called the asymptotic steady state solution

We call $v(x,t)$ the transient solution.

ANOTHER SPECIAL CASE:

IBVP
$$\begin{cases} \text{PDE} & \dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial x^2} + g(x,t) \quad 0<x<L,\ t>0 \\ \text{B.C.} & \dfrac{\partial u}{\partial x}(0,t) = A(t) \\ & u(L,t) = B(t) \\ \text{I.C.} & u(x,0) = f(x) \end{cases}$$

<u>Rmk:</u> The general solution to this IBVP is via GREEN'S FUNCTIONS covered later on.

Here we are going to try a trick which sometimes works. Again, use linear superposition

$$(\dagger) \qquad u(x,t) = v(x,t) + K(x,t)$$

(hopefully) each of these yields solvable problems.

Let's be specific to see how this plays out.

$$L = 1 \qquad \nu = 5$$

$$q(x,t) = t^2$$

$$A = \sin t$$

$$B = 2$$

$$f(x) = e^x$$

The general idea is to generate a solution $v(x,t)$ that correspond to a PDE that's non-homogeneous, but with homogeneous B.C.

Using (#)

$$\frac{\partial}{\partial t}(v+k) = \nu \frac{\partial^2}{\partial x^2}(v+k) + q(x,t)$$

$$\frac{\partial}{\partial x}(v+K)(0,t) = A(t)$$

$$(v+K)(L,t) = B(t)$$

$$(v+K)(x,0) = f(x)$$

We choose $K(x,t)$ so that the "v problem" has homogeneous B.C.

$$K(x,t) = xA(t) - LA(t) + B(t)$$

(a very good guess!)

Note that

$$\frac{\partial K}{\partial x}(0,t) = A(t)$$

$$K(L,t) = B(t)$$

With this $K(x,t)$, the IVBP becomes:

$$③\begin{cases} \dfrac{\partial v}{\partial t} - \nu \dfrac{\partial^2 v}{\partial x^2} = q - [x A'(t) - L A'(t) + B'(t)] \\ \qquad \equiv Q \\ \dfrac{\partial v}{\partial x}(0,t) = 0 \\ v(L,t) = 0 \\ v(x,0) = f(x) - [x A(0) - L A(0) + B(0)] \equiv F(x) \end{cases}$$

homogeneous!

We'll call this the "v problem".

<u>Rmk:</u> this form of K leads to a "v problem" that has homogeneous B.C.

To proceed,

We will use the associated SL problem:

$$\begin{cases} \varphi'' + \lambda^2 \varphi = 0 \\ \varphi'(0) = 0 \quad \varphi(L) = 0 \end{cases}$$

the B.C. were chosen to be consistent with those of the "v problem". Hence,

$$\varphi_n(x) = \cos\left[\frac{(n+\frac{1}{2})\pi}{L} x\right] \quad n = 0, 1, \ldots$$

$$\lambda_n = \frac{(n+\frac{1}{2})\pi}{L} \quad n = 0, 1, \ldots$$

$$\int_0^L \varphi_n^2 \, dx = \frac{L}{2}$$

Use this e'function family to write

$$\boxed{v(x,t) = \sum_{n=0}^{\infty} \psi_n(t) \varphi_n(x)} \quad (1)$$

Also expand

$$Q(x,t) = \sum_{m=0}^{\infty} a_m(t)\varphi_m(x) \quad (2)$$

Using orthogonality to find

$$a_m(t) = \frac{2}{L}\int_0^L Q(x,t)\,\varphi_m(x)\,dx$$

So $a_m(t)$ are thus known.

Substitute (2) & (1) into (3) "v problem"

The PDE gives:

$$\sum_{n=0}^{\infty}\left[\psi_n'\varphi_n - \nu\,\psi_n \varphi_{n\,xx}\right] = \sum_{n=0}^{\infty} a_n(t)\,\varphi_n(x).$$

We want to find equation satisfied by $\psi_n(t)$:

know that $\varphi_{n\,xx} = -\lambda_n^2 \varphi_n(x)$

$$\therefore \sum_{n=0}^{\infty} \left[\psi_n'(t) + \nu \lambda_n^2 \psi_n(t)\right] \varphi_n(x) - \sum_{n=0}^{\infty} a_n(t)\,\varphi_n(x) = 0$$

Multiply B.S. by $\varphi_m(x)$ and integrate from 0 to L (w.r.t weight = 1), We get:

$$\therefore \quad \psi_n'(t) + \nu \lambda_n^2 \psi_n = a_n \qquad (\ast)$$

$n = 0, 1, \ldots$

The equation for the unknown $\psi_n(t)$.

$(\ast)$ is a linear first order ODE

We can solve it:

$$\psi_n = C_n e^{-\nu\lambda_n^2 t} + \int_0^t Q_n(s) e^{-\nu\lambda_n^2 (t-s)}\, ds$$

C_n are constants, set by I.C. of 3

Apply I.C. of 3

$$F(x) = v(x,0) = \sum_{n=0}^{\infty} \psi_n(0)\,\varphi_n(x)$$

$$= \sum_{n=0}^{\infty} C_n \varphi_n(x)$$

$$\therefore \quad C_n = \frac{2}{L}\int_0^L F(x)\,\varphi_n(x)\,dx$$

Let's go back to the explicit case:

$Q = t^2$, $A = \sin t$, $B = 2$, $f(x) = e^x$

$L = 1$ $\quad \nu = 5$

$\therefore\ u(x,t) = v(x,t) + \underbrace{x\sin t - \sin t + 2}_{K(x,t)}$

The ③ problem:

PDE $\quad \dfrac{\partial v}{\partial t} = 5\dfrac{\partial^2 v}{\partial x^2} + Q(x,t)$

$$Q(x,t) = t^2 - [x\cos t - \cos t]$$

$$= t^2 - (x+1)\cos t$$

B.C. $\quad \dfrac{\partial v}{\partial x}(0,t) = 0 \qquad v(1,0) = 0$

I.C. $\quad v(x,0) = e^x - 2$

$$Q(x,t) = \sum_{n=0}^{\infty} a_n(t)\,\varphi_n(x)$$

$$a_n(t) = 2\int Q(x,t)\,\varphi_n(x)\,dx$$

$$= 2\int_0^1 \left(t^2 - (x+1)\right)\varphi_n(x)\,dx$$

$$a_n(t) = t^2 - 2\cos t \int_0^1 (x+1)\,\varphi_n(x)\,dx$$

$$a_n(t) = t^2 - \frac{8\cos t}{\pi^2(4n^2+4n+1)}\left[2\pi n(-1)^n\left(1+\frac{\pi}{2}\right)-1\right]$$

$n = 0, 1 \ldots$

$$\frac{d\psi_n}{dt} + 5\lambda_n^2 \psi_n = a_n(t)$$

$$\psi_n(t) = c_n e^{-5\lambda_n^2 t} + \int_0^t a_n(s) e^{-5\lambda_n^2 (t-s)}\, ds$$

$$c_n = 2\int_0^1 (e^{-x} - 2)\, \varphi_n(x)\, dx$$

$$c_n = \frac{2\pi(1-e)(-1)^n}{\left[1+\pi^2\left(n+\frac{1}{2}\right)^2\right]} - \frac{4}{\pi\left(n+\frac{1}{2}\right)\left[1+\pi^2\left(n+\frac{1}{2}\right)^2\right]}$$

$n = 0, 1, \ldots$

(needs to be checked...)

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