SOLUTION TO THE HEAT EQUATION (PARABOLIC)

Let T be the temperature of a rod (so 1 space dimension).

$$T = T_0 + u(x,t)$$

T_0 is the known ambient temperature,

$u(x,t)$ is the fluctuating temperature.

The x represents space

t is time $t \geq 0$

The bar has finite extent $0 \leq x \leq \ell$

The bar has a known constant thermal diffusivity $\nu \geq 0$ (a constant)

PDE $\quad \dfrac{\partial u}{\partial t} = \nu \dfrac{\partial^2 u}{\partial x^2} \qquad t>0$

$\qquad\qquad\qquad\qquad\qquad\qquad 0 \le x \le l$

I.C. $\quad u(x,0) = f(x) \qquad 0 \le x \le l$

B.C. $\quad u(0,t) = u(l,t) = 0$

Dirichlet (Homogeneous) B.C.

Homogeneous PDE, i.e. No source/sink.

Rmk: - The bar temperature initially $(t=0)$ is

$T = T_0 + f(x)$

- The bar temperature for all time at $x=0$ & $x=l$ is $T = T_0$

METHOD OF SOLUTION

SEPARATION OF VARIABLES

Assume a solution of the form:

$$(*) \quad u(x,t) = \varphi(x)\psi(t)$$

"Separable form"

then substitute into the IBVP (initial boundary value problem): PDE + BC + IC

Substituting (*) into PDE:

$$\frac{\partial}{\partial t}\varphi(x)\psi(t) = \nu\frac{\partial^2}{\partial x^2}\varphi(x)\psi(t)$$

rearrange:

$$(**) \quad \varphi(x)\frac{\partial\psi}{\partial t} = \nu\frac{\partial^2\varphi}{\partial x^2}\psi(t)$$

If the seperation of Variables procedure is going to succeed, we should be able to write (**) in the form

$$G(x) = H(t)$$

i.e. the sides are exclusively functions of either x or t. So (**) divided by $u = \varphi\psi$:

$$\frac{1}{\psi}\frac{\partial\psi}{\partial t} = \nu \frac{\partial^2\varphi}{\partial x^2}\frac{1}{\varphi}$$

So we can proceed.

$$\frac{1}{\nu}\frac{1}{\psi}\psi_t = \frac{1}{\varphi}\varphi_{xx}$$

The most general solution for ψ & φ requires that b.s. equal to a constant, i.e.

$$\frac{1}{\nu}\frac{\psi_t}{\psi} = \frac{\varphi_{xx}}{\varphi} = -k^2, \text{ a constant}$$

$$(\$) \quad \psi_t = -k^2\nu\psi \quad \text{or} \quad \psi_t + k^2\nu\psi = 0$$

$$\varphi_{xx} = -k^2\varphi \quad \text{or} \quad \varphi_{xx} + k^2\varphi = 0$$

We also need seperability of B.C.

$$u(0,t) = 0 \Rightarrow \varphi(0)\psi(t) = 0 \quad \varphi(0) = 0$$

$$u(l,t) = 0 \Rightarrow \varphi(l)\psi(t) = 0 \quad \varphi(l) = 0$$

So B.C. seperate, we can proceed.

$$\begin{cases} \varphi_{xx} + k^2\varphi = 0 \\ \varphi(0) = \varphi(\ell) = 0 \end{cases}$$

so $\varphi(x) = A\cos(kx) + B\sin(kx)$

$\varphi(0) = 0 = A$

$\varphi(\ell) = 0$ implies $k = \frac{n\pi}{\ell}$ $n = 1, 2, \ldots$

$\therefore$ $\varphi_n(x) = B_n \sin\frac{n\pi x}{\ell}$ $n = 1, 2, \ldots$

Looking ahead,

$$u(x,t) = \sum_{n=1}^{\infty} B_n(t) \sin\frac{n\pi x}{\ell}$$

or $$u(x,t) = \sum_{n=1}^{\infty} B_n(t)\varphi_n(x)$$

Go back to solving ($):

$\frac{1}{\psi}\psi_t = -k^2\nu \Rightarrow \psi(t) = De^{-k^2\nu t}$

$\therefore$

$$u(x,t) = \sum_{n=1}^{\infty} D_n e^{-\left(\frac{n\pi}{\ell}\right)^2 \nu t} \varphi_n(x)$$

How do we find D_n? We'll use the I.C.

$$u(x,0) = f(x)$$

$$\sum_{n=1}^{\infty} D_n \varphi_n(x) = f(x)$$

so multiply b.s. by $\varphi_m(x)$ & integrate:

$$\int_0^{\ell} dx\, \varphi_m(x) \sum_{n=1}^{\infty} D_n \varphi_n(x) = \int_0^{\ell} dx\, f(x)\, \varphi_m(x)$$

$$Z\, D_m = \int_0^{\ell} dx\, f(x)\, \varphi_m(x)$$

Z is the normalization constant

$$D_m = \frac{1}{Z} \int_0^{\ell} f(x)\varphi_m(x)\,dx$$

$m = 1, 2, \ldots\ldots$

$$Z = \int_0^{\ell} \sin^2\frac{n\pi x}{\ell}\,dx = \frac{\ell}{2}$$

$$D_m = \frac{2}{\ell} \int_0^{\ell} f(x)\varphi_m(x)\,dx$$

So we have solved for

$$u(x,t) = \sum_{n=1}^{\infty} D_n e^{-\left(\frac{n\pi}{\ell}\right)^2 \nu t} \sin\frac{n\pi x}{\ell}$$

Examine the behavior of solution in the homework.

EFFECT OF B.C.

We used Dirichlet B.C., meaning we prescribed u at the boundaries. In fact we prescribed $u = 0$ at boundaries. We say that

We are giving the IBVP Homogeneous Dirichlet B.C.

Other Boundary Conditions for heat equation:

at $x=a$, the boundary, you can specify

(A) $\frac{\partial u}{\partial x}(x=a,t) = h(t)$

Neumann (non Homogeneous if $h(t) \neq 0$)

if $h(t)=0$ we say that the boundary condition models an insulated end of the bar.

(B) $\frac{\partial u}{\partial x}(x=a,t) + c\,u(x=a,t) = g(t)$

Mixed or Robin B.C.

if $g(t)=0$ then homogeneous Robin B.C.

c is a constant

if $g(t)=0$ the Flux of heat is zero at

boundary.

You can have different boundary conditions at each end of the domain

(c) Can have periodic b.c.

$$u(x=a,t) = u(x=b,t) \quad \forall t$$

ex) PDE $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}$ $\quad t>0 \quad 0<x<l$

B.C. $\begin{cases} u(0,t) = 0 \\ \frac{\partial u}{\partial x}(l,t) = 0 \end{cases}$

I.C. $u(x,0) = f(x)$

assume: $u(x,t) = \psi(t)\,\varphi(x)$ subst into PDE

$$\psi_t + \nu k^2 \psi = 0 \qquad \varphi_{xx} + k^2 \varphi = 0$$

$$\varphi(0) = 0 \quad \frac{\partial \varphi}{\partial x}(l) = 0$$

as before, but different B.C. on $\varphi(x)$.

$\psi \sim e^{-\nu k^2 t}$, as before

Let's focus on $\begin{cases} \varphi_{xx} + k^2\varphi = 0 \\ \varphi(0) = \varphi'(l) = 0 \end{cases}$

$\varphi(x) = A\cos kx + B\sin kx$

$\varphi'(x) = -Ak\sin kx + Bk\cos kx$

$\varphi(0) = 0 = A$

$\varphi'(l) = 0 = Bk\cos kl$

$$k_n = \frac{(n+\frac{1}{2})\pi}{l} \quad n = 0, 1, \ldots$$

$$\varphi_n(x) = \sin k_n x = \sin\left[\frac{(n+\frac{1}{2})\pi x}{l}\right]$$

$n = 0, 1, \ldots$

$$u(x,t) = \sum_{n=0}^{\infty} a_n e^{-k_n^2 \nu t} \sin\left[\frac{(n+\frac{1}{2})\pi x}{\ell}\right]$$

$$a_n = \frac{2}{\ell}\int_0^{\ell} f(x) \sin\left[\frac{(n+\frac{1}{2})\pi x}{\ell}\right] dx$$

$n = 0, 1 \ldots$

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