FOURIER SIN/COS SERIES

$(\star)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n x}{L}\right) + b_n \sin\left(\frac{2\pi n x}{L}\right)$$

$$a_n = \frac{2}{L}\int_0^L f(x) \cos\frac{2n\pi x}{L}\, dx$$

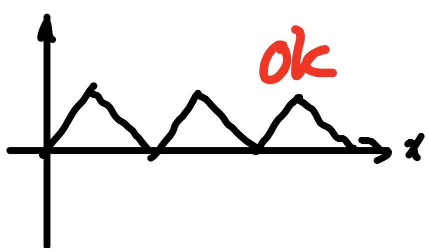
$$b_n = \frac{2}{L}\int_0^L f(x) \sin\frac{2n\pi x}{L}\, dx$$

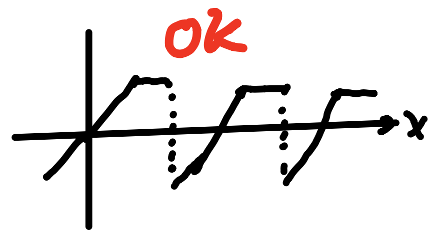
$n = 0, 1 \ldots$

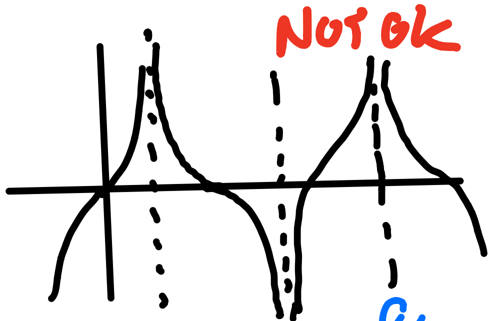
THIS SERIES REPRESENTATION APPLIES to $f(x)$ SUCH THAT:

* $f(x)$ continuous or a function with countable finite discontinuities

e.g.

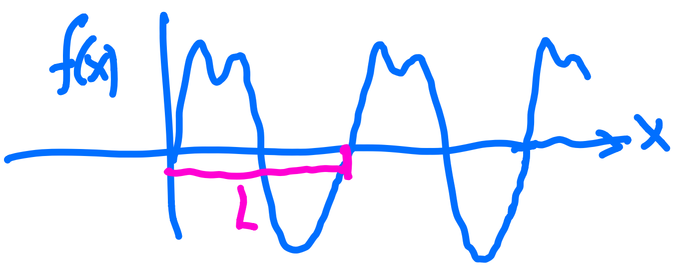






* $f(x)$ is Periodic, i.e. $f(x+L) = f(x) \;\; \forall x$

L is called the period, the smallest constant translation for which $f(x)$ repeats itself:



* Also
$$\int_{x_0}^{x_0+L} |f(x)|^2 dx < \infty$$

we say that $f(x)$ is

L_2 Periodic function

We can always write $f(x)$ as follows:

$$f(x) = f_o(x) + f_e(x)$$

$f_o(x)$ is ODD $\qquad f_e(x)$ is EVEN

$$f_o(x) = \frac{1}{2}\left[f(x) - f(-x)\right]$$

$$f_e(x) = \frac{1}{2}\left[f(x) + f(-x)\right]$$

COMPLEX FOURIER SERIES REPRESENTATION

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n x}{L}} \in \mathbb{C}$$

$c_n \in \mathbb{C}$

$$c_n = \frac{1}{L}\int_0^L f(x)\, e^{-\frac{2i\pi n x}{L}}\, dx$$

In terms of the coefficients a_n, b_n in (✱)
the c_n's can be seen to be:

$$c_n = \frac{1}{2}[a_n - ib_n], \quad c_{-n} = \frac{1}{2}[a_n + ib_n] \quad n = 0, 1, \ldots$$

Note that if $f(x)$ is real, then

$$c_{-n} = c_n^* \quad \text{(complex conjugates)}$$

A Few Useful Facts:

If $f = f_e \Rightarrow b_n = 0$ (even functions have cosine series)

If $f = f_o \Rightarrow a_n = 0$ (odd functions have sine series)

$$a_0 = \frac{1}{L}\int_0^L f(x)\,dx = \langle f(x) \rangle_L$$

Hence a_0 always represents the average of $f(x)$ over the period.

If $\langle f(x) \rangle_L = 0 \Rightarrow a_0 = 0$

PARSEVAL'S THEOREM

$$\frac{1}{L}\int_0^L |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

$$= \left(\frac{1}{2}a_0\right)^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

Take $\quad f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{2i\pi n x}{L}} \qquad g(x) = \sum_{n=-\infty}^{\infty} k_n e^{\frac{2\pi i n x}{L}}$

$$f g^* = \sum_{n=-\infty}^{\infty} c_n g^* e^{\frac{2\pi i n x}{L}}$$

complex conjugate

$$\frac{1}{L}\int_{x_0}^{x_0+L} f g^* dx = \sum_{n=-\infty}^{\infty} c_n \underbrace{\frac{1}{L}\int_{x_0}^{x_0+L} g^*(x) e^{\frac{2\pi i n x}{L}} dx}_{k_n^*}$$

$$\frac{1}{L}\int_{x_0}^{x_0+L} f g^* dx = \sum_{n=-\infty}^{\infty} c_n k_n^*$$

if $g = f \Rightarrow$

$$\frac{1}{L}\int_{x_0}^{x_0+L} |f|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

//

CONVERGENCE

$$S_N = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\frac{2\pi n x}{L} + b_n \sin\frac{2\pi n x}{L}$$

If f is L_2 periodic

$$\lim_{N\to\infty} \int_{x_0}^{x_0+L} |f(x) - S_N(x)|^2 dx = 0$$

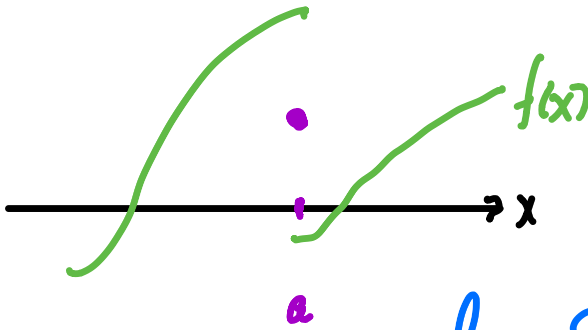
L_2 convergence

if, in addition, $f(x)$ is continuous

$$\lim_{N\to\infty} |f(x) - S_N(x)| = 0 \quad \forall x$$

uniform convergence.

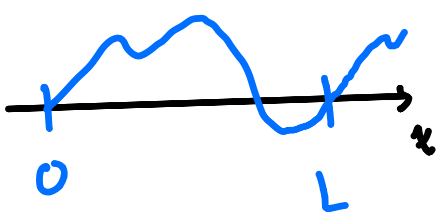
What does $S_N(x)$ converge to at discontinuities?

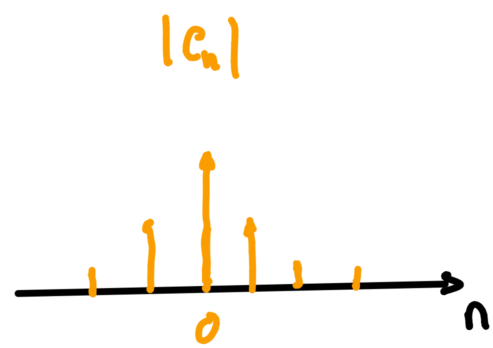


$$\lim_{N\to\infty} S_N(a) = \frac{1}{2}\left[f(a+\varepsilon) + f(a-\varepsilon)\right]$$

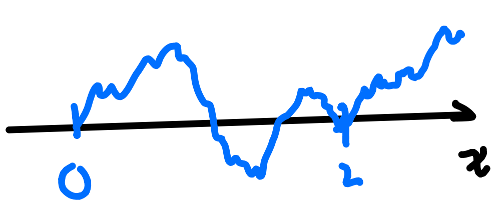
Useful Properties

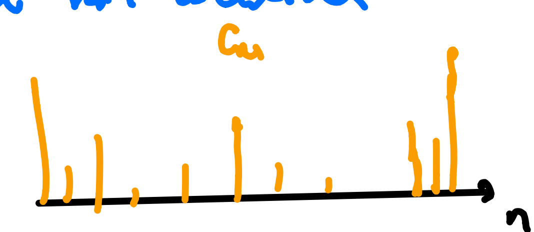
S'pose $f(x)$ is smooth and non-localized



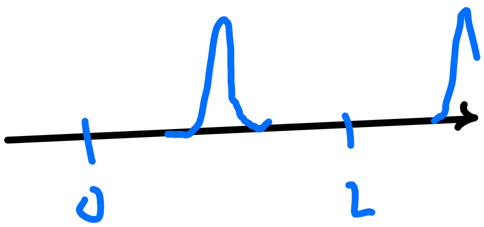


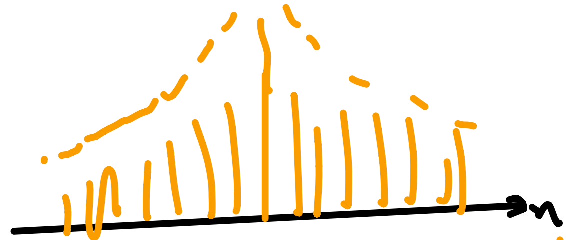
S'pose $f(x)$ is "rough" and non-localized





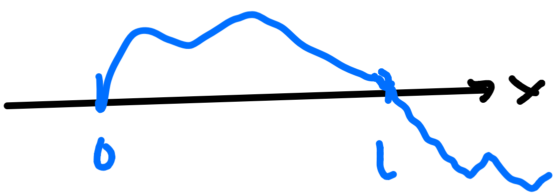
$f(x)$ compact

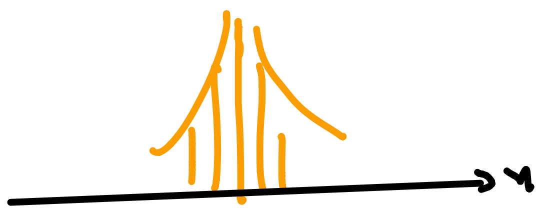




a lot of c_n's are important

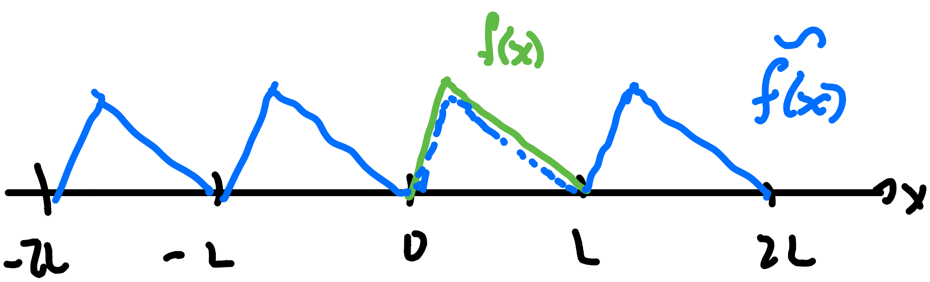
$f(x)$ not compact





only a few c_n's are important

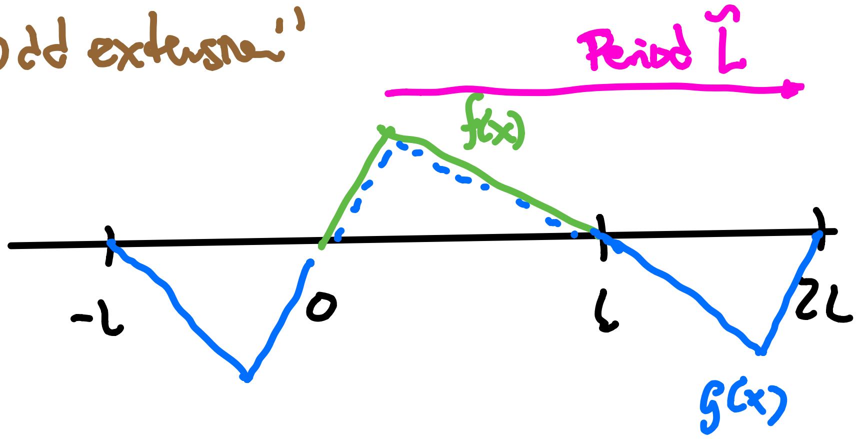
PERIODIC EXTENSIONS



$\tilde{f}(x)$ is a periodic extension of $f(x)$

$$S_\infty^f(x_0) = S_\infty^f(x_0 + nL) \qquad n \in \mathbb{Z}$$

"Odd extension"



$g(x) = f(x)$ in $0 \le x \le L$

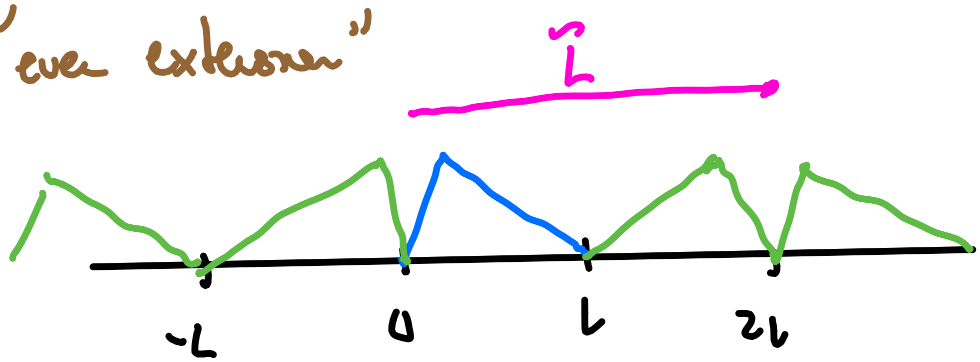
$g(x)$ has period $2L = \tilde{L}$

$g(x)$ will generate S^g_∞ unlike S^f_∞

but it will agree with $f(x)$ in $x \in [0, L]$

It will be a sine series.

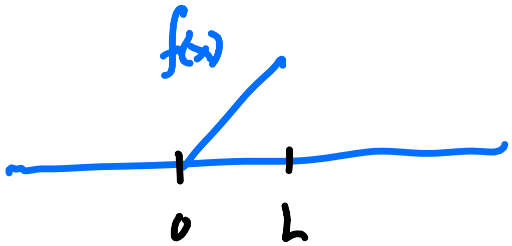
"even extension"



$\tilde{L} = 2L$

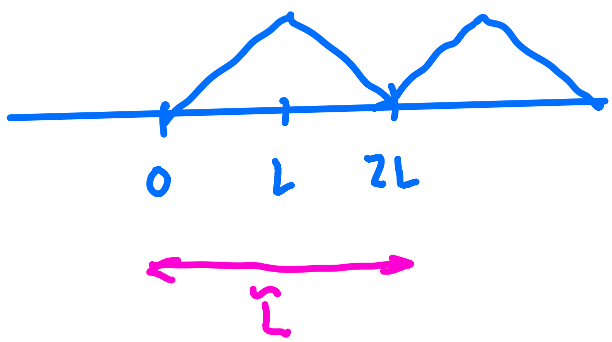
Will have a cosine series

ex)

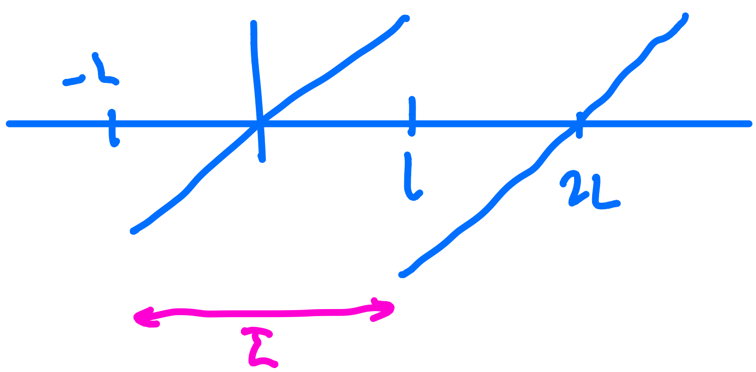


To generate a series that has uniform convergence. Compute the series

for

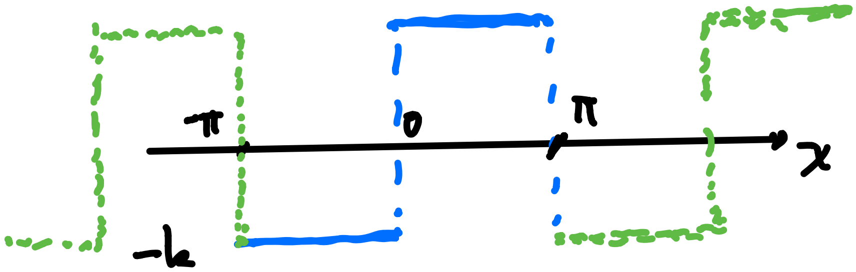


Instead of



ORDER OF CONVERGENCE

Consider a "square wave":



L = 2π
period

$$f(x) = \begin{cases} -k & -\pi < x < 0 \\ k & 0 \leq x \leq \pi \end{cases}$$

It is square integrable:

$$\int_{-\pi}^{\pi} |f(x)|^2 dx = k^2 \int_{-\pi}^{0} dx + k^2 \int_{0}^{\pi} dx = 2\pi k^2 < \infty$$

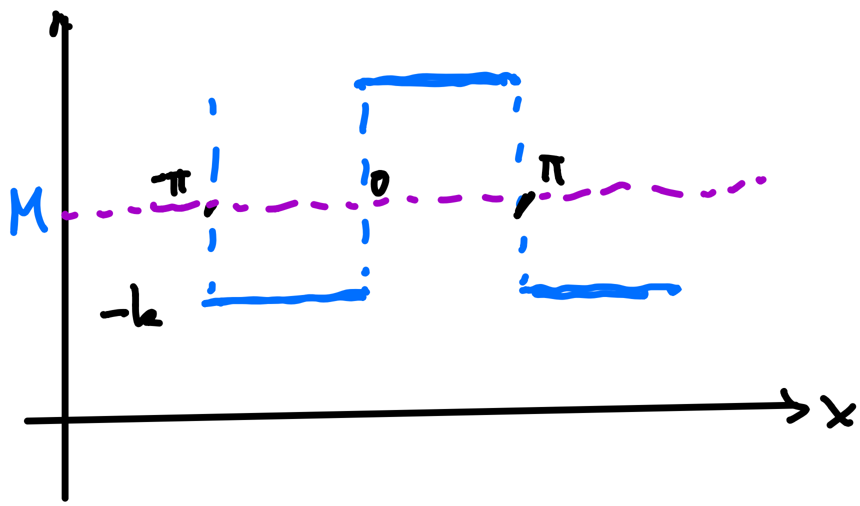
it's piece-wise continuous with a finite number of finite discontinuities. Also,

$$f(x) = -f(-x) \quad \text{odd}$$

we thus expect a sine series.

also, $\langle f \rangle_L = 0 \Rightarrow a_0 = 0$

Aside: S'pse we knew Fourier Series for $f(x)$ as above, what would the Fourier Series of $g(x)$ as below be?



$g(x) = M + f(x)$

so Fourier series of $g(x)$ is the same as for $f(x)$ plus the constant M. //

Compute Fourier Series of $f(x)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx \qquad n = 1, 2 \ldots$$

$$b_n = \frac{2k}{n\pi}(1 - \cos n\pi)$$

$$1 - \cos n\pi = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ even} \end{cases}$$

i.e., $b_1 = \frac{4k}{\pi}$ $b_2 = 0$, $b_3 = \frac{4k}{3\pi}$ etc

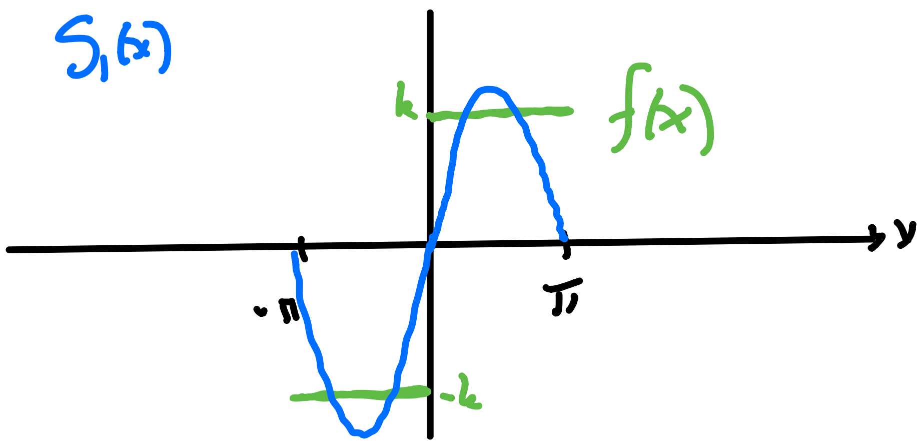
$$(*) \quad f(x) = \sum_{m=0}^{\infty} \frac{4k}{\pi} \frac{1}{(2m+1)} \sin\left[(2m+1)x\right]$$

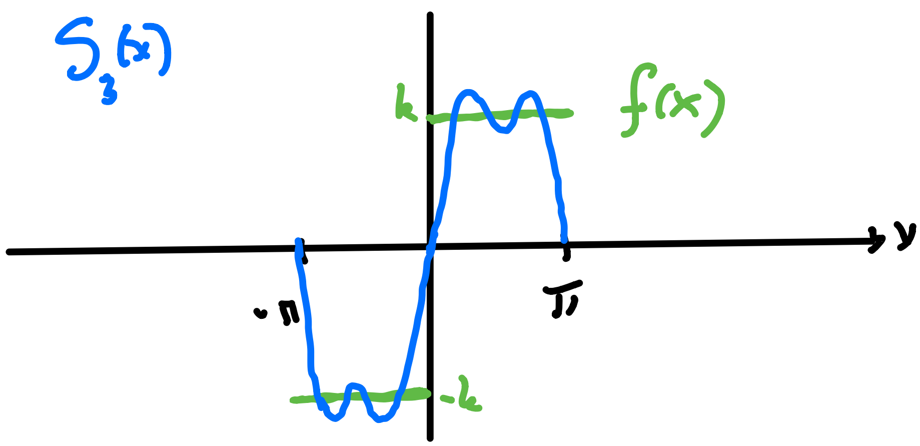
since coefficients drop $\propto \frac{1}{m}$

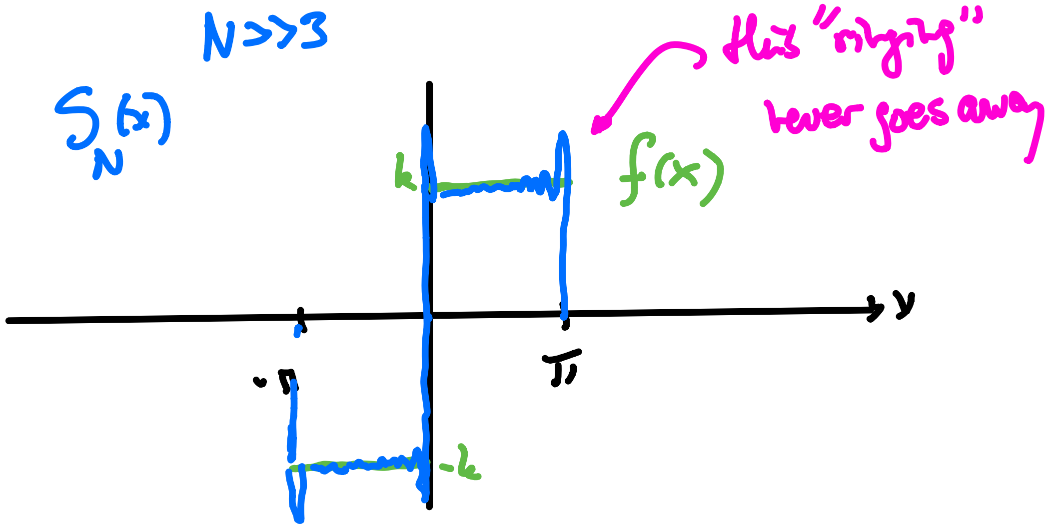
The series convergence is slow, you need 10 times more terms for the error to drop by 10, for example.

If you take a finite Fourier Sum:

(see Mathematica Demo)







The ringing saturates to roughly 18% overshoot as $N \to \infty$

THIS IS CALLED GIBBS PHENOMENON

ex) Take Pulse $k = \frac{\pi}{4}$ and period 2π

From ($\yen$)

$$S_N(x) = \sin x + \frac{1}{3}\sin 3x + \dots \frac{1}{N-1}\sin[(N-1)x]$$

substitute $x = 0$

$$S_N(0) = 0$$

More generally

$$\lim_{N\to\infty} S_N(x=a) = \lim_{\varepsilon\to 0} \frac{f(a+\varepsilon)+f(a-\varepsilon)}{2}$$

Now compute

(*) $$S_N\left(\frac{2\pi}{2N}\right) = \frac{\pi}{2}\left[\frac{2}{N}\operatorname{sinc}\left(\frac{1}{N}\right) + \frac{2}{N}\operatorname{sinc}\left(\frac{3}{N}\right) \right.$$

$$\left. \cdots + \frac{2}{N}\operatorname{sinc}\left(\frac{[N-1]}{N}\right)\right]$$

$\operatorname{sinc}(x) \equiv \frac{\sin x}{x}$ The sinc function

since

$$S_N\left(\frac{2\pi}{2N}\right) = \sin\frac{\pi}{N} + \frac{1}{3}\sin\frac{3\pi}{N} + \cdots \frac{1}{N-1}\sin\left[\frac{[N-1]\pi}{N}\right]$$

(*) is the midpoint numerical quadrature rule of the integral $\int_0^1 \operatorname{sinc} x \, dx$ with spacing $\frac{2}{N}$

$$\therefore \lim_{N\to\infty} S_N\left(\frac{2\pi}{2N}\right) = \frac{\pi}{2}\int_0^1 \operatorname{sinc} x\, dx = \frac{1}{2}\int_0^{\pi} \frac{\sin t}{t}\, dt$$

$$= \frac{\pi}{4} + \frac{\pi}{2}(0.089490\ldots)$$

so about 18% overshoot.

you get similar result at $-\frac{2\pi}{2N}\ldots$ //

Going back to $(\ast)$

$$f(x) = A \sum_{m=0}^{\infty} \frac{1}{(2m+1)} \sin\left[(2m+1)x\right]$$

$$A = \frac{4h}{\pi}$$

we note that each term in the series has

the wavelength $\lambda_n = \frac{2\pi}{n}$ $n = 1, 3, 5 \ldots$ unit of length

the wavenumber $2m+1 = \frac{2\pi}{\lambda_m}$ $m = 0, 1 \ldots$

$$\lambda_m = \frac{2\pi}{2m+1}$$

units of 1/length

So the terms oscillate faster (in x) as $m \to \infty$

and $\lambda_m \searrow 0$.

Also note that each term in series $\propto \dfrac{1}{m}$

So we say this series converges but at a slow rate.

Rmk: What's fast? $\dfrac{1}{m^p}$ $p>1$

A useful calculation concerning rates:

Take $f(x)$ continuous and its derivatives continuous. $f(x)$ is L_2, period, also odd. Period is 2π:

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin n\pi x \, dx$$

Integrate by parts:

$$= \overset{0}{\cancel{-\frac{1}{n\pi} f(x)\cos n\pi x \Big|_{-\pi}^{\pi}}} + \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x) \sin n\pi x \, dx$$

Integrate by parts again:

$$b_n = \cancelto{0}{\frac{f'(x)\sin n\pi x}{n^2 \pi}\Bigg|_{-\pi}^{\pi}} - \frac{1}{n^2 \pi}\int_{-\pi}^{\pi} f''(x)\sin n\pi x \, dx$$

Stopping here we note that $|\sin n\pi x| = 1$

$$|b_n| = \left| \frac{1}{n^2 \pi}\int_{-\pi}^{\pi} f''(x)\sin n\pi x \, dx \right|$$

$$\leq \frac{1}{n^2}\,\frac{1}{\pi}\int_{-\pi}^{\pi} |f''(x)|\,dx \int_{-\pi}^{\pi} |\sin n\pi x|\,dx$$

$$\leq \frac{1}{n^2} M$$

where M is a constant.

$\therefore$ if $f''(x)$ in $x \in [-\pi, \pi]$ is large M is large

but $b_n \propto \frac{1}{n^2}$ indicating series converges "fast"

Fourier Bessel Expansion

Consider

$$(*) \quad x^2 y'' + x y' + (\mu^2 x^2 - \nu^2) y = 0$$

$$\nu \geq 0$$

is of S.L. form, i.e.

$$\left[\begin{array}{lll} p = x & q = -\frac{\nu^2}{x^2} & w = x \\ & \lambda = \mu^2 & \end{array}\right.$$

$$\therefore \quad \mathcal{L}y + \lambda w y = 0 \; .$$

The solution of $(*)$

$$y = \begin{cases} c_1 J_\nu(\mu x) + c_2 J_{-\nu}(\mu x) & \nu \notin \mathbb{Z} \\ c_1 J_\nu(\mu x) + c_2 Y_\nu(\mu x) & \nu \in \mathbb{Z} \end{cases}$$

The J_ν, $J_{-\nu}$ are Bessel functions. They oscillate.

J_ν is bounded at $x = 0$

$J_{-\nu}$ is unbonded at $x=0$
Y_ν is unbounded at $x=0$

Y_ν are Bessel functions of the "second kind", they also oscillate and slowly decay.

So in ODE's with B.C. $y(0)=a$
a number, possibly 0

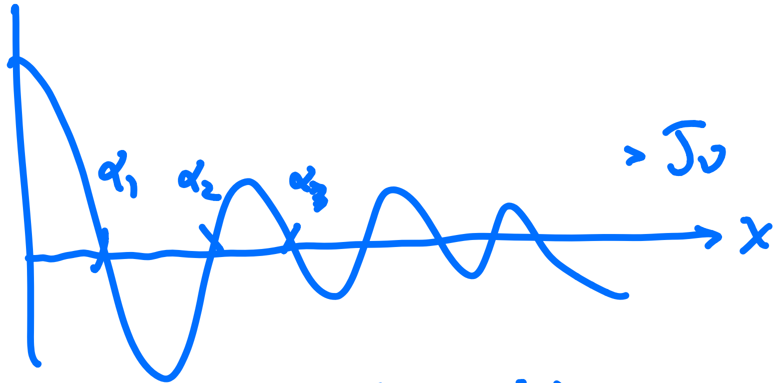
then $y = c_1 J_\nu(\mu x)$

since $J_{-\nu}(0)$ & $Y_\nu(0)$ are unbounded

i.e. $c_2 = 0$

(c) if B.C. $y(l)=0$

$J_\nu(\alpha_n) = 0 \quad \mu_n = \frac{\alpha_n}{l} \quad n=1,2\ldots$



α_s are zeros of $J_\nu(x)$

if B.C. $y'(l)=0$

$$J_\nu'(\beta_n)=0 \qquad \mu_n = \frac{\beta_n}{l} \quad n=1,2\ldots$$

β_n are roots of $J_\nu(x)$

if B.C. $k\,y(l) + l\,y'(l) = 0 \quad k \geq 0$

$$k\,J_\nu(\gamma_n) + \gamma_n\,J_\nu(\gamma_n) = 0$$

$$\mu_n = \frac{\gamma_n}{l}$$

The $J_\nu, J_{-\nu}, Y_\nu$ are orthogonal families of functions with respect to the weight $w(x) = x$ for $0 \le x \le l$

ex) Suppose $f(x)$ is defined $0 \le x \le l$

$f(x)$ is finite at $x=0$

then $$f(x) = \sum_{n=1}^{\infty} c_n J_\nu(\mu_n x)$$

$$c_n = \frac{1}{Q} \int_0^l x f(x) J_\nu(\mu_n x)\, dx$$

$$Q = \int_0^l x J_\nu^2(\mu_n x)\, dx$$

//