USING FOURIER TRANSFORMS TO SOLVE PDE'S:

Consider the following Cauchy Problem:

Find $u(x,t)$

$$\begin{cases} \text{PDE} & \dfrac{\partial u}{\partial t} = D\dfrac{\partial^2 u}{\partial x^2} \quad x \in \mathbb{R}^1, \; t > 0 \\ \text{IC} & u(x,0) = f(x) \end{cases}$$

Assume $\int_{-\infty}^{\infty} |u(x,t)|\, dx < \infty$

and $\int_{-\infty}^{\infty} |f(x)|\, dx < \infty$

let $\left\{ u(x,t) = \dfrac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(k,t) e^{-ikx}\, dk \right.$

$$\Big(\hat{u}(k,t) = \int_{-\infty}^{\infty} u(x,t)\, e^{ikx}\, dx$$

substitute into PDE

$$\frac{\partial}{\partial t} \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{u}(k,t)\, e^{-ikx}\, dk$$

$$= D \frac{\partial^2}{\partial x^2} u(x,t) .$$

We use $\mathcal{F}(d^2 f/dx^2) = -k^2 \mathcal{F}(f)$ in what follows:

$$\int_{-\infty}^{\infty} \frac{\partial \hat{u}(k,t)}{\partial t} e^{-ikx}\, dk$$

$$= -D \int_{-\infty}^{\infty} k^2 \hat{u}(k,t)\, e^{-ikx}\, dk$$

i.e. $\mathcal{F}\left(\frac{\partial^2}{\partial x^2} u\right) = (ik)^2 \mathcal{F}(u)$

$$\int_{-\infty}^{\infty}\left[\frac{\partial \hat{u}}{\partial t}(k,t) + Dk^2\hat{u}(k,t)\right]e^{-ikx}\,dk = 0$$

$$\Rightarrow \quad \frac{d\hat{u}(k,t)}{dt} = -Dk^2\hat{u}(k,t) \qquad (*)$$

for each k.

$(*)$ is a seperable ODE that requires an initial condition:

$$\hat{f}(k) \equiv \mathcal{F}(u(x,0)) = \hat{u}(k,0) = \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx$$

so

$$(\#)\quad \begin{cases} \dfrac{d\hat{u}}{dt}(k,t) = -Dk^2\hat{u}(k,t) & t>0 \\ \hat{u}(k,0) = \hat{f}(k) \end{cases}$$

for each k: Solving $(\#)$ we obtain

$$\hat{u}(k,t) = \hat{f}(k)e^{-Dk^2t}$$

$\therefore \quad u(x,t) = \mathcal{F}^{-1}\left(\hat{U}(k,t)\right)$

$$(\ddagger) \quad u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{F}(k) e^{-k^2 Dt} e^{-ikx}\, dk$$

$(\ddagger)$ can be seen as a convolution of two functions, one is $f(x)$ and $\mathcal{F}^{-1}\left(e^{-k^2Dt}\right)$. To see this:

if $\quad \hat{S}(k) = \hat{F}(k)\,\hat{W}(k)$

then $\mathcal{F}^{-1}\left(\hat{S}(k)\right)$ is a convolution

$$S(x) = \int_{-\infty}^{\infty} f(\xi)\, W(x-\xi)\, d\xi$$

So, in $(\ddagger)$

$$\hat{S}(k,t) = \hat{F}(k)\,\hat{W}(k,t)$$

where $\hat{W}(k,t) = e^{-k^2Dt}$

$$\hat{f}(k) = \mathcal{F}^{-1}(f(x))$$

$$S(x,t) = f(x) * W(x,t)$$, where

$$W(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{W}(k,t)\, e^{-ikx}\, dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k^2Dt}\, e^{-ikx}\, dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k^2Dt}\,[\cos kx - i\sin kx]\, dk$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-k^2Dt}\cos kx\, dk$$

since $e^{-k^2Dt}\sin kx$ is ODD.

let $z \equiv k\sqrt{Dt}$

then

$$w(x,t) = \int_{-\infty}^{\infty} e^{-z^2} \frac{1}{\pi\sqrt{Dt}} \cos\left(\frac{xz}{\sqrt{Dt}}\right) dz$$

$$= \frac{1}{\pi\sqrt{Dt}} \int_{-\infty}^{\infty} e^{-z^2} \cos\left(\frac{xz}{\sqrt{Dt}}\right) dz$$

This integral appears in Tables of integrals (it is computed using residues in the complex plane)

$$w(x,t) = \frac{1}{\sqrt{4\pi Dt}} e^{-x^2/4Dt}$$

A normal distribution with variance $4Dt$.

The width of $w(x,t)$ increases as $t \to \infty$

The amplitude of $w(x,t) \propto \frac{1}{\sqrt{Dt}}$ as $t \to \infty$

$$u(x,t) = \int_{-\infty}^{\infty} f(\xi)\, w(x-\xi,t)\, d\xi$$

$$= \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} f(\xi)\, e^{-(x-\xi)^2/4Dt}\, d\xi$$

HEAT EQUATION ON AN INFINITE 2D DOMAIN

$u = u(x,y,t)$, the Cauchy problem is:

$$\text{HE} \begin{cases} \text{PDE} & \dfrac{\partial u}{\partial t} = D\,(u_{xx} + u_{yy}) \quad \text{in } \mathbb{R}^2,\ t>0 \\ \text{I.C.} & u(x,y,0) = f(x,y) \quad t=0 \end{cases}$$

We use a 2D Fourier Transform in x, y:

$$\hat{U}(\lambda,\mu,t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x,y,t)\, e^{i(\lambda x+\mu y)}\, dx\, dy$$

$$u(x,y,t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{U}(\lambda,\mu,t)\, e^{-i(\lambda x+\mu y)}\, d\lambda\, d\mu$$

Take $\mathcal{F}T$ of (HE):

(A) $$\frac{\partial \hat{U}}{\partial t}(\lambda,\mu,t) - k^2 D\, \hat{U}(\lambda,\mu,t) = 0 \quad t>0$$

$$k^2 = \mu^2 + \lambda^2$$

(B) $$\hat{f}(\lambda,\mu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{i(\lambda x+\mu y)}\, dx\, dy$$

$$= \mathcal{F}(u(x,y,0))$$

Solve IVP consisting of (A) & (B):

$$\hat{U}(\lambda,\mu,t) = \hat{F}(\lambda,\mu)\, e^{-k^2 D t}$$

Take $\mathcal{F}^{-1}(\hat{U}(\lambda,\mu,t)) = u(x,y,t)$:

$$u(x,y,t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{F}(\lambda,\mu)\, e^{-k^2 D t}\, e^{-i(\lambda x+\mu y)}\, d\lambda\, d\mu$$

FOURIER TRANSFORM FOR WAVE EQUATION IN 1 SPACE DIMENSION:

Cauchy Problem for waves:

PDE $\dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2}$ $\quad x \in \mathbb{R},\ t > 0$

I.C. $u(x,0) = f(x)$ $\quad \dfrac{\partial u}{\partial t}(x,0) = 0$

this could be non-zero.

FT pair $\begin{cases} u(x,t) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} \hat{U}(k,t)\, e^{-ikx}\, dk \\ \hat{U}(k,t) = \displaystyle\int_{-\infty}^{\infty} u(x,t)\, e^{ikx}\, dx \end{cases}$

FT the PDE & I.C.

$$\frac{1}{c^2}\int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial t^2} e^{ikx}\, dx = \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial x^2} e^{ikx}\, dx \quad \text{or}$$

$$\frac{1}{c^2}\,\frac{d^2\hat{U}}{dt^2} = -k^2 \hat{U} \quad \text{or}$$

$$(\bigstar)\quad \frac{d^2\hat{U}}{dt^2} + \omega^2 \hat{U} = 0$$

$$\omega^2 = \frac{k^2}{c^2}$$

The solution of $(\bigstar)$

$\hat{U}(k,t) = A(k)\cos\omega t + B(k)\sin\omega t$

We use I.C. to determine $A(k)$, $B(k)$.

We know that $\mathcal{F}(u(x,0)) = \mathcal{F}(f(x))$

$\mathcal{F}(u_t(x,0)) = 0$, i.e. no initial velocity.

Let $\hat{F}(k) = \mathcal{F}(f(x))$

$\hat{U}(k,t)$ when $t=0$ should be $\hat{F}(k)$:

$$\hat{U}(k,0) = A(k)\cos\omega 0 + B(k)\sin\omega 0$$
$$= A(k) = \hat{F}(k)$$

Since $\mathcal{F}(u_t(x,0)) = 0$

then $B(k) = 0$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-ikx}\hat{f}(k)\cos(kct)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{-ikx}\cos(kct)\int_{-\infty}^{\infty} dx'\, e^{ikx'} f(x')$$

We will manipulate this answer to show that the solution has 2 waves, propagating in opposite directions: this is known as the D'Alembert Solution to the wave equation:

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx'\, f(x')\int_{-\infty}^{\infty} dk\left(e^{-ikx}e^{ikx'}\cos(kct)\right)$$

$$\cos(kct) = \frac{1}{2}\left(e^{ikct} + e^{-ikct}\right)$$

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx' f(x') \int_{-\infty}^{\infty} \frac{dk}{2}\left[e^{-ik(x-x'-ct)} + e^{-ik(x-x'+ct)}\right]$$

Recall the Fourier transform pair for the Dirac Delta Function

$$u(x,t) = \frac{1}{2}\int_{-\infty}^{\infty} dx' f(x')\left[\delta(x-x'-ct) + \delta(x-x'+ct)\right]$$

Use the sifting property of Dirac Delta Function to evaluate the 2 integrals

$$u(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right]$$

D'Alembert's Solution

The solution consists of 2 counter propagating waves, with shape determined by the initial disturbance $u(x,0) = f(x)$ //

SEMI INFINITE DOMAIN

We might use Laplace transform for this case, but we might wind up with a challenge going from transform space to real space. A trick that allows us to use Fourier transforms is illustrated below:

Consider

$$\begin{cases} \text{PDE} & u_t = \nu u_{xx} \quad x>0 \ t>0 \\ \text{I.C.} & u(x,0) = \phi(x) \quad x>0 \end{cases}$$

Use the "Method of Reflection", through the $x=0$ boundary:

The idea is to solve another problem on the entire space domain (over the whole real line), which is consistent with the solution for $x>0$:

For the above problem the ODD extension works:

Let $\begin{cases} \psi(x) = \phi(x) & \text{for } x>0 \\ \psi(0) = 0 \\ \psi(x) = -\phi(-x) & \text{for } x<0 \end{cases}$

Solve for $v(x,t)$ $x \in \mathbb{R}, t>0$

obeying: $\begin{cases} v_t = \nu v_{xx} & x \in \mathbb{R}^1 \quad t>0 \\ v(x,0) = \psi(x) & x \in \mathbb{R}^1 \end{cases}$

Then $u(x,t) = v(x,t)$ for $x \geq 0$ $t \geq 0$:

Assume $\int_{-\infty}^{\infty} |v|\,dx < \infty$, $\int_{-\infty}^{\infty} |\psi|\,dx < \infty$

so we can use Fourier transforms: from previous notes, found that

$$(\dagger) \qquad v(x,t) = \int_{-\infty}^{\infty} w(x-y,t)\,\psi(y)\,dy$$

$$w(x,t) = \mathcal{F}^{-1}\left(e^{-\nu k^2 t}\right)$$

From $(\dagger)$ we can find $u(x,t)$ in terms of a convolution: splitting the integral,

$$v(x,t) = \int_{-\infty}^{0} w(x-y,t)\,\psi(y)\,dy$$

$$+ \int_{0}^{\infty} w(x-y,t)\,\psi(y)\,dy$$

Change of variable of integration on

let $y = -y' \Rightarrow dy = -dy'$

$$v(x,t) = -\int_{\infty}^{0} w(x+y,t)\,\psi(-y)\,dy + \int_{0}^{\infty} w(x-y,t)\,\psi(y)\,dy$$

$-\psi(-x) = \phi(x)$ for $x<0$

$$= + \int_{0}^{\infty} w(x+y,t)\,[-\phi(y)]\,dy$$

$\psi(y) = \phi(y)$

$$+ \int_{0}^{\infty} w(x-y,t)\,\phi(y)\,dy$$

$$\therefore u(x,t) = \int_{0}^{\infty} dy\,\big[w(x-y,t) - w(x+y,t)\big]\,\phi(y)$$

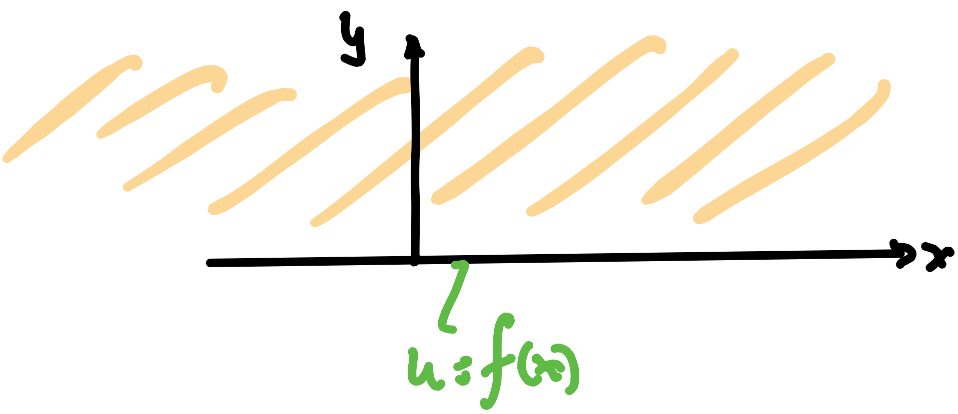
here $w(x,t) = \dfrac{1}{\sqrt{4\pi\nu t}}\, e^{-x^2/4\nu t}$

SOLVING THE ELLIPTICAL PROBLEM ON SEMI-INFINITE DOMAINS

Consider

PDE $\quad u_{xx} + u_{yy} = 0 \quad \left|\begin{array}{l} x \in \mathbb{R}^1 \\ 0 < y < \infty \end{array}\right.$

B.C. $\quad u(x,0) = f(x) \quad x \in \mathbb{R}^1$



The "Dirichlet Problem on a half Plane"

$$\hat{U}(k,y) = \int_{-\infty}^{\infty} u(x,y)\, e^{ikx}\, dx \quad (\#)$$

$$u(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{U}(k,y)\, e^{-ikx}\, dk \quad (\#\#)$$

Differentiate (#) twice wrt to y:

$$\frac{\partial^2 \hat{U}}{\partial y^2} = \frac{\partial^2}{\partial y^2} \int_{-\infty}^{\infty} u(x,y)\, e^{ikx}\, dx$$

use the PDE, i.e. $u_{xx} = -u_{yy}$

$$\frac{\partial^2 \hat{U}}{\partial y^2} = - \int_{-\infty}^{\infty} u_{xx}\, e^{ikx}\, dx$$

Use the derivative rule for F.T.

Assume $\lim_{x \to \pm\infty} u(x,y) = 0$ for $0 < y < \infty$

Assume $\lim_{x \to \pm\infty} \frac{\partial u}{\partial x}(x,y) = 0 \quad 0 < y < \infty$

So when we integrate by parts twice (in x):

$$\frac{\partial^2 \hat{U}}{\partial y^2} = k^2 \int_{-\infty}^{\infty} u(x,y)\, e^{ikx}\, dx = k^2 \hat{U}$$

or

$\therefore$ A $\dfrac{d^2\hat{U}}{dy^2} - k^2\hat{U} = 0$ $k \in \mathbb{R}^1$

$0 < y < \infty$

at $y=0$ $u(x,0) = f(x)$

B $\hat{f}(k) = \int_{-\infty}^{\infty} f(x)e^{ikx}\,dx$

B constitutes data for the solution of A

Solving A:

$$\hat{U} = A(k)e^{ky} + B(k)e^{-ky}$$

To get a second solvability condition:

Assume $\hat{U}(k,y)$ is bounded as $y \to +\infty$

$\therefore\ \hat{U}(k,y) = B(k)e^{-ky}$ C

However, Recall that $k \in \mathbb{R}^1$, so for $k<0$ C is unbounded for $y \geq 0$.

So we modify (C) :

$$\hat{u}(k,y) = B(k)\, e^{-|k|y}$$

valid for $-\infty < k < \infty$.

Use $\hat{u}(k, y=0) = \hat{f}(k)$

$$\therefore \hat{u}(k,y) = \hat{f}(k)\, e^{-|k|y}$$

To find $u(x,y)$ use $\mathcal{F}^{-1}(\hat{u}(k,y))$:

$$u(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{-ikx} \left[\hat{f}(k)\, e^{-|k|y} \right] \quad (€)$$

is the general solution.

We can recast (€) as follows:

$$u(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk\, e^{-ikx} \left[\int_{-\infty}^{\infty} f(\xi)\, e^{ik\xi}\, d\xi \right] e^{-|k|y}$$

We can write (€) as a convolution:

From a Table of transforms:

$$\mathcal{F}\left(\frac{1}{\sqrt{\pi}} \frac{a}{x^2+a^2} \right) = e^{-|k|a}$$

this result can be obtained using complex variable integration.

Write

$$(\$) \quad u(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(k)\, \hat{G}(k,y) e^{-ikx}\, dk$$

where $\hat{G}(k,y) = e^{-|k|y}$

$\therefore$ We can use the convolution theorem to rewrite $(\$)$ as

$$u(x,y) = \int_{-\infty}^{\infty} f(\xi)\, g(x-\xi, y)\, d\xi$$

where $\mathcal{F}^{-1}\left(\hat{G}(k,y)\right) = \frac{1}{\sqrt{\pi}} \frac{y}{x^2+y^2} = g(x,y)$

$$\therefore \quad u(x,y) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y f(\xi)}{(x-\xi)^2 + y^2} \, d\xi$$

Poisson's Integral Formula for the solution of the elliptic problem on a half-plane.