CLASSICAL LINEAR PARTIAL DIFFERENTIAL EQUATIONS (PDE)

PARABOLIC PDE (for example the Heat Equation)

$$\text{(HE)} \qquad \Delta \varphi = \frac{1}{\alpha} \frac{\partial \varphi}{\partial t} + f(\underset{\sim}{r}, t)$$

$$t \geq t_0 \quad , \alpha > 0$$

$+$ B.C.

$\underset{\sim}{r}$ is space, in 2D $\underset{\sim}{r} = (x, y)$, 3D $\underset{\sim}{r} = (x, y, z)$

t is time, t_0 is the initial time,

α is the diffusivity constant.

$f(\underset{\sim}{r}, t)$ is the known forcing (sources/sinks)

Δ is the Laplacian

in 2D (cartesian) the Laplacian becomes

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

in 3D (cartesian)

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad , \text{etc.}$$

$\varphi(\underset{\sim}{r}, t)$ represents heat, but also models other fields.

There are solutions to the HE that

$$\frac{\partial \varphi(\underset{\sim}{r}, t)}{\partial t} = 0, \text{ i.e. stationary.}$$

For these $\varphi = \varphi(\underset{\sim}{r})$ only depend on space.

These stationary fields satisfy the Elliptic Problem (for example, Poisson Equation:

$$\text{(PE)} \qquad -\Delta \varphi = f(\underset{\sim}{r})$$

if $f(\underset{\sim}{r}) = 0$

(LE) $\Delta\varphi = 0$

The Laplace Equation

There is a 4^{th} order counterpart to Laplace's equation (also elliptic)

$$\nabla^4\varphi = \Delta^2\varphi = f(r)$$

The BiHarmonic Equation

Rmk: $f(r)$ can be zero in Biharmonic equation.

HYPERBOLIC EQUATIONS (The Wave Equation)

(WE) $\Delta\varphi = \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} + f(\underset{\sim}{r}, t)$

c is the wave speed, a real quantity. f is source/sink.

SOME PDES ARE MIXTURES OF THE ABOVE TYPES:

The Advection-Diffusion Equation

$$\frac{\partial \varphi}{\partial t} + \underset{\sim}{u} \cdot \nabla \varphi = \alpha \Delta \varphi$$

Here $\underset{\sim}{u} = \underset{\sim}{u}(\underset{\sim}{r}, t)$ is prescribed

Rmk: if $\alpha \Delta \varphi = 0$ then $\frac{\partial \varphi}{\partial t} + \underset{\sim}{u} \cdot \nabla \varphi = 0$ "Advection Equation"

Schrödinger Equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \psi + V(\underset{\sim}{r})\psi \equiv H\psi$$

$\hbar = \frac{h}{2\pi}$, here h is Planck's Constant

m mass

$V(\underset{\sim}{r})$ is the potential (prescribed)

ψ is complex.

Rmk: since time is complex, the solutions are not like those of a parabolic problem.

The Helmholtz Equation

$$\text{(HME)} \quad \Delta\psi + k^2\psi = f(\underset{\sim}{r})$$

$\psi = \psi(\underset{\sim}{r})$, $k^2 \geq 0$ a constant

f is a source/sink

Hyperbolic Problems are characterized by wave-like solutions. These propagate with finite speed.

The fact that this equation is second order is not a necessary or condition for wave-like behavior. The equation of the form

$$\varphi_t + a\varphi_x = 0$$

(in 1 space dimension) has

wave solutions. This is a special case of the advection equation

$$\frac{\partial \varphi}{\partial t} + \underset{\sim}{u} \cdot \nabla \varphi = 0$$

where $\underset{\sim}{u} = a\hat{i}$ a constant, and $\nabla = \partial_x \hat{i}$

So Advection equation has wave-like solutions.

Parabolic Equations have "heat-like" solutions. They dissipate and spread, they do not conserve energy. They are related to the advection-diffusion equation

$$\frac{\partial \varphi}{\partial t} + \underset{\sim}{u} \cdot \nabla \varphi = \alpha \Delta \varphi + f$$

when $\underset{\sim}{u}$ is negligible. The solutions "propagate" at infinite speeds.

Elliptic Equation solutions do not depend on time. They can arise from the advection diffusion equation when $\frac{\partial \varphi}{\partial t} = 0$ and $\underset{\sim}{u} = 0$, i.e.

$$-\Delta \varphi = f$$

or $\quad \varphi = -\Delta^{-1} f$

$-\Delta^{-1}$ is the "inverse Laplacian operator", found via Green's function.

We think of solutions to Poisson/Laplace equations as being non-local.

Rmk: The above problems are LINEAR and have analytical solutions for simple problems.
There are NONLINEAR counterparts and only exceptional examples have analytic solutions.

Rmk: We will discuss linear problems with two types of boundary conditions:

(A) BOUNDED DOMAINS (SPACE &/OR TIME) Initial/Boundary Value Problems.

(B) UNBOUNDED DOMAINS (SPACE &/OR TIME) Cauchy Problems.