INTEGRAL TRANSFORMS

AND FOURIER TRANSFORMS, IN PARTICULAR.

An integral transform

$$f(x) \longleftrightarrow \hat{F}(p)$$

is of the form

$$\hat{F}(p) = \int_a^b K(p,x)\, f(x)\, dx \quad (\dagger)$$

K is called the kernel

ex) The Laplace Transform:

$$\hat{F}(s) = \int_0^\infty e^{-st} f(t)\, dt \equiv \mathcal{L}(f(t))$$

other transforms: Fourier, Hilbert, Hankel, Bessel, etc.

INTUITION

The transform is a linear transformation.

In what follows, we'll consider a finite dimensional linear transformation:

Recall that a linear transformation from $\mathbb{R}^n \to \mathbb{R}^n$, say, is:

$$(*) \qquad b = Ax$$

$$x \in \mathbb{R}^n, \; b \in \mathbb{R}^n \quad A \in \mathbb{R}^{n \times n}$$

$(*)$ says that b is a linear combination of the columns of $A = \begin{pmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{pmatrix}$,

i.e. $\quad b = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$

x_i's are the coordinates or coefficients

But now think of the x_i's as being (in magnitude) the "weights" of the a_i's in defining b, i.e. how important each a_i is in building b.

If we now compare

$$(\$) \qquad \hat{F}(p) = \int_a^b K(p,x)\, f(x)\, dx$$

to the problem $b = Ax$, and consider a Riemann sum approximation to the integral ($), we can also see that the Riemann sum could be considered a linear transformation of $f(x)$ to $\hat{F}(p)$:

$$b(p_j) \approx \sum_{i=1}^{N} K(p_j, x_i)\, f(x_i)\, \Delta x_i$$

$$\Delta x_i = x_{i+1} - x_i$$

and $\quad b - a = \sum_{i=1}^{N} \Delta x_i$

$$\Delta p_j = p_{j+1} - p_j$$

and $p_b - p_a = \sum_{j=1}^{N} \Delta p_j$

then $b = A f$

where $b_j = b(p_j)$

$A = (a_{ji}) \Delta x_i$

$f = f(x_i)$

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Focus on Fourier Transform

$$\hat{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} f(t)\, dt \quad (\$)$$

We write this as

$$\hat{F}(\omega) = \mathcal{F}(f(t))$$

We also have the inverse transform

$$f(t) = \mathcal{F}^{-1}(\hat{F}(\omega))$$

$$f(t) = \mathcal{F}^{-1}(\hat{F}(\omega)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \hat{F}(\omega)\, d\omega$$

Can we relate the Fourier Transform to the Fourier Series?

Yes, shown informally below:

Recall that for $g(t)$ square integrable and periodic

$$(*) \quad g(t) = \sum_{k=-\infty}^{\infty} c_k e^{i\omega_k t}$$

$$g(t+T) = g(t)$$

T is the period $(T = 2\ell)$

$$\omega_k = \frac{2\pi k}{T} \qquad = \frac{\pi k}{\ell}$$

(if t is time then ω_k has units of $1/\text{time}$, T has units of time)

ω_k are called the "frequencies".

(*) says that, for some t, a linear superposition (of possibly) infinite orthogonal functions

$$e^{i\omega_k t} = \cos(\omega_k t) + i\sin(\omega_k t)$$

$$k = 0, \pm 1, \pm 2 \ldots$$

times the (complex) coefficients c_k converge to g at t.

We note that the distance between each frequency ω_k and ω_{k+1} is constant

$$\Delta\omega \equiv \omega_{k+1} - \omega_k = \frac{2\pi}{T}$$

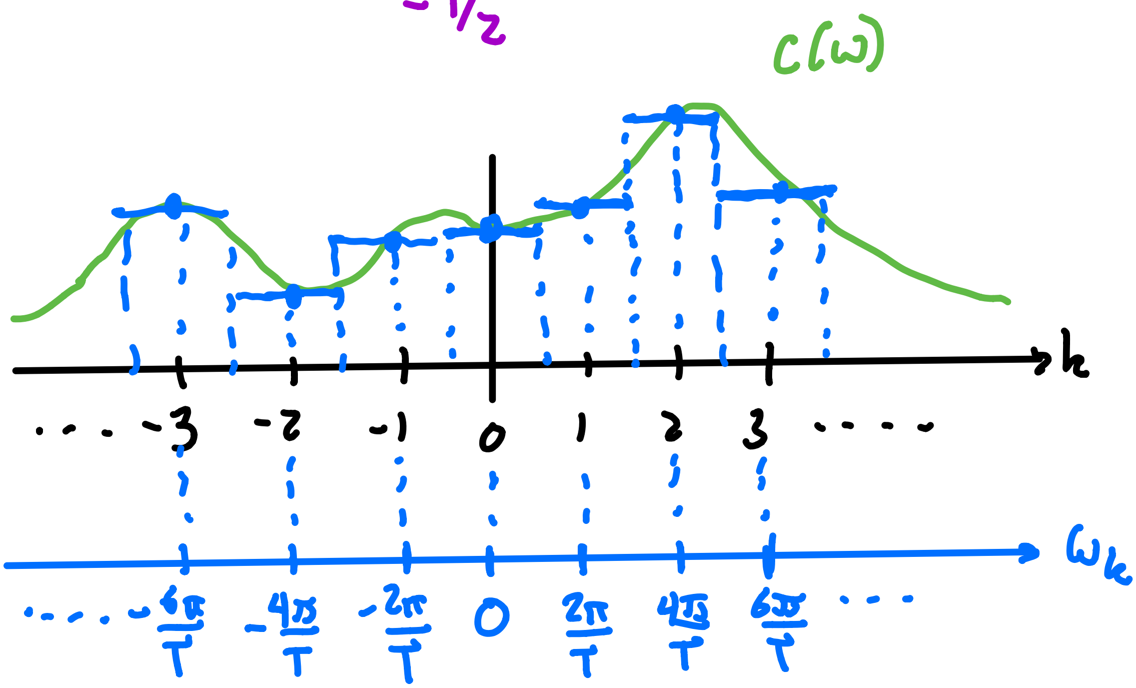
To find the coefficients c_k

$$(**) \quad c_k = \frac{1}{T}\int_{-T/2}^{T/2} g(t) e^{-i\omega_k t}\, dt$$

$$= \frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} g(t)\, e^{-i\omega_k t}\, dt$$

Substitute (**) into (*)

$$g(t) = \sum_{k=-\infty}^{\infty} \frac{\Delta\omega}{2\pi}\int_{-T/2}^{T/2} g(u)\, e^{-i\omega_k u}\, du\; e^{i\omega_k t}$$



Take some k box, with area centered about k, with area

$$\frac{2\pi}{T} c_k$$

we then multiply this area by $e^{i\omega_k t}$

$$\frac{2\pi}{T} c_k e^{i\omega_k t}$$

let $T \to \infty$ then $\Delta\omega = \frac{2\pi}{T} \to d\omega$

the distance between ω_k and ω_{k+1} becomes infinitesimally small, i.e. we get a continuum of frequencies and of "spectral" components $c(\omega)$.

At the same time we let $k \to \pm\infty$

so $$\sum_{k=-\infty}^{\infty} \frac{\Delta\omega}{2\pi} g(\omega_k) e^{i\omega_k t}$$

$$\to \frac{1}{2\pi} \int_{-\infty}^{\infty} g(\omega) e^{i\omega t} d\omega$$

where $g(\omega_k) = \int_{-T/2}^{T/2} g(u) e^{-i\omega_k u} du$

$$\therefore g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{i\omega t} \int_{-\infty}^{\infty} du\, g(u) e^{-i\omega u}$$

$$\mathcal{F}\left(f(t)\right) = f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$$\mathcal{F}^{-1}\left(f(\omega)\right) = f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\omega) e^{+i\omega t} d\omega$$

In the above, though we started with L_2 periodic functions, $g(t)$ must only be absolutely integrable (over the whole real line) to have a Fourier transform, i.e.

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty$$

so $f(t)$ is "L_1" function.

ex) Calculate the Fourier Transform of the Dirac Delta Function

The Dirac Delta function is actually a distribution, rather than a function.

$$\delta(x-a)$$

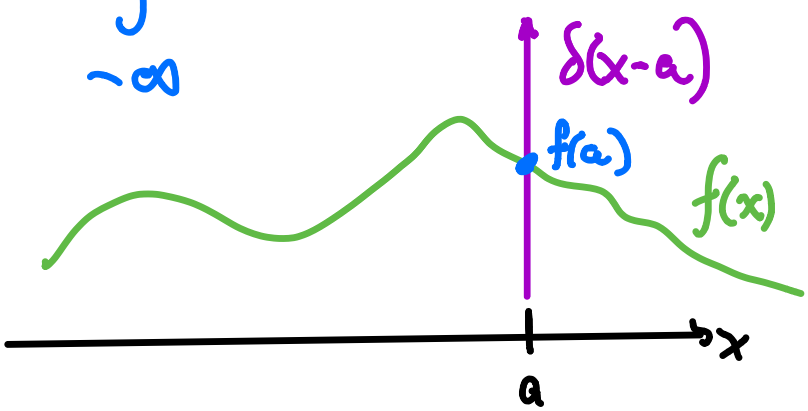
This is a Dirac Delta function situated at $x=a$. It has unbounded height and is infinitely thin (in x). Formally

$$\int_{-\infty}^{\infty} \delta(x-a)\, dx = 1$$

It has the sifting property:

Given a function $f(x)$

$$\int_{-\infty}^{\infty} \delta(x-a) f(x)\, dx = f(a) \quad (\ddagger)$$



The Dirac Delta function has a Fourier Transform:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{i\omega t}\int_{-\infty}^{\infty} du\, f(u)\, e^{-i\omega u}$$

$$= \int_{-\infty}^{\infty} du\, f(u)\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-u)}\, d\omega\right\}$$

Compare this expression to $(\mp)$

we deduce that

$$\delta(t-u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} e^{-i\omega u}\, d\omega$$

$$\therefore \mathcal{F}\big(\delta(t-u)\big) = e^{-i\omega u}$$

Shift Theorem

$$e^{-i\alpha\omega}\hat{F}(\omega) = \mathcal{F}(f(t-\alpha))$$

(show this by change of variables)

ex) $\mathcal{F}^{-1}\left(e^{-i\omega t_0} e^{-\omega^2/4a^2}\right)$

Need to know $\mathcal{F}^{-1}\left(e^{-\omega^2/4a^2}\right)$

from Table $\mathcal{F}^{-1}\left(e^{-\omega^2/4a^2}\right) = (2a^2)^{1/2} e^{-a^2t^2}$

$\therefore \mathcal{F}^{-1}\left(e^{-i\omega t_0} e^{-\omega^2/4a^2}\right) = (2a^2)^{1/2} e^{-a^2(t-t_0)^2}$ //

Derivatives:

$$\mathcal{F}\left(f^{(n)}(t)\right) = (i\omega)^n \hat{\mathcal{F}}(f)$$

$f^{(n)} \equiv \dfrac{d^n f}{dt^n}$ and $\hat{\mathcal{F}}(f) = \mathcal{F}(f(t))$

(show this by integration by parts)

Convolution

$$h(t) = \int_{-\infty}^{t} f(\tau)\, g(t-\tau)\, d\tau = \int_{-\infty}^{t} g(\tau)\, f(t-\tau)\, d\tau$$

or $h(t) \equiv f(t) * g(t)$

$$\mathcal{F}(h(t)) = (2\pi)^{1/2}\, \mathcal{F}(f(t))\, \mathcal{F}(g(t))$$

or $\hat{H}(\omega) = (2\pi)^{1/2}\, \hat{F}(\omega)\, \hat{G}(\omega)$.

To show this:

first: $\mathcal{F}(f(t-a)) = \int_{-\infty}^{\infty} f(t-a) \frac{e^{-i\omega t}}{\sqrt{2\pi}}\, dt$

Let $s = t-a$

$ds = dt$

$$= \int_{-\infty}^{\infty} f(s) \frac{e^{-i\omega(s+a)}}{\sqrt{2\pi}}\, ds = e^{-i\omega a} \int_{-\infty}^{\infty} \frac{f(s) e^{-i\omega s}}{\sqrt{2\pi}}\, ds$$

$$= e^{-i\omega a}\,\mathcal{F}\big(f(t)\big)$$. Now, use this in:

$$\text{So}\quad \mathcal{F}\big(h(t)\big) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt\, e^{-i\omega t}\int_{-\infty}^{t} f(\tau)\,g(t-\tau)\,d\tau$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dt \int_{-\infty}^{t} d\tau\, f(\tau)\,g(t-\tau)\,e^{-i\omega t}$$

$$= \int_{-\infty}^{\infty} d\tau\, f(\tau)\,\hat{G}(\omega)\,e^{-i\omega\tau}$$

$$= \hat{G}(\omega)\int_{-\infty}^{\infty} d\tau\, f(\tau)\,e^{-i\omega\tau} = \hat{G}(\omega)\sqrt{2\pi}\,\hat{F}(\omega)$$

$$\therefore\quad \mathcal{F}\big(g * f\big) = \sqrt{2\pi}\,\hat{G}(\omega)\,\hat{F}(\omega)$$

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Parseval's Theorem (Energy Theorem)

$$\int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} d\omega \, |\hat{F}(\omega)|^2$$

i.e. $$\int_{-\infty}^{\infty} dt\, |f(t)|^2 dt = \int_{-\infty}^{\infty} dt \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega \hat{F}^* e^{-i\omega t} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega' \hat{F} e^{i\omega' t}$$

$$= \int_{-\infty}^{\infty} d\omega \hat{F}^*(\omega) \int_{-\infty}^{\infty} d\omega' \hat{F}(\omega') \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} dt e^{i(\omega' - \omega)t}}_{\delta(\omega-\omega')}$$

$$= \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \; \hat{F}^*(\omega) \hat{F}(\omega') \, \delta(\omega - \omega')$$

$$= \int_{-\infty}^{\infty} \hat{F}^*(\omega) F(\omega)\, d\omega = \int_{-\infty}^{\infty} |\hat{F}(\omega)|^2 d\omega$$

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